The distance-$t$ chromatic index of graphs

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Workshop on Graphs and Matroids
Problem definition

Let $G = (V, E)$ be a (simple) graph.

The distance between two edges in $G$ is the number of vertices in a shortest path between them, i.e. distance in the line graph $L(G)$ of $G$. (So adjacent edges have distance 1.)

A distance-$t$ matching of $G$ is a set of edges no two of which are within distance $t$ in $G$. 

$t = 1$
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A distance-$t$ matching of $G$ is a set of edges no two of which are within distance $t$ in $G$. 

$t = 2$
A distance-$t$ edge-colouring is an assignment of colours to edges of $G$ such that each colour class induces a distance-$t$ matching.

The distance-$t$ chromatic index $\chi'_t(G)$ of $G$ is the least integer $k$ such that there exists a distance-$t$ edge-colouring of $G$ using $k$ colours.

Remarks:

- $\chi'_1(G)$ is the chromatic index $\chi'(G)$ of $G$.
- A distance-2 matching is an induced matching and so $\chi'_2(G)$ is the strong chromatic index $s\chi'(G)$ of $G$.
- $\chi'_t(G) = \chi((L(G))^t)$ where $(L(G))^t$ is the $t^{th}$ power of the line graph.
A proposed practical motivation for $\chi'_t$:

- Timeslot assignment (TDMA) for wireless sensor networks.
  - Each matching in the colouring corresponds to a set of simultaneous pairwise transmissions among sensors in a particular timeslot.
  - The distance requirement models the range of network interference that results from transmission between two sensors.
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Scope of current work

Two main settings (with $\Delta$ large):

1. $\chi'_t(G)$ for graphs $G$ of maximum degree $\Delta$:

   $$\chi'_t(\Delta) := \max\{\chi'_t(G) : \Delta(G) \leq \Delta\}.$$
Scope of current work

Two main settings (with $\Delta$ large):

1. $\chi'_t(G)$ for graphs $G$ of maximum degree $\Delta$:

   $$\chi'_t(\Delta) := \max\{\chi'_t(G) : \Delta(G) \leq \Delta\}.$$

2. $\chi'_t(G)$ when $G$ is also prescribed to have girth at least $g$:

   $$\chi'_t(\Delta, g) := \max\{\chi'_t(G) : \Delta(G) \leq \Delta, \text{girth}(G) \geq g\};$$

   particularly, when does $\chi'_t(\Delta, g)$ becomes $o(\chi'_t(\Delta))$ in terms of $g$?
Background

$t = 1$.

Vizing's Theorem implies that $\chi'_1(\Delta) = \Delta + 1$ and $\chi'_1(\Delta, g) \geq \Delta$ for all $g$. 
Background

$t = 2$.

Erdős and Nešetřil proposed the problem of determining $\chi'_2(\Delta)$ in 1985. They suggested as extremal the multiplied 5-cycle $\implies \chi'_2(\Delta) \geq 1.25 \Delta^2$. Molloy and Reed (1997) showed $\chi'_2(\Delta) \leq 1.998 \Delta^2$ for large enough $\Delta$. 

NB: Faudree, Gyárfás, Schelp, Tuza (1990) conjectured $\chi'_2(\Delta, 4) = \Delta^2$. Mahdian (2000) showed $\chi'_2(\Delta, 5) = \mathcal{O}(\Delta^2/\log \Delta)$ (and in fact the stronger result for all $C_4$-free graphs). A probabilistic construction shows $\chi'_2(\Delta, g) = \Omega(\Delta^2/\log \Delta)$ for all $g \geq 5$. 

R. J. Kang (CWI)
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The complete bipartite graphs $K_{\Delta,\Delta} \implies \chi'_2(\Delta, 4) \geq \Delta^2$.

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A probabilistic construction shows $\chi'_2(\Delta, g) = \Omega(\Delta^2 / \log \Delta)$ for all $g \geq 5$. 
A table for $\chi'_t(\Delta)$ and $\chi'_t(\Delta, g)$ ($\Delta$ large)

<table>
<thead>
<tr>
<th>$t \backslash g$</th>
<th>3 (lower/upper)</th>
<th>4</th>
<th>5</th>
<th>$\ldots$</th>
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<tbody>
<tr>
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A distance-$t$ version of the Erdős-Nešetřil problem

Consider the following upper bound:

$$\chi'_t(\Delta) \leq 1 + \Delta((L(G))^t) \leq 1 + 2 \sum_{j=1}^{t} (\Delta - 1)^j < 2\Delta^t.$$ 

**Problem**

*For each $t \geq 3$, is $\limsup_{\Delta \to \infty} \chi'_t(\Delta)/\Delta^t$ less than $2 - \varepsilon$ for some $\varepsilon > 0$?*

NB: Molloy and Reed solved the $t = 2$ case with $\varepsilon > 0.002$.

We next show $\limsup_{\Delta \to \infty} \chi'_t(\Delta)/\Delta^t$ is positive for every fixed $t \geq 3$. 
Two constructive lower bounds

**Proposition (K and Manggala)**

For arbitrarily large $\Delta$, there exists a bipartite, $\Delta$-regular graph of girth 6 such that $\chi'_3(G) = \Delta^3 - \Delta^2 + \Delta$.

$t = 3$, $\Delta = 3$: point-line incidence graph of the Fano plane.
Two constructive lower bounds

Proposition (K and Manggala)

Fix $t \geq 2$. For arbitrarily large $\Delta$, there exists a $\Delta$-regular graph such that $\chi'_t(G) > \Delta^t / (2(t - 1)^{t-1})$.

$t = 4, \Delta = 6$. 
A table for $\chi'_t(\Delta)$ and $\chi'_t(\Delta, g)$ ($\Delta$ large)

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<tr>
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Main theorem I

**Theorem (Kaiser and K)**

For each $t \geq 2$, $2 - \limsup_{\Delta \to \infty} \frac{\chi'_t(\Delta)}{\Delta^t} \geq 0.00008$.

I.e. the $t$-E-N problem affirmed with a *uniform* choice of $\varepsilon$ for all $t$. 
Main theorem I: proof idea

**Theorem (Kaiser and K)**

For each \( t \geq 2 \),
\[
2 - \limsup_{\Delta \to \infty} \frac{\chi'_t(\Delta)}{\Delta^t} \geq 0.00008.
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This relies on colouring graphs with sparse neighbourhood counts.
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For each $t \geq 2$, $2 - \limsup_{\Delta \to \infty} \frac{\chi'_t(\Delta)}{\Delta^t} \geq 0.00008$.

This relies on colouring graphs with sparse neighbourhood counts.

**Lemma (Molloy and Reed (1997))**

Let $\delta, \varepsilon > 0$ be such that $\varepsilon < \frac{\delta}{2(1-\varepsilon)} e^{-\frac{3}{1-\varepsilon}}$ and let $\hat{\Delta}_0$ be large enough. If $\hat{G} = (\hat{V}, \hat{E})$ is a graph with maximum degree at most $\hat{\Delta} \geq \hat{\Delta}_0$ such that at most $(1 - \delta)\left(\frac{\hat{\Delta}}{2}\right)$ edges span each $N(\hat{v}), \hat{v} \in \hat{V}$, then $\chi(\hat{G}) \leq (1 - \varepsilon)\hat{\Delta}$.

Thus the $t$-E-N problem can be resolved by showing neighbourhood counts in $(L(G))^t$ with $\Delta(G) \leq \Delta$ are at most $(1 - \delta) \cdot 2\Delta^{2t}$. 
Main theorem I: proof idea

Assume $G = (V, E)$ is $\Delta$-regular. Let $e \in E$ be arbitrary.
Set $\hat{N} := N_{L(G)}(e)$.

Set $\hat{S} := E(L(G)^t[\hat{N}])$ and, for contradiction, assume $|\hat{S}| > (1 - \delta) \cdot 2\Delta^{2t}$.
Main theorem 1: proof idea

Assume $G = (V, E)$ is $\Delta$-regular. Let $e \in E$ be arbitrary. Set $\hat{N} := N_{L(G)}(e)$.

Consider $\tau(e, f) := \max\{0, (\# ef\text{-walks with } \leq t + 1 \text{ edges}) - 1\}$.

Set $\hat{S} := E(L(G)^t[\hat{N}])$ and, for contradiction, assume $|\hat{S}| > (1 - \delta) \cdot 2\Delta^{2t}$.

Claim

$$\sum_{e,f \in \hat{N}} \tau(e, f) + \text{Esc} < 4\delta \cdot \Delta^{2t}.$$
Main theorem I: proof idea

Claim
\[ \sum_{e,f \in \hat{N}} \tau(e,f) + \text{Esc} < 4\delta \cdot \Delta^{2t}. \]

Set
\[ A^* := A_1 \cup \cdots \cup A_{t-1} \cup B_t, \]
\[ \sigma_t(u,v) := \#uv\text{-walks with } \leq t \text{ edges and first edge in } \hat{N}. \]

Claim
\[ \sum_{u,v \in A^*} \sigma_t(u,v) > \alpha \cdot \Delta^{2t-1}. \]
Main theorem 1: $t = 3$

For $t = 3$, we can extend the argument of Molloy and Reed for $t = 2$, which applies Jensen’s Inequality twice for a lower bound on the number of $C_4$s in $N_{(L(G))^3}(e)$, $\forall e \in V$.

Theorem (Kaiser and K)

$$2 - \limsup_{\Delta \to \infty} \chi'_3(\Delta)/\Delta^3 \geq 0.0002.$$
A table for $\chi'_t(\Delta)$ ($\Delta$ large)

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Remarks:
- The general proof gives an alternative solution to the E-N problem, albeit with a much weaker constant.
- It remains possible that $\limsup_{\Delta \to \infty} \chi'_t(\Delta)/\Delta^t = o(1)$ as $t \to \infty$. 
Main theorem II

Theorem (Kaiser and K)

For $t \geq 2$, all graphs $G$ of girth at least $2t + 1$ and maximum degree at most $\Delta$ have $\chi'_t(G) = O(\Delta^t / \log \Delta)$. 
Main theorem II

Theorem (Kaiser and K)

For $t \geq 2$, all graphs $G$ of girth at least $2t + 1$ and maximum degree at most $\Delta$ have $\chi'_t(G) = O(\Delta^t / \log \Delta)$.

By a probabilistic construction, this bound is tight up to a constant factor dependent upon $t^1$.

Proposition (Kaiser and K)

There is a function $f = f(\Delta, t) = (1 + o(1))\Delta^t / (t \log \Delta)$ (as $\Delta \to \infty$) such that, for every $g \geq 3$ and every $\Delta$, there is a graph $G$ of girth at least $g$ and maximum degree at most $\Delta$ with $\chi'_t(G) \geq f(\Delta, t)$.

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$^1$If girth at least $3t - 2$, the upper bound can be strengthened to $O(\Delta^t / (t \log \Delta))$. 

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A table for $\chi'_t(\Delta, g)$ ($\Delta$ large)

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Open problems

1. Is there some $\varepsilon > 0$ such that $\limsup_{\Delta \to \infty} \chi'_t(\Delta)/\Delta^t \geq \varepsilon$ for all $t$?

2. Is it true that $\limsup_{\Delta \to \infty} \chi'_t(\Delta, 2t)/\Delta^t > 0$ for all $t \geq 4$?
Thank you!