DG-ALGEBRAS AND DERIVED A_∞-ALGEBRAS

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INITIAL QUESTION

Let A be a differential graded algebra over a commutative ring k, possibly unbounded and with homological grading.

Its homology $H_*(A)$ is a graded k-algebra.

QUESTION

Is there some additional structure on $H_*(A)$ which allows us to recover the quasi-isomorphism type of A from $H_*(A)$?

If k is a field (or, more generally, $H_*(A)$ is k-projective), a minimal A_∞-algebra structure on $H_*(A)$ provides an answer.

A_∞-ALGEBRAS

DEFINITION (STASHEFF)

An A_∞-algebra is a graded k-module A together with a unit element $1_A \in A_0$ and k-linear maps $m_j : A^{\otimes j} \to A[2-j]$ for $j \geq 1$, satisfying

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0$$

for $n \geq 1$ (and a unit condition).

- A map $f : A \to B$ of A_∞-algebras is a family of k-linear maps $f_j : A^{\otimes j} \to B[1-j]$ satisfying appropriate relations.
- An A_∞-algebra is *minimal* if $m_1 = 0$.
- dgas are A_∞-algebras with m_1 the differential, m_2 the multiplication, and all other $m_j = 0$.

Minimal models

Theorem (Kadeishvili)

Let A be a dga with $H_*(A)$ k-projective. There exist a minimal A_∞-algebra structure on $H_*(A)$ and a quasi-isomorphism $f : H_*(A) \to A$, where the m_2 of $H_*(A)$ is the algebra multiplication.

One can recover the quasi-isomorphism type of A from the minimal A_∞-algebra $H_*(A)$.

In other words: The higher multiplications $m_j, j \geq 3$, on the graded algebra $(H_*(A), m_2)$ provide the data we are asking for.

Resolutions

The statement of Kadeishvili's theorem does in general not hold if $H_*(A)$ is not k-projective.

What to do instead?

Look for higher multiplications on a resolution of $H_*(A)$!

We consider $(\mathbb{N}, \mathbb{Z})$-bigraded k-modules with the $\mathbb{N}$-grading the 'horizontal' direction and the $\mathbb{Z}$-grading the 'vertical' direction.

For E and F bigraded k-modules,

- $E[st]_{ij} = E_{i-s,t-j}$
- $(E \otimes F)_{uv} = \bigoplus_{\substack{i+p=u \\ j+q=v}} E_{ij} \otimes_k E_{pq}$

Definition of dA_∞-algebras

Definition

A *derived A_∞-algebra* (or *dA_∞-algebra*) is a $(\mathbb{N}, \mathbb{Z})$-bigraded k-module E with a unit element $1_E \in E_{0,0}$ and structure maps

$$m_{ij} \colon E^{\otimes j} \to E[i, 2-(i+j)] \text{ with } i \geq 0, j \geq 1$$

satisfying

$$\sum_{\substack{i+p=u \\ r+q+t=v \\ r+1+t=j}} (-1)^{rq+t+pj} m_{ij}(\mathbf{1}^{\otimes r} \otimes m_{pq} \otimes \mathbf{1}^{\otimes t}) = 0$$

for all $u \geq 0$ and $v \geq 1$ (and a unit condition).

A dga may be viewed as a dA_∞-algebra concentrated in horizontal degree 0, with m_{01} the differential, m_{02} the multiplication, and all other $m_{ij} = 0$.

Structure maps & Formulas for dA_∞-algebras

A dA_∞-algebra has structure maps starting with:

$m_{01}\colon E \to E[0,1]$ $\quad$ $m_{11}\colon E \to E[1,0]$ $\quad$ $m_{21}\colon E \to E[2,-1]$

$m_{02}\colon E^{\otimes 2} \to E$ $\quad$ $m_{12}\colon E^{\otimes 2} \to E[1,-1]$

$m_{03}\colon E^{\otimes 3} \to E[0,-1]$

The first six relations are:

$$
\begin{aligned}
m_{01}m_{01} &= 0\\
m_{01}m_{02} &= m_{02}(\mathbf{1}\otimes m_{01}) + m_{02}(m_{01}\otimes \mathbf{1})\\
m_{01}m_{11} &= m_{11}m_{01}\\
m_{02}(m_{02}\otimes \mathbf{1}) &= m_{01}m_{03} + m_{02}(\mathbf{1}\otimes m_{02}) + m_{03}(m_{01}\otimes \mathbf{1}^{\otimes 2})\\
&\quad + m_{03}(\mathbf{1}\otimes m_{01}\otimes \mathbf{1}) + m_{03}(\mathbf{1}^{\otimes 2}\otimes m_{01})\\
m_{11}m_{02} &= m_{01}m_{12} + m_{12}(\mathbf{1}\otimes m_{01}) + m_{12}(m_{01}\otimes \mathbf{1})\\
&\quad + m_{02}(\mathbf{1}\otimes m_{11}) + m_{02}(m_{11}\otimes \mathbf{1})\\
m_{11}m_{11} &= m_{01}m_{21} + m_{21}m_{01}
\end{aligned}
$$

Some terminology for dA_∞-algebras

- A map of dA_∞-algebras is a family of k-module maps $f_{ij}\colon E^{\otimes j} \to E[i, 1-(i+j)]$ satisfying appropriate relations.
- A dA_∞-algebra is *minimal* if $m_{01} = 0$.
- A map $f\colon E \to F$ is an E_2*-equivalence* if it induces isomorphisms in the iterated homology with respect to m_{01} and m_{11}. This is possible since we require

$$m_{01}m_{01} = 0 \quad \text{and} \quad m_{01}m_{21} + m_{21}m_{01} = m_{11}m_{11}.$$

Main theorems

Theorem 1

Let A be a dga over a commutative ring k. There exists a k-projective minimal dA_∞-algebra E together with an E_2-equivalence $E \to A$ of dA_∞-algebras.

- This *minimal dA_∞-algebra model E* of A is well defined up to E_2-equivalences between k-projective minimal dA_∞-algebras.
- (E, m_{11}, m_{02}) is a k-projective resolution of the graded k-algebra $H_*(A)$.

Theorem 2

The quasi-isomorphism type of A can be recovered from E.

Proof of Theorem 1 (sketch)

Definition

A bidga B is a dA_∞-algebra with $m_{ij}^B = 0$ if $i + j \geq 3$. Maps of bidgas have $f_{ij} = 0$ for $i + j \geq 2$.

- Equivalently: Monoids in the category of bicomplexes.
- dgas are bidgas concentrated in horizontal degree 0

1st step of proof

Given A, there is a bidga B and an E_2-equivalence $B \to A$ of bidgas such that $H_*(B, m_{01}^B)$ is k-projective.

2nd step of proof

Set $E = H_*(B, m_{01}^B)$. There exist a minimal dA_∞-structure on E and an E_2-equivalence $E \to B$.

Application and example: Ext-algebras

Let M be a k-module and P be a k-projective resolution of M.

The endomorphism dga $A = \mathrm{Hom}_k(P, P)$ of P has homology

$$H_*(A) = \mathrm{Ext}_k^{-*}(M, M).$$

A minimal dA_∞-algebra model of A is a resolution of the Yoneda algebra $\mathrm{Ext}_k^*(M, M)$ together with structure maps m_{ij}. This data encodes the quasi-isomorphism type of the endomorphism dga.

Example

Let $k = \mathbb{Z}$ and $M = \mathbb{Z}/p$. Then $H_*(A) = \Lambda^*_{\mathbb{Z}/p}(w)$ with $|w| = -1$.

EXAMPLE: $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p)$

Let $E = \Lambda^*_{\mathbb{Z}}(a, b)$ with $|a| = (0, -1)$ and $|b| = (1, 0)$.

$$
\begin{array}{cccccc}
 & & & 0 & & 1 \\
0 & \mathbb{Z}/p\{\iota\} & \dashleftarrow & \mathbb{Z}\{\iota\} & \xleftarrow{\cdot p} & \mathbb{Z}\{b\} \\
-1 & \mathbb{Z}/p\{w\} & \dashleftarrow & \mathbb{Z}\{a\} & \xleftarrow{\cdot p} & \mathbb{Z}\{ab\}
\end{array}
$$

- The given data specifies the m_{11} and m_{02} of a minimal dA_∞-algebra.
- m_{12} satisfies $m_{12}(a \otimes b) = \iota$, $m_{12}(a \otimes ab) = a$, $m_{12}(ab \otimes b) = -b$, $m_{12}(ab \otimes ab) = -ab$ and vanishes on the other generators of $E \otimes E$.
- All other m_{ij} vanish.

This is a complete description of a minimal dA_∞-model for A.

Derived Hochschild cohomology class

Let A be a dga with minimal dA_∞-model E. The complex

$$C^{qt}(E) = \bigoplus_{r+s=q} \mathrm{Hom}_k(E^{\otimes r}, E[s,t])$$

has a differential $C^{qt} \to C^{q+1,t}$ induced from m_{11} and a Hochschild differential. Its cohomology is the *derived Hochschild cohomology* $\mathrm{dHH}^{qt}(H_*(A))$.

Proposition

$\gamma_A := [m_{03} + m_{12} + m_{21}] \in \mathrm{dHH}^{3,-1}(H_*(A))$ *is a well defined cohomology class depending only on the quasi-isomorphism type of A.*

If $H_*(A) = 0$ for $* < 0$, then

$$\mathrm{dHH}^{3,-1}(H_*(A)) \to \mathrm{dHH}^3(H_0(A), H_1(A))$$

maps γ_A to the first k-invariant of A.

Twisted chain complexes

Definition

A *twisted chain complex* E is an $(\mathbb{N}, \mathbb{Z})$-graded k-module with differentials $d_i^E : E \to E[i, 1-i]$ for $i \geq 0$ satisfying

$$\sum_{i+p=u} (-1)^i d_i d_p = 0 \text{ for } u \geq 0.$$

Maps are families of k-module maps $f_i : E \to F[i, -i]$ satisfying $\sum_{i+p=u} (-1)^i f_i d_p^E = \sum_{i+p=u} d_i^F f_p$.
Composition of maps: $(gf)_u = \sum_{i+p=u} g_i f_p$

Slogan

A_∞-algebras	$\leftrightarrow$	chain complexes
dA_∞-algebras	$\leftrightarrow$	twisted chain complexes

If E is a dA_∞-algebra, then $(E, m_{i1}, i \geq 0)$ is a twisted chain complex.

dA_∞-STRUCTURES AND THE TENSOR COALGEBRA

Let E be a dA_∞-algebra, $SE = E[0,1]$, and $\overline{T}SE = \bigoplus_{j\geq 1} SE^{\otimes j}$

$$
\begin{aligned}
m_{ij} \quad &\leftrightarrow \quad \widetilde{m}^1_{ij} \colon SE^{\otimes j} \to SE[i, 1-i] \\
&\leftrightarrow \quad \widetilde{m}^q_{ij} = \sum_{\substack{r+1+t=q \\ r+s+t=j}} \mathbf{1}^{\otimes r} \otimes \widetilde{m}^1_{is} \otimes \mathbf{1}^{\otimes t} \colon SE^{\otimes j} \to SE^{\otimes q}[i, 1-i] \\
&\leftrightarrow \quad \widetilde{m}_i \colon \overline{T}SE \to \overline{T}SE[i, 1-i] \text{ with components } \widetilde{m}^q_{ij}.
\end{aligned}
$$

LEMMA

dA_∞-relations $\Leftrightarrow$ $(\overline{T}SE, \widetilde{m}_i)$ is a twisted chain complex

LEMMA

dA_∞-algebra maps $E \to F$ correspond to maps $(\overline{T}SE, \widetilde{m}^E_i) \to (\overline{T}SF, \widetilde{m}^F_i)$ of twisted chain complexes

Proof of Theorem 2 (sketch)

- The category of modules over a dA_∞-algebra E is enriched in twisted chain complexes.
- The endomorphism object $\underline{\mathrm{Hom}}_E(E, E)$ is a monoid in tCh_k.
- $\mathrm{Tot}\,\underline{\mathrm{Hom}}_E(E, E)$ is a dga.
- If E has E_2-homology concentrated in horizontal degree 0, there is an E_2-equivalence $E \to \mathrm{Tot}\,\underline{\mathrm{Hom}}_E(E, E)$.
- If E and F are E_2-quasi-isomorphic, then $\mathrm{Tot}\,\underline{\mathrm{Hom}}_E(E, E)$ and $\mathrm{Tot}\,\underline{\mathrm{Hom}}_F(F, F)$ are quasi-isomorphic as dgas.

Proof of Theorem 2.

Apply the last statement to $E \to A$. □

In other words:

The dA_∞-algebras with E_2-homology concentrated in horizontal degree 0 model quasi-isomorphism types of dgas.