GRADED LOG SYMMETRIC RING SPECTRA

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Plan for the talk

1. Review Rognes' definition of *log symmetric ring spectra*, that is, commutative symmetric ring spectra with logarithmic structures.
2. Give a model category framework for log symmetric ring spectra and explain what it is good for.
3. Introduce a new notion of *graded units* for ring spectra that is sensitive to units in non-zero degrees of the homotopy groups (joint with Christian Schlichtkrull).
4. Extend (1) (and (2)) to *graded log symmetric ring spectra* which are better suited for the study of periodic ring spectra.

Log rings

Definition

A *pre-log ring* (A, M, α) is a commutative ring A together with a commutative monoid M and a monoid homomorphism $\alpha\colon M \to (A, \cdot)$.

- We often just write (A, M) for (A, M, α).
- Example: The trivial pre-log ring $(A, 1)$.
- Example: The canonical pre-log ring $(\mathbb{Z}[M], M)$.

Definition

A pre-log ring (A, M, α) is a *log ring* if the map $\widetilde{\alpha}$ in the pullback square

$$\begin{array}{ccc} \alpha^{-1}(\mathrm{GL}_1 A) & \xrightarrow{\widetilde{\alpha}} & \mathrm{GL}_1 A \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & (A, \cdot) \end{array}$$

is an isomorphism.

- Example: The trivial log ring $(A, \mathrm{GL}_1 A)$.

THE ASSOCIATED LOG RING

EXAMPLE

If (B, N, β) is a log ring and $f\colon A \to B$ is a ring homomorphism, the pullback

$$\begin{array}{ccc} f_*N & \longrightarrow & (A, \cdot) \\ \big\downarrow & & \big\downarrow{\scriptstyle (f,\cdot)} \\ N & \xrightarrow{\beta} & (B, \cdot) \end{array}$$

defines the *direct image log ring* (A, f_*N).

CONSTRUCTION

For a pre log ring (A, M), the *associated log ring* (A, M^a) has

$$M^a = M \coprod_{\alpha^{-1}\mathrm{GL}_1 A} \mathrm{GL}_1(A) \to (A, \cdot).$$

The resulting *logification functor* comes with a natural map $(A, M) \to (A, M^a)$. Logification is left adjoint to the forgetful functor from log rings to pre-log rings.

Example: Discrete valuation rings

Example

Let A be a discrete valuation ring with uniformizer π and fraction field K. Writing $M = \langle \pi \rangle$ for the free commutative monoid on π, we get a pre-log ring (A, M). The associated log ring (A, M^a) has

$$M^a = \langle \pi \rangle \times \mathrm{GL}_1 A \cong A \setminus \{0\} \hookrightarrow (A, \cdot)$$

- Embedded in log rings, the localization map $A \to K$ factors as
  $$(A, \mathrm{GL}_1 A) \to (A, M^a) \to (K, GL_1 K)$$
  through a non-trivial log ring.
- (A, M^a) is the direct image log ring of $(K, \mathrm{GL}_1 K)$ along $A \to K$.
- We are interested in topological analogs of the middle term, and their log-THH, log-TAQ, log-TC, and log algebraic K-theory. (See John Rognes' talk.)

THE RINGS VS. RING SPECTRA DICTONARY

ALGEBRA	HOMOTOPY THEORY
commutative ring A	commutative symmetric ring spectrum A
commutative monoid M	commutative $\mathcal{I}$-space monoid M (aka E_∞ space)
multiplikative monoid $(A, \cdot)$ of a commutative ring A	commutative $\mathcal{I}$-space monoid $\Omega^{\mathcal{I}}(A)$ associated with A (aka multiplicative E_∞ space)
units $\mathrm{GL}_1 A = A^\times \subset (A, \cdot)$	invertible path components $\mathrm{GL}_1^{\mathcal{I}} A \subset \Omega^{\mathcal{I}}(A)$ with $\pi_0(\mathrm{GL}_1^{\mathcal{I}} A) \cong \mathrm{GL}_1(\pi_0 A)$

Log symmetric ring spectra

Definition (Rognes)

A *pre-log symmetric ring spectrum* (A, M) is a commutative symmetric ring spectrum A together with a commutative $\mathcal{I}$-space monoid M and a map of commutative $\mathcal{I}$-space monoids $\alpha\colon M \to (A, \cdot)$.

We write $\mathcal{CL}\mathrm{Sp}^{\Sigma}_{pre}$ for the resulting category.

Definition (Rognes)

A pre-log symmetric ring spectrum (A, M) is a *log symmetric ring spectrum* if the map $\widetilde{\alpha}$ in the (homotopy) pullback square

$$\begin{array}{ccc} \alpha^{-1}(\mathrm{GL}_1^{\mathcal{I}} A) & \xrightarrow{\widetilde{\alpha}} & \mathrm{GL}_1^{\mathcal{I}} A \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & \Omega^{\mathcal{I}} A \end{array}$$

is an $\mathcal{I}$-equivalence.

Examples for log symmetric ring spectra

- If (A, M) is a log ring in the algebraic sense, the Eilenberg-Mac Lane spectrum HA and the constant discrete commutative $\mathcal{I}$-space monoid cM form a log symmetric ring spectrum (HA, cM).
- One can form direct (and inverse) image log structures as before.
- Let A be a commutative symmetric ring spectrum. Any map $K \to (\Omega^{\mathcal{I}} A)(\mathbf{n})$ of spaces can be extended to a map of commutative $\mathcal{I}$-space monoids

  $$\mathbb{C}F_{\mathbf{n}}^{\mathcal{I}} K \to \Omega^{\mathcal{I}} A$$

  from the free commutative $\mathcal{I}$-space monoid on K. One can consider the logification of this pre-log structure.
- Special case: Pre-log structures generated by a map $a\colon S^k \to (\Omega^{\mathcal{I}} A)(\mathbf{n})$ representing $[a] \in \pi_k A$.

The pre-log model structure

Since

$$\mathbb{S}^{\mathcal{I}}[-]\colon \mathcal{C}\mathcal{S}^{\mathcal{I}} \rightleftarrows \mathcal{C}\mathrm{Sp}^{\Sigma}\colon \Omega^{\mathcal{I}}$$

is a Quillen adjunction and $\mathcal{C}\mathcal{L}\mathrm{Sp}^{\Sigma}_{pre}$ is a comma category, we get an *injective level model structure* on $\mathcal{C}\mathcal{L}\mathrm{Sp}^{\Sigma}_{pre}$. A map $(f, f^{\flat})\colon (B, N) \to (A, M)$ given by

$$\begin{array}{ccc} B & & N \longrightarrow \Omega^{\mathcal{I}} B \\ {\scriptstyle f}\downarrow & & {\scriptstyle f^{\flat}}\downarrow \qquad\quad \downarrow{\scriptstyle \Omega^{\mathcal{I}} f} \\ A & & M \longrightarrow \Omega^{\mathcal{I}} A \end{array}$$

is a

- cofibration (resp. weak equivalence) if f and $f^{\flat}$ are cofibrations (resp. weak equivalences)
- fibration if f and $N \to M \times_{\Omega^{\mathcal{I}} A} \Omega^{\mathcal{I}} B$ are fibrations.

So (B, N) is fibrant if B is and $N \to \Omega^{\mathcal{I}} B$ is a fibration. If (B, N) is fibrant, $\widetilde{\alpha}\colon \alpha^{-1}(\mathrm{GL}_1^{\mathcal{I}} A) \to \mathrm{GL}_1^{\mathcal{I}} A$ is a fibration.

The log model structure

Proposition

The model category $\mathcal{CLS}p^{\Sigma}_{pre}$ can be localized to get a new model category $\mathcal{CLS}p^{\Sigma}_{log}$ with the same cofibrations.

- *The fibrant objects are the log symmetric ring spectra and*
- *the fibrant replacement is logification.*

Unfortuneately, it turns out that this localization is to weak for many purposes: It allows too many commutative $\mathcal{I}$-space monoids, or too many fibrations between those.

Exact maps of monoids

In logarithmic geometry, the focus is often on monoids M with special properties, like

- M finitely generated or
- $M \to M^{gp}$ injective (M is *integral*).

It is not clear what the topological analogs should be.

The situation is better in the relative context:

Definition

A map $N \to M$ of commutative monoids is exact if

$$\begin{array}{ccc} N & \longrightarrow & N^{gp} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{gp} \end{array}$$

is a pullback square.

Replete commutative $\mathcal{I}$-space monoids over M

Definition (Rognes)

- A map $\varepsilon \colon N \to M$ in $\mathcal{CS}^{\mathcal{I}}$ is *virtual surjective* if $\pi_0(\varepsilon^{gp}) \colon \pi_0(N^{gp}) \to \pi_0(M^{gp})$ is surjective.
- It is *replete* if it is virtual surjective and

$$\begin{array}{ccc} N & \longrightarrow & N^{gp} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M^{gp} \end{array}$$

  is a homotopy pullback square.

- A virtual surjective $N \to M$ has a *repletion* $N \to N^{rep} \to M$ with $N^{rep} \to M$ replete and $N^{gp} \to (N^{rep})^{gp}$ an $\mathcal{I}$-equivalence.
- Virtual surjectivity ensures that repletion gives something replete. It always holds in augmented contexts.
- Repletion is crucial for log-THH and log-TAQ.

Group completion model structure

We already used:

Definition

A commutative $\mathcal{I}$-space monoid M is *group complete* if the commutative monoid $\pi_0(M)$ is a group.

Proposition

The positive $\mathcal{I}$*-local model structure on* $\mathcal{CS}^{\mathcal{I}}$ *can be localized to get the* group completion model structure $\mathcal{CS}^{\mathcal{I}}_{gp}$ *with*

- *fibrant objects the group complete positive fibrant ones and*
- *fibrant replacement the group completion map.*

Fibrations in $\mathcal{CS}^{\mathcal{I}}_{gp}$

Proposition

A virtual surjective $\mathcal{I}$-local fibration $N \to M$ in $\mathcal{CS}^{\mathcal{I}}$ is replete if and only if it is a fibration in $\mathcal{CS}^{\mathcal{I}}_{gp}$

Corollary

The acyclic cofibration / fibration factorization in $\mathcal{CS}^{\mathcal{I}}_{gp}$ models repletion.

Example

The fibrant objects in the comma category

$$(M \downarrow \mathcal{CS}^{\mathcal{I}}_{gp} \downarrow M) = (\mathcal{CS}^{\mathcal{I}}_{gp})^M_M$$

are the objects $M \to N \to M$ for which $N \to M$ is a replete $\mathcal{I}$-local fibration, and fibrant replacement is repletion.

The replete model structure

Proposition

Localizing $\mathcal{CLS}p^{\Sigma}_{log}$ *with respect to*

$$\{(\mathbb{S}[P], P) \to (\mathbb{S}[Q], Q) \mid P \to Q \text{ gen. acy. cof. for } \mathcal{CS}^{\mathcal{I}}_{gp}\}$$

gives the replete *model structure* $\mathcal{CLS}p^{\Sigma}_{rep}$ *on log symmetric ring spectra.*

Example

Let (A, M) be a log symmetric ring spectrum. If (B, N) is fibrant in the category

$$(\mathcal{CL}\mathrm{Sp}^{\Sigma}_{rep})^{(A,M)}_{(A,M)}$$

of replete augmented (A, M)-algebras, then (B, N) is a log symmetric ring spectrum with $N \to M$ replete.

Log modules

Lemma

If $(B, N) \to (A, \mathrm{GL}_1^{\mathcal{I}} A)$ *is a fibration in* $\mathcal{CLS}p^{\Sigma}_{rep}$*, then* (B, N) *is a trivial log symmetric ring spectrum.*

Corollary

There is a Quillen equivalence between

$$(\mathcal{CLS}p^{\Sigma}_{rep})^{(A,\mathrm{GL}_1^{\mathcal{I}} A)}_{(A,\mathrm{GL}_1^{\mathcal{I}} A)} \quad \textit{and} \quad (\mathcal{CS}p^{\Sigma})^A_A.$$

Theorem (Basterra-Mandell)

There are Quillen equivalences relating $Sp((\mathcal{CS}p^{\Sigma})^A_A)$ *and* Mod_A*.*

This motivates

Definition

$\mathrm{Mod}_{(A,M)} = \mathrm{Sp}((\mathcal{CL}\mathrm{Sp}^{\Sigma}_{rep})^{(A,M)}_{(A,M)})$, where $\mathrm{Sp}(\text{-})$ is stabilization.

Corollary

$\mathrm{Mod}_{(A,\mathrm{GL}_1^{\mathcal{I}} A)}$ *is Quillen equivalent to* Mod_A*.*

LOG ALGEBRAIC K-THEORY

One might define the *log algebraic K-theory* $K(A, M)$ of (A, M) as the Waldhausen K-theory of the subcategory of compact (=small) cofibrant objects in $\mathrm{Mod}_{(A,M)}$.

COROLLARY

$K(A, \mathrm{GL}_1 A) \simeq K(A)$.

QUESTION

When does the map

$$(A, M) \to (A \wedge_{\mathbb{S}[M]} \mathbb{S}[M^{\mathrm{gp}}], M^{\mathrm{gp}})$$

induce an equivalence in algebraic K-theory?

This is work in progress, joint with John Rognes.

Non-connective ring spectra

- Let E be a commutative symmetric ring spectrum, and let $e \to E$ be its connective cover. Then $\mathrm{GL}_1^{\mathcal{I}} e \to \mathrm{GL}_1^{\mathcal{I}} E$ is an equivalence – the units only see the connective cover.

This has undesirable effects for log structures:

Example

Let $u \in \pi_2 KU = \mathbb{Z}[u^{\pm 1}]$ be the Bott class. Then $(KU, \langle u \rangle)^a$ is not the trivial log structure, although u is a unit.

- Ideally, u should generate a non-trivial log structure on ku such that $\langle u \rangle \to \Omega^{\mathcal{I}}(ku) \to \Omega^{\mathcal{I}}(KU)$ generates the trivial log structure on KU.

Conclusion

Need a notion of graded units of E that respects units in non-zero degrees of the graded ring $\pi_* E$.

Towards graded units

Problem

$\mathrm{GL}_1^{\mathcal{I}} A$ was build from $\Omega^{\mathcal{I}} A = (\mathbf{n} \mapsto \Omega^{\mathbf{n}} A_{\mathbf{n}})$,
and the $\Omega^n A_n$ don't carry any information about $\pi_i A, i < 0$.
They cannot see if a class in $\pi_i A, i > 0$, is a unit.

- We need to take all $\Omega^k A_m$ into account to form the graded units.
- These graded units of A should be an object in a category of *graded commutative spaces* whose π_0 is the graded monoid $\mathrm{GL}_1(\pi_* A)$.

Idea

Replace $\mathcal{I}$ by a symmetric monoidal category $\mathcal{J}$ with objects $(\mathbf{k}, \mathbf{m})$ that serves as an organizing device for all $\Omega^k A_m$, just as $\mathcal{I}$ organizes the $\Omega^n A_n$.

The category $\mathcal{J}$

Definition

Let $\mathcal{J}$ be the category that has

- as objects the pairs $(\mathbf{k}, \mathbf{m})$ with $\mathbf{k}, \mathbf{m} \in \operatorname{Ob} \mathcal{I}$ finite sets,
- as morphisms $(\mathbf{k}, \mathbf{m}) \to (\mathbf{l}, \mathbf{n})$ the triples (φ, ψ, τ) with
  - $\varphi \colon \mathbf{k} \to \mathbf{l}$ and $\psi \colon \mathbf{m} \to \mathbf{n}$ injective maps and
  - τ a bijection $\mathbf{l} \setminus \varphi(\mathbf{k}) \to \mathbf{n} \setminus \psi(\mathbf{m})$.

Particularly, $\mathcal{J}$ has no morphisms $(\mathbf{k}, \mathbf{m}) \to (\mathbf{l}, \mathbf{n})$ unless $k - m = l - n$.

- Composition is defined in the obvious way.
- $\mathcal{J}$ is symmetric monoidal with respect to concatenation.

Lemma

$\mathcal{J}$ is equivalent to the Grayson-Quillen construction on the category of finite sets and bijections.

Corollary

$B\mathcal{J} \simeq QS^0$*, and* $\pi_* BJ \cong \pi_*^{st} S^0$.

$\mathcal{J}$ AND SYMMETRIC SPECTRA

- The functor $\mathrm{Ev}_m\colon \mathrm{Sp}^\Sigma \to \mathcal{S}_*, X \mapsto X_m$ has a left adjoint F_m given by

$$(F_m K)_n = \coprod_{\alpha\colon \mathbf{m}\to\mathbf{n} \in \mathcal{I}} K \wedge S^{\mathbf{n}-\alpha} \cong \Sigma_n^+ \wedge_{1\times\Sigma_{n-m}} K \wedge S^{n-m}$$

- The free symmetric spectra on spheres assemble to a functor

$$F_- S^- \colon \mathcal{J}^{\mathrm{op}} \to \mathrm{Sp}^\Sigma, \qquad (\mathbf{k},\mathbf{m}) \mapsto F_m S^k$$

  with $(\varphi,\psi,\tau)^*\colon F_n S^l \to F_m S^k$ adjoint to

$$S^{\mathbf{l}} \xrightarrow{\cong} S^{\mathbf{k}} \wedge S^{\mathbf{l}-\varphi} \xrightarrow{\cong} S^{\mathbf{k}} \wedge S^{\mathbf{n}-\psi} \hookrightarrow (F_m S^k)_n.$$

- The functor $F_- S^-\colon (\mathcal{J}^{\mathrm{op}},\sqcup) \to (\mathrm{Sp}^\Sigma,\wedge)$ is strong symmetric monoidal.

Commutative $\mathcal{J}$-space monoids

Definition

- A *$\mathcal{J}$-space* is a functor from $\mathcal{J}$ to (unbased) spaces. $\mathcal{J}$-spaces form a symmetric monoidal category $(\mathcal{S}^{\mathcal{J}}, \boxtimes)$.
- A *commutative $\mathcal{J}$-space monoid* is a commutative monoid in $\mathcal{S}^{\mathcal{J}}$. Commutative $\mathcal{J}$-space monoids form a category $\mathcal{C}\mathcal{S}^{\mathcal{J}}$.

We view objects of $\mathcal{C}\mathcal{S}^{\mathcal{J}}$ as graded commutative spaces.

Lemma

$$\Omega^{\mathcal{J}}(X) = \Big((\mathbf{k}, \mathbf{m}) \mapsto \mathrm{Map}_{Sp^{\Sigma}}(F_{\mathbf{m}} S^{\mathbf{k}}, X) = \Omega^k X_m\Big)$$

is the right adjoint in an adjunction $\mathbb{S}^{\mathcal{J}}[-] \colon \mathcal{S}^{\mathcal{J}} \rightleftarrows Sp^{\Sigma} \colon \Omega^{\mathcal{J}}$.

Corollary

There is an adjunction of commutative monoids

$$\mathbb{S}^{\mathcal{J}}[-] \colon \mathcal{C}\mathcal{S}^{\mathcal{J}} \rightleftarrows \mathcal{C}Sp^{\Sigma} \colon \Omega^{\mathcal{J}}.$$

MODEL STRUCTURES ON $\mathcal{J}$-SPACES

THEOREM (S.-SCHLICHTKRULL)

$\mathcal{S}^{\mathcal{J}}$ has a positive $\mathcal{J}$-local model structure which lifts to $\mathcal{CS}^{\mathcal{J}}$.

- *Fibrant objects are homotopy constant in positive degrees, i.e., $X(\mathbf{k},\mathbf{m}) \to X(\mathbf{k}+\mathbf{1},\mathbf{m}+\mathbf{1})$ is a weak equivalence if $m \geq 1$.*
- *f is a $\mathcal{J}$-equivalence if $\operatorname{hocolim}_{\mathcal{J}} f$ is a weak equivalence.*

The $\mathcal{J}$-local model structure is the localization of a level model structure with respect to set of maps W such that

$$\mathbb{S}^{\mathcal{J}}[W] = \{(F_1 S^1 \xrightarrow{\lambda} F_0 S^0) \wedge F_m S^k | k \geq 0, m \geq 1\}.$$

COROLLARY

$\mathbb{S}^{\mathcal{J}}[-]\colon \mathcal{CS}^{\mathcal{J}} \rightleftarrows \mathcal{C}Sp^{\Sigma} \colon \Omega^{\mathcal{J}}$ is a Quillen adjunction with respect to the $\mathcal{J}$-local positive and the stable positive model structure.

Let A be a positively fibrant commutative symmetric ring spectrum. The set of invertible path components

$$\mathrm{GL}_1^{\mathcal{J}} A \subset \Omega^{\mathcal{J}} A \quad \text{has} \quad \pi_{0,*}(\mathrm{GL}_1^{\mathcal{J}} A) = \mathrm{GL}_1(\pi_* A).$$

It defines the *graded units* of A.

DEFINITION

- A *graded pre-log symmetric ring spectrum* (A, M) is a commutative symmetric ring spectrum A together with a commutative $\mathcal{J}$-space monoid M and a map of commutative $\mathcal{J}$-space monoids $\alpha\colon M \to (A, \cdot)$.
- A graded pre-log symmetric ring spectrum (A, M) is a *graded log symmetric ring spectrum* if the induced map $\alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}} A) \to \mathrm{GL}_1^{\mathcal{J}} A$ is a $\mathcal{J}$-equivalence.

Examples for graded log ring spectra

Let A be positively fibrant. An $a \in \pi_k A$ generates a map $\langle a \rangle^{\mathrm{gr}} = \mathbb{C}F^{\mathcal{J}}_{\mathbf{k+1,1}}(\mathrm{pt}) \to \Omega^{\mathcal{J}} A$ from a free commutative $\mathcal{J}$-space. We get a graded pre-log symmetric ring spectrum $(A, \langle a \rangle^{\mathrm{gr}})$.

Example

$(KU, \langle u \rangle^{\mathrm{gr}})$ generates the trivial graded log structure, while $(ku, \langle u \rangle^{\mathrm{gr}})$ generates a non-trivial log structure.

Let $i \colon e \to E$ be the connective cover of a non-connective ring spectrum E. Forming the direct image log structure gives a factorization

$$(e, \mathrm{GL}_1^{\mathcal{J}}\, e) \to (e, i_* \mathrm{GL}_1^{\mathcal{J}}\, E) \to (E, \mathrm{GL}_1^{\mathcal{J}}\, E)$$

in graded log symmetric ring spectra.

Example

The map $ku \to KU$ gives a factorization

$$(ku, \mathrm{GL}_1^{\mathcal{J}}\, ku) \to (ku, i_* \mathrm{GL}_1^{\mathcal{J}}\, KU) \to (KU, \mathrm{GL}_1^{\mathcal{J}}\, KU).$$

Model structures for graded log ring spectra

- Most things about the model category description of log symmetric rings spectra apply also to *graded log ring spectra*.
- Main difficulty: Since the unit in $\mathcal{CS}^{\mathcal{J}}$ is not terminal, the bar construction doesn't apply to give group completion.