# Logarithmic structures on $K$-theory spectra

Steffen Sagave

Universität Bonn

August 2012

http://www.math.uni-bonn.de/people/sagave/



## Preamble

### Definition

A *pre-log structure* on a commutative ring $A$ is a commutative monoid $M$ together with a monoid map $\alpha \colon M \to (A, \cdot)$.
The triple $(A, M, \alpha)$ is a called a *pre-log ring*.

- Any $a \in A$ generates a pre-log structure $\langle a \rangle \to (A, \cdot)$.

### Definition

A pre-log ring $(A, M, \alpha)$ is a *log ring* if the map $\widetilde{\alpha}$ in the pullback

$$\begin{array}{ccc} \alpha^{-1}(A^{\times}) & \xrightarrow{\widetilde{\alpha}} & A^{\times} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha} & (A, \cdot) \end{array}$$

is an isomorphism. The logification of a pre log-structure $(M, \alpha)$ is $M^a = M \coprod_{\alpha^{-1}(A^{\times})} A^{\times} \to (A, \cdot)$. It is a log structure.

- $A \setminus \{0\} \hookrightarrow (A, \cdot)$ is a log structure if $A$ is an integral domain
- $A^{\times} \hookrightarrow (A, \cdot)$ defines the trivial log structure on $A$



## Overview

### Central Aim

- Introduce pre-log and log structures on structured ring spectra.
- Outline the logarithmic versions of TAQ and THH.

### Guiding Example

Consider the connective complex $K$-theory spectrum $ku$ and the Bott class $u \in \pi_*(ku) \cong \mathbb{Z}[u]$. What is a good notion of a pre-log structure generated by $u$?

### Contents of the talk

1. Motivation: Why is this interesting?
2. Log ring spectra: Rognes' definition and a modification
3. Logarithmic topological Andre-Quillen homology
4. Logarithmic topological Hochschild homology
(joint work in progress with J. Rognes and C. Schlichtkrull)



## Algebraic $K$-theory of local fields

Let $A$ be a complete discrete valuation ring with field of fractions $K$ and residue field $k$. Assume that char $K = 0$ and that $k$ is perfect with char $k = p > 2$.

### Example

$A = \mathbb{Z}_p, K = \mathbb{Q}_p$, and $k = \mathbb{F}_p$

In their computation of $K_*(K, \mathbb{Z}/p^v)$, Hesselholt-Madsen use a map of cofibration sequences

$$\begin{array}{ccccccc} K(k) & \longrightarrow & K(A) & \longrightarrow & K(K) & \longrightarrow & \Sigma K(k) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{THH}(k) & \to & \mathrm{THH}(A) & \to & \mathrm{THH}(A|K) & \to & \Sigma\,\mathrm{THH}(k). \end{array}$$

The $\mathrm{THH}(A|K)$ in the bottom sequence is **not** equivalent to $\mathrm{THH}(K)$. (It is defined as the THH of a certain linear Waldhausen category.)

## Log differentials and tame ramification

For a pre-log ring $(A, M, \alpha)$, there is an $A$-module $\Omega^1_{(A,M)}$ of log Kähler differentials (to be defined later). For $m \in M$, it has additional generators $d\log m$ with $d\alpha(m) = \alpha(m) d\log m$.

Hesselholt-Madsen show $\pi_1(\mathrm{THH}(A|K)) \cong \Omega^1_{(A, A\setminus\{0\})}$.
This suggests that $\mathrm{THH}(A|K)$ is a candidate for the THH of the log ring $(A, A \setminus \{0\}) = (A, \langle \text{uniformizer} \rangle^a)$.

One motivation behind the algebraic geometry of log rings is to extend the range of smooth and étale maps. In particular, tamely ramified extensions become log étale.

Let $(A, K, k)$ and $(B, L, l)$ be DVRs as above, and assume that $A \to B$ is a finite tamely ramified extension. Then

$$HB \wedge_{HA} \mathrm{THH}(A|K) \simeq_p \mathrm{THH}(B|L).$$

Thinking of these as log THH-terms, this is compatible with $(A, A \setminus \{0\}) \to (B, B \setminus \{0\})$ being formally log étale.



## Algebraic $K$-theory of topological $K$-theory

Let $ku_p$ be the $p$-complete connective complex $K$-theory spectrum. Blumberg-Mandell established homotopy cofiber sequences

$$\begin{array}{ccccccc} K(\mathbb{Z}_p) & \longrightarrow & K(ku_p) & \longrightarrow & K(KU_p) & \longrightarrow & \Sigma K(\mathbb{Z}_p) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{THH}(\mathbb{Z}_p) & \to & \mathrm{THH}(ku_p) & \to & \mathrm{THH}(ku_p|KU_p) & \to & \Sigma\,\mathrm{THH}(\mathbb{Z}_p), \end{array}$$

and similarly for the $p$-complete Adams summand $\ell_p$.

Again, $\mathrm{THH}(ku_p|KU_p)$ is not equivalent to $\mathrm{THH}(KU_p)$.
(It is defined as the THH of a certain Waldhausen category.)

Understanding $\mathrm{THH}(ku_p|KU_p)$ is desirable: It captures information about the algebraic $K$-theory of the non-connective ring spectrum $KU_p$.



## Tame ramification for structured ring spectra

The above analogy between DVRs and $K$-theory spectra suggests that $\mathrm{THH}(ku_p|KU_p)$ and $\mathrm{THH}(\ell_p|L_p)$ may arise as the log THH of appropriate *log ring spectra* $(ku_p, ?)$ and $(\ell_p, ?)$.

On homotopy groups, the inclusion $\ell_p \to ku_p$ induces the map

$$\mathbb{Z}_p[v_1] \to \mathbb{Z}_p[u], \qquad v_1 \mapsto u^{p-1}.$$

Thinking of $v_1$ and $u$ as uniformizers, $\ell_p \to ku_p$ is tamely ramified on homotopy groups.

Computations of $\mathrm{THH}(\ell_p)$ and $\mathrm{THH}(ku_p)$ by McClure-Staffeldt and Ausoni show that $\ell_p \to ku_p$ behaves like a tamely ramified extension on THH.

Hence a suitable extension of $\ell_p \to ku_p$ to a map of log ring spectra $(\ell_p, ?) \to (ku_p, ?)$ should be a *formally log étale map*.



## Pre-log ring spectra

In the context of structured ring spectra, $E_\infty$ ring spectra play the role of commutative rings, and $E_\infty$ spaces play the role of commutative monoids.

**Definition (Rognes)**
A *pre-log ring spectrum* $(A, M, \alpha)$ is an $E_\infty$ ring spectrum $A$ together with an $E_\infty$ space $M$ and an $E_\infty$ map $\alpha\colon M \to \Omega^\infty_\otimes(A)$.
Here $\Omega^\infty_\otimes(A)$ is the underlying multiplicative $E_\infty$ space of $A$.

By definition, $\mathrm{GL}_1(A) \subseteq \Omega^\infty_\otimes A$ is the sub $E_\infty$ space of invertible path components (corresponding to $\pi_0(A)^\times \subseteq \pi_0(A)$).

**Definition (Rognes)**
A pre-log ring spectrum $(A, M, \alpha)$ is a log ring spectrum if $\widetilde{\alpha}\colon \alpha^{-1}(\mathrm{GL}_1(A)) \to \mathrm{GL}_1(A)$ is a weak equivalence.

- $\mathrm{GL}_1(A) \hookrightarrow \Omega^\infty_\otimes A$ defines the trivial log structure on $A$

## Log structures on $ku$

What are interesting log structures on $ku$?

Example

Let $A$ be an integral domain with quotient field $K$. Let $K^\times \to (K, \cdot)$ be the trivial log structure. Forming the pullback of

$$(A, \cdot) \to (K, \cdot) \leftarrow K^\times,$$

we obtain the log ring $(A, A \setminus \{0\})$ considered earlier.

Example

Replacing $A$ by $ku$ and $K$ by $KU$, the pullback of

$$\Omega^\infty_\otimes(ku) \to \Omega^\infty_\otimes(KU) \leftarrow \mathrm{GL}_1(KU),$$

only provides the trivial log structure $\mathrm{GL}_1(ku) \to \Omega^\infty_\otimes(ku)$ on $ku$.

Problem

Both $\Omega^\infty_\otimes(ku) \to \Omega^\infty_\otimes(KU)$ and $\mathrm{GL}_1(ku) \to \mathrm{GL}_1(KU)$ are equivalences, but $\pi_*(KU) = \mathbb{Z}[u^{\pm 1}]$ has more units than $\pi_*(ku)$.



## Graded and ungraded monoids

It would be better to use a homotopical version of the adjunction

$$\begin{array}{c}(\mathbb{Z}\text{-graded com. monoids})\\ \uparrow\downarrow \\ (\mathbb{Z}\text{-graded com. rings})\end{array}$$

The adjunction

$$\Sigma^\infty(-)_+ \colon (E_\infty \text{ spaces}) \rightleftarrows (E_\infty \text{ ring spectra}) : \Omega^\infty_\otimes$$

models only its "ungraded" counterpart.

To do so, we will next define the "underlying graded multiplicative $E_\infty$ space" of an $E_\infty$ ring spectrum and its subobject of "graded units". These capture information about the graded commutative monoids $(\pi_*(A), \cdot)$ and $(\pi_*(A), \cdot)^\times$.



## Commutative $\mathcal{J}$-space monoids

Definition

Let $\mathcal{J}$ be the category with objects $(\mathbf{m_1}, \mathbf{m_2})$ where the $\mathbf{m_i} = \{1, \dots, m_i\}$ are finite sets. A morphism $(\alpha_1, \alpha_2, \rho) \colon (\mathbf{m_1}, \mathbf{m_2}) \to (\mathbf{n_1}, \mathbf{n_2})$ is a pair of injective maps $\alpha_i \colon \mathbf{m_i} \to \mathbf{n_i}$ together with a bijection $\rho \colon \mathbf{n_1} \setminus \alpha_1 \to \mathbf{n_2} \setminus \alpha_2$.

$\mathcal{J}$ is symmetric monoidal under entry-wise concatenation.

Remark

$\mathcal{J}$ is equivalent to Quillen's localization construction $\Sigma^{-1}\Sigma$ on the category of finite sets and bijections $\Sigma$. Hence $B\mathcal{J} \simeq QS^0$.

Definition

A *commutative $\mathcal{J}$-space monoid* $M$ is a functor $M \colon \mathcal{J} \to \mathcal{S}$ to the category of unbased spaces together with a unit $1 \in M(\mathbf{0}, \mathbf{0})$ and coherent multiplication maps

$$M(\mathbf{n_1}, \mathbf{n_2}) \times M(\mathbf{n'_1}, \mathbf{n'_2}) \to M(\mathbf{n_1} \sqcup \mathbf{n'_1}, \mathbf{n_2} \sqcup \mathbf{n'_2}).$$



## $\mathcal{J}$-spaces and symmetric spectra

Let $\mathcal{CS}^\mathcal{J}$ be the category of commutative $\mathcal{J}$-space monoids. The category $\mathcal{J}$ is chosen so that there is an adjunction

$$\mathbb{S}^\mathcal{J}[-] \colon \mathcal{CS}^\mathcal{J} \rightleftarrows \mathcal{C}\mathrm{Sp}^\Sigma \colon \Omega^\mathcal{J} \qquad (*)$$

relating $\mathcal{CS}^\mathcal{J}$ to the category of commutative symmetric ring spectra $\mathcal{C}\mathrm{Sp}^\Sigma$. For $A \in \mathcal{C}\mathrm{Sp}^\Sigma$ we have

$$\Omega^\mathcal{J}(A)(\mathbf{m_1}, \mathbf{m_2}) = \Omega^{m_2}(A_{m_1}).$$

A map $(\alpha_1, \alpha_2, \rho) \colon (\mathbf{m_1}, \mathbf{m_2}) \to (\mathbf{n_1}, \mathbf{n_2})$ sends $f \colon S^{m_2} \to A_{m_1}$ to

$$S^{\mathbf{n_2}} \xrightarrow[(\alpha_2)_*]{\cong} S^{\mathbf{m_2}} \wedge S^{\mathbf{n_2} \setminus \alpha_2} \xrightarrow{f \wedge \rho_*^{-1}} A_{\mathbf{m_1}} \wedge S^{\mathbf{n_1} \setminus \alpha_1} \xrightarrow{\sigma} A_{\mathbf{m_1} \sqcup (\mathbf{n_1} \setminus \alpha_1)} \xrightarrow[(\alpha_1)_*]{\cong} A_{\mathbf{n_1}}$$

The $\Omega^\mathcal{J}(A)$ captures information about $\pi_i(A)$ for all $i \in \mathbb{Z}$. The adjunction $(*)$ will serve as the homotopical version of

$$(\mathbb{Z}\text{-graded com. monoids}) \rightleftarrows (\mathbb{Z}\text{-graded com. rings}).$$

## GRADED $E_\infty$ SPACES

THEOREM (S-SCHLICHTKRULL)

*The category $\mathcal{CS}^{\mathcal{J}}$ admits a cofibrantly generated proper simplicial model structure where*

- *$M \to N$ is a weak equivalence if* $\text{hocolim}_{\mathcal{J}} M \to \text{hocolim}_{\mathcal{J}} N$ *is a weak equivalence in $\mathcal{S}$*
- *$M$ is fibrant if all $(\alpha_1, \alpha_2, \rho)\colon (\mathbf{m_1}, \mathbf{m_2}) \to (\mathbf{n_1}, \mathbf{n_2})$ with $m_1 \geq 1$ induce a weak equivalence of (fibrant) spaces $M(\mathbf{m_1}, \mathbf{m_2}) \to M(\mathbf{n_1}, \mathbf{n_2})$.*

COROLLARY

*$(\mathbb{S}^{\mathcal{J}}[-], \Omega^{\mathcal{J}})$ is a Quillen adjunction with respect to this model structure and the positive stable model structure on $\mathcal{C}Sp^{\Sigma}$.*

THEOREM (S-SCHLICHTKRULL)

*There is a chain of Quillen equivalences relating $\mathcal{CS}^{\mathcal{J}}$ and the category of $E_\infty$-spaces over $B\mathcal{J} \simeq QS^0$.*

We view $\Omega^{\mathcal{J}}(A)$ as the underlying graded $E_\infty$ space of $A$.



## GRADED PRE-LOG RING SPECTRA

DEFINITION

A *graded pre-log ring spectrum* $(A, M, \alpha)$ is a commutative symmetric ring spectrum $A$ together with a commutative $\mathcal{J}$-space monoid $M$ and a map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ in $\mathcal{CS}^{\mathcal{J}}$.

DEFINITION

A graded pre-log ring spectrum $(A, M, \alpha)$ is a log ring spectrum if $\widetilde{\alpha}\colon \alpha^{-1}(\mathrm{GL}_1^{\mathcal{J}}(A)) \to \mathrm{GL}_1^{\mathcal{J}}(A)$ is a weak equivalence in $\mathcal{CS}^{\mathcal{J}}$.

The previous definition uses

DEFINITION

Let $A$ be a positive fibrant in $\mathcal{C}\mathrm{Sp}^{\Sigma}$. The *graded units* of $A$ is the sub commutative $\mathcal{J}$-space monoid $\mathrm{GL}_1^{\mathcal{J}}(A) \subseteq \Omega^{\mathcal{J}}(A)$ corresponding to the submonoid $(\pi_*(A))^\times$ of $(\pi_*(A), \cdot)$.



## LOG STRUCTURES ON $K$-THEORY SPECTRA

We view the connective cover $ku \to KU$ as a map of positive fibrant objects in $\mathcal{C}\mathrm{Sp}^{\Sigma}$ and form the following pullback in $\mathcal{CS}^{\mathcal{J}}$:

$$\begin{array}{ccc} i_* \mathrm{GL}_1^{\mathcal{J}}(KU) & \longrightarrow & \Omega^{\mathcal{J}}(ku) \\ \downarrow & & \downarrow \\ \mathrm{GL}_1^{\mathcal{J}}(KU) & \longrightarrow & \Omega^{\mathcal{J}}(KU) \end{array}$$

We get a sequence of graded log ring spectra

$$(ku, \mathrm{GL}_1^{\mathcal{J}}(ku)) \to (ku, i_* \mathrm{GL}_1^{\mathcal{J}}(KU)) \to (KU, \mathrm{GL}_1^{\mathcal{J}}(KU)).$$

So passing to graded log ring spectra with trivial log structures, $ku \to KU$ factors through the "intermediate localization" $(ku, i_* \mathrm{GL}_1^{\mathcal{J}}(KU))$.

This is analogous to the factorization

$$(A, A^\times) \to (A, A \setminus \{0\}) \to (K, K^\times)$$

for an integral domain $A$ with quotient field $K$.



## LOG STRUCTURES AND LOCALIZATION

For an integral domain $A$ with quotient field $K$, we can reconstruct $K$ from the log ring $(A, A \setminus \{0\})$ by forming

$$A[(A \setminus \{0\})^{-1}] \cong \mathbb{Z}[(A \setminus \{0\})^{\mathrm{gp}}] \otimes_{\mathbb{Z}[A \setminus \{0\}]} A.$$

This has a homotopical counterpart:

THEOREM (S)

*The following square is a homotopy pushout in $\mathcal{C}Sp^{\Sigma}$:*

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{J}}[i_* \mathrm{GL}_1^{\mathcal{J}}(KU)] & \longrightarrow & ku \\ \downarrow & & \downarrow \\ \mathbb{S}^{\mathcal{J}}[(i_* \mathrm{GL}_1^{\mathcal{J}}(KU))^{gp}] & \longrightarrow & KU \end{array}$$

- Works more generally for connective $A \in \mathcal{C}\mathrm{Sp}^{\Sigma}$ and $x \in \pi_*(A)$ with $A \simeq A[1/x]_{\geq 0}$.
- For log structures defined with usual $E_\infty$-spaces, this cannot hold because $\Sigma^\infty(M_+)$ is always connective.

## Logarithmic derivations

Let $(A, M)$ be a log ring and let $X$ be an $A$-module. Let $A \oplus X$ be the square zero extension of $A$ by $X$.

Definition

The square zero extension of $(A, M)$ by $X$ is the log ring $(A \oplus X, M \times (X, +))$ with structure map $(m, x) \mapsto \alpha(m)(1 \oplus x)$.

Definition

A *derivation* of $(A, M)$ with values in $X$ is map of log rings $(A, M) \to (A \oplus X, M \times (X, +))$ over $(A, M)$.

The $A$-module of *log Kähler* differentials is characterized by

$$\mathrm{Hom}_A(\Omega^1_{(A,M)}, X) \cong \mathrm{Der}((A, M), X).$$

More explicitly,

$$\Omega^1_{(A,M)} \cong \Omega^1_A \oplus (A \otimes M^{\mathrm{gp}}) / (d\alpha(m) \sim \alpha(m) \otimes m)$$

(It is a standard convention to set $d\log m = 1 \otimes m$.)



## Derivations of graded log ring spectra

In similar fashion as described by Rognes for log ring spectra, these notions have counterparts in graded log ring spectra:

Let $(A, M)$ be a graded pre-log ring spectrum, let $X$ be an $A$-module, and let $A \vee X$ be the square zero extension.

There is a pre-log ring spectrum $(A \vee X, M \boxtimes (1 + X)^{\mathcal{J}})$ where $(1+X)^{\mathcal{J}}$ is a $\mathcal{J}$-space monoid model for the connective cover of the underlying spectrum of $X$. ($\boxtimes$ is the coproduct in $\mathcal{CS}^{\mathcal{J}}$.)

Definition

Let $(R, P) \to (A, M)$ be a map of graded pre-log ring spectra. The space $\mathrm{Der}_{(R,P)}((A, M), X)$ of graded log derivations with values in $X$ is the space of maps of graded log ring spectra

$$(A, M) \to (A \vee X, M \boxtimes (1 + X)^{\mathcal{J}})$$

under $(R, P)$ and over $(A, M)$.



## Log TAQ

Let $(R, P) \to (A, M)$ be a map of graded pre-log ring spectra. Its *logarithmic topological André-Quillen homology* is the $A$-module $\mathrm{TAQ}^{(R,P)}(A, M)$ characterized by the property

$$\mathrm{Mod}_A(\mathrm{TAQ}^{(R,P)}(A, M), X) \simeq \mathrm{Der}_{(R,P)}((A, M), X).$$

The inclusion of the $p$-complete Adams summand extends to

$$(\ell_p, i_* \mathrm{GL}_1^{\mathcal{J}}(L_p)) \to (ku_p, i_* \mathrm{GL}_1^{\mathcal{J}}(KU_p))$$

Theorem (S)

*This map is formally log étale, i.e.,* $\mathrm{TAQ}^{(\ell_p, i_* \mathrm{GL}_1^{\mathcal{J}}(L_p))}(ku_p, i_* \mathrm{GL}_1^{\mathcal{J}}(KU_p))$ *is contractible.*

- The proof is non-computational.
- One may view this of an instance of *tame ramification.*



## Definition of log THH

(from here on this is joint work in progress with John Rognes and Christian Schlichtkrull)

Let $(A, M)$ be a graded pre-log ring spectrum with $A$ and $M$ cofibrant.

- $\mathrm{THH}(A)$ can be defined as $A \otimes S^1 \cong B^{\mathrm{cy}}_{\wedge}(A)$.
- Can also form $B^{\mathrm{cy}}_{\boxtimes}(M) \cong M \otimes S^1$ for $M \in \mathcal{CS}^{\mathcal{J}}$. The map $\alpha\colon M \to \Omega^{\mathcal{J}}(A)$ induces a map $\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\boxtimes}(M)] \to \mathrm{THH}(A)$.
- The homotopy pullback of

  $$M \to M^{\mathrm{gp}} \leftarrow (B^{\mathrm{cy}}_{\boxtimes}(M))^{\mathrm{gp}}$$

  defines the *replete bar construction* $B^{\mathrm{rep}}_{\boxtimes}(M)$ on $M$.

Definition

The *logarithmic topological Hochschild homology* of $(A, M)$ is

$$\mathrm{THH}(A, M) = \mathbb{S}^{\mathcal{J}}[B^{\mathrm{rep}}_{\boxtimes}(M)] \wedge^{\mathbb{L}}_{\mathbb{S}^{\mathcal{J}}[B^{\mathrm{cy}}_{\boxtimes}(M)]} \mathrm{THH}(A)$$

THEOREM (ROGNES-S-SCHLICHTKRULL)

*Let A be a commutative symmetric ring spectrum, and let $x \in \pi_n(A)$ be a homotopy class of even positive degree n such that $\pi_*(A) \cong \pi_0(A)[x]$. Then there is a homotopy cofiber sequence of* THH($A$)*-modules*

$$\mathrm{THH}(\pi_0(A)) \to \mathrm{THH}(A) \to \mathrm{THH}(A, i_* \, \mathrm{GL}_1^{\mathcal{J}}(A[1/x])) \to \Sigma\, \mathrm{THH}(\pi_0(A))$$

- This applies for example to $ku, ku_p, \ell_p$.