Exercises for "Algebraic Topology"

Sheet 1*, September 13, 2017

Definition. Let A be an abelian group and let M be a set. The *A-linearization of M* is the set

$$A[M] = \{f\colon M \to A \mid f^{-1}(A \setminus \{0\}) \text{ is finite}\}.$$

We view $A[M]$ as an abelian group via the pointwise addition of functions.

Exercise 1.1. Let k be a field and let M be a set. Viewing k as an abelian group by only remembering its additive structure, we obtain an abelian group $k[M]$.

(i) Show that the abelian group structure on $k[M]$ extends to a k-vector space structure in a canonical way.

(ii) Show that $k[M]$ has a basis which admits a bijection from M. (This implies that the dimension of $k[M]$ equals $|M|$ if M is finite.)

Exercise 1.2. Let M be a set and let $\iota\colon M \to \mathbb{Z}[M]$ be the map of sets sending m to the function $f\colon M \to \mathbb{Z}$ with $f(m) = 1$ and $f(n) = 0$ if $n \neq m$. Moreover, let A be an abelian group and let $\psi\colon M \to A$ be a map of sets.

Show that there exists a unique group homomorphism $\tilde{\psi}\colon \mathbb{Z}[M] \to A$ such that $\tilde{\psi} \circ \iota = \psi$.

Definition. A subset $C \subseteq \mathbb{R}^n$ is *convex* if $t \cdot x + (1-t) \cdot y \in C$ holds for all $x, y \in C$ and for all $t \in \mathbb{R}$ with $0 \leq t \leq 1$. It is easy to see that the intersection of convex subsets is convex again. The *convex hull* of a subset $M \subseteq \mathbb{R}^n$ is the intersection of all convex subsets of $\mathbb{R}^n$ containing M.

Exercise† 1.3. Show that the standard n-simplex

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, t_0 + \dots + t_n = 1\}$$

is the convex hull of $\{e_0, \dots, e_n\}$ where $e_i \in \mathbb{R}^{n+1}$ has 1 in entry i and 0 in all other entries.

Exercise† 1.4. Recall that a map of sets $\alpha\colon \{0, \dots, m\} \to \{0, \dots, n\}$ induces a continuous map $\alpha_*\colon \Delta^m \to \Delta^n$ given by $\alpha_*(t_0, \dots, t_m) = (\sum_{i \in \alpha^{-1}(0)} t_i, \dots, \sum_{i \in \alpha^{-1}(n)} t_i)$. In the lecture, we considered the map $\delta_i\colon \Delta^{n-1} \to \Delta^n$ induced by the unique order preserving injection $\{0, \dots, n-1\} \to \{0, \dots, n\}$ that does not hit i.

For $n \geq 0$ and $0 \leq i \leq n$, we now define $\sigma_i\colon \Delta^{n+1} \to \Delta^n$ to be the map induced by the unique order preserving surjection $\{0, \dots, n+1\} \to \{0, \dots, n\}$ that hits i twice. Consider the following three squares:

$$\begin{tikzcd} \Delta^n \arrow[r, "\delta_i"] \arrow[d, "\delta_{i+1}"'] & \Delta^{n+1} \arrow[d, "\sigma_i"] \\ \Delta^{n+1} \arrow[r, "\sigma_i"] & \Delta^n \end{tikzcd} \qquad \begin{tikzcd} \Delta^n \arrow[r, "\delta_j"] \arrow[d, "\sigma_{i-1}"'] & \Delta^{n+1} \arrow[d, "\sigma_i"] \\ \Delta^{n-1} \arrow[r, "\delta_j"] & \Delta^n \end{tikzcd} \qquad \begin{tikzcd} \Delta^n \arrow[r, "\delta_j"] \arrow[d, "\sigma_i"'] & \Delta^{n+1} \arrow[d, "\sigma_i"] \\ \Delta^{n-1} \arrow[r, "\delta_{j-1}"] & \Delta^n \end{tikzcd}$$

Show that the first square commutes, that the second square commutes if $j < i$, and that the third square commutes if $j > i + 1$.

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* Please return Wednesday September 20, 2017. Exercises marked with † count for the bonus.

Definition. Let $(A^i)_{i\in I}$ be a family of abelian groups indexed by a set I. Recall that the direct sum $\bigoplus_{i\in I} A^i$ can be defined as the abelian group consisting of families of group elements $(a_i)_{i\in I}$ with $a_i \in A_i$ for all $i \in I$ and $a_i = 0$ for all but finitely many $i \in I$. The addition of the direct sum is defined by $(a_i)_{i\in I} + (b_i)_{i\in I} = (a_i + b_i)_{i\in I}$, and the familiy given by the zero elements of the groups A_i provides the zero element for the direct sum. There are canonical homomorphisms $\iota_{A^i} \colon A^i \to \bigoplus_{i\in I} A^i$ sending $a \in A^i$ to the tuple with $a_i = a$ and $a_j = 0$ if $j \neq i$.

Exercise† **1.5.** Let C^i be a family of chain complexes indexed by a set I. Then setting

$$(\bigoplus_{i\in I} C^i)_n = \bigoplus_{i\in I} (C^i)_n \qquad \text{and} \qquad \partial((a_i)_{i\in I}) = (\partial(a_i))_{i\in I}$$

defines a chain complex $\bigoplus_{i\in I} C^i$, and the homomorphisms $\iota_{(C^i)_n} \colon (C^i)_n \to \bigoplus_{i\in I}(C^i)_n$ from the previous definition form a chain map $\iota_{C^i} \colon C^i \to \bigoplus_{i\in I} C^i$.

(i) Show that $\bigoplus_{i\in I} C^i$ has the following universal property: Given a chain complex D and a family of chain maps $f_i \colon C^i \to D$, there is a unique chain map

$$f\colon \bigoplus_{i\in I} C^i \to D$$

with $f_i = f \circ \iota_{C^i}$ for every $i \in I$.

(ii) Show that there is an isomorphism

$$\textstyle\bigoplus_{i\in I} H_n(C^i) \to H_n\left(\bigoplus_{i\in I} C^i\right)$$

whose composite with $\iota_{H_n(C^i)}$ is the map $H_n(C^i) \to H_n\left(\bigoplus_{i\in I} C^i\right)$ induced by the chain map ι_{C^i}.

Exercises for "Algebraic Topology"

Sheet 2*, September 20, 2017

Definition. Let k be a field. A *chain complex of k-vector spaces C* is a sequence of k-vector spaces and k-linear maps

$$\dots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \geq 1$. In analogy with the chain complexes of abelian groups considered in the lecture, one can form homology k-vector spaces $H_n(C)$ for $n \geq 0$, rather than just homology groups. (One can unify the definition of chain complexes of abelian groups and chain complexes of k-vector spaces by considering chain complexes of modules over a ring.)

Exercise† 2.1. Let k be a field and let C be a chain complex of k-vector spaces such that only finitely many of the C_n are non-trivial vector spaces, and such that each C_n is a finite dimensional k-vector space.

Show that the following equation holds:

$$\sum_{n\geq 0}(-1)^n \dim_k(C_n) = \sum_{n\geq 0}(-1)^n \dim_k(H_n(C))$$

This integer is the *Euler characteristic* of C.

Exercise† 2.2. Let $f\colon A \to B$ be a homomorphism of abelian groups, let $Q = B/\operatorname{im} f$ be the quotient of B by the image of f, and let $q\colon B \to Q$ be the canonical homomorphism to the quotient.

(i) Show that if $p\colon B \to P$ is a homomorphism of abelian groups with $p \circ f = 0$, then there exists a unique homomorphism $p'\colon Q \to P$ with $p' \circ q = p$.

(ii) Suppose that $q'\colon B \to Q'$ is a homomorphism of abelian groups such that $q' \circ f = 0$ and such that for any homomorphism of abelian groups $p\colon B \to P$ with $p \circ f = 0$, there exists a unique homomorphism $p'\colon Q' \to P$ with $p' \circ q' = p$. Show that there exists a unique isomorphism $r\colon Q \to Q'$ such that $r \circ q = q'$.

(This shows that the property established in (i) characterizes the pair (Q, q) up to unique isomorphism. A pair with this property is called "the" cokernel of f.)

(iii) Let C and D be chain complexes of abelian groups and let $f\colon C \to D$ be a chain map. For $n \geq 0$, let $E_n = D_n/\operatorname{im} f_n$ and let $q_n\colon D_n \to E_n$ be the quotient map. Show that there exists a unique family of maps $\partial^E_n\colon E_n \to E_{n-1}$ such that

$$\dots \xrightarrow{\partial^E_3} E_2 \xrightarrow{\partial^E_2} E_1 \xrightarrow{\partial^E_1} E_0$$

is a chain complex and the q_n define a chain map $q\colon D \to E$.

* Please return Wednesday September 27, 2017. Exercises marked with † count for the bonus.

Exercise† 2.3. We consider chain complexes of abelian groups C, D and E that are defined as follows: For each $n \geq 0$, we set

$$C_n = \mathbb{Z}, \qquad D_n = \mathbb{Z}/2, \qquad \text{and} \qquad E_n = \mathbb{Q}\,.$$

For each of these chain complexes, the differential is given by

$$\partial_n = \begin{cases} \text{multiplication with } 2 & \text{if } n \text{ is even} \\ \text{multiplication with } 0 & \text{if } n \text{ is odd} \end{cases}$$

Compute the homology groups $H_n(C), H_n(D)$, and $H_n(E)$ for all $n \geq 0$.

Exercise 2.4. Consider the boundary

$$\partial\Delta^3 = \{(t_0, t_1, t_2, t_3) \in \Delta^3 \,|\, t_i = 0 \text{ for some } i\}$$

of the standard 3-simplex Δ^3. It has a canonical triangulation with

- vertices $V_0 = (1, 0, 0, 0), \ldots, V_3 = (0, 0, 0, 1)$,
- edges $E_{ij} = \{(t_0, t_1, t_2, t_3) \,|\, t_k = 0 \text{ for all } k \neq i \text{ and } k \neq j\}$ for $1 \leq i < j \leq 3$, and
- 2-dimensional faces $T_i = \{(t_0, t_1, t_2, t_3) \,|\, t_i = 0\}$ for $i = 0, \ldots, 3$.

(i) Generalize the "motivating example" from lecture 1 by constructing differentials ∂_2 and ∂_1 in a chain complex C of $\mathbb{Q}$-vector spaces of the form

$$\cdots \to 0 \to \mathbb{Q}[\{T_0, \ldots, T_3\}] \xrightarrow{\partial_2} \mathbb{Q}[\{E_{ij} \,|\, 0 \leq i < j \leq 3\}] \xrightarrow{\partial_1} \mathbb{Q}[\{V_0, \ldots, V_3\}]$$

In particular, show that $\partial_1 \circ \partial_2 = 0$ holds.

(ii) Compute the dimensions of the $\mathbb{Q}$-vector spaces $H_i(C)$ for $0 \leq i \leq 2$.

(iii) Compute the Euler characteristic of C (in the sense of Exercise 2.1).

Remark. One can also consider the singular homology groups of the topological space $\partial\Delta^3$. They are defined in terms of the singular chain complex of $\partial\Delta^3$. However, by definition the singular chain groups $C_n(\partial\Delta^3; \mathbb{Q})$ are infinite dimensional $\mathbb{Q}$-vector spaces for all $n \geq 0$. Later in this course we will show the non-obvious result that the homology groups of the above chain complex C are isomorphic to the singular homology groups of $\partial\Delta^3$ with coefficients in $\mathbb{Q}$.

Exercises for "Algebraic Topology"

Sheet 3*, September 27, 2017

Exercise 3.1. (The five lemma) Let

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r, "\beta_1"'] & B_2 \arrow[r, "\beta_2"'] & B_3 \arrow[r, "\beta_3"'] & B_4 \arrow[r, "\beta_4"'] & B_5 \end{tikzcd}$$

be a commutative diagram of abelian groups and group homomorphisms in which both rows are exact. Verify the following statements:

(i) If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.
(ii) If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.
(iii) If f_2 and f_4 are isomorphisms, f_1 is surjective, and f_5 is injective, then f_3 is an isomorphism.

Definition. Let A and B be abelian groups. Then we write

$$\mathrm{Hom}(A, B) = \{f\colon A \to B \mid f \text{ is a group homomorphism}\}$$

for the set of all group homomorphisms from A to B. For $f, g \in \mathrm{Hom}(A, B)$, we define $f + g$ to be the group homomorphism given by $(f + g)(a) = f(a) + g(a)$ and note that this addition turns $\mathrm{Hom}(A, B)$ into an abelian group with zero element the group homomorphism $A \to B$ with constant value 0. We also note that a group homomorphism $i\colon B' \to B$ induces a group homomorphism

$$i_*\colon \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, B), \quad f \mapsto i \circ f$$

and that for the trivial abelian group 0, the group $\mathrm{Hom}(A, 0)$ is as well trivial.

Exercise 3.2. Let A be an abelian group, let

$$0 \to B' \xrightarrow{i} B \xrightarrow{p} \overline{B} \to 0$$

be a short exact sequence of abelian groups, and consider the induced sequence of abelian groups

$$0 \to \mathrm{Hom}(A, B') \xrightarrow{i_*} \mathrm{Hom}(A, B) \xrightarrow{p_*} \mathrm{Hom}(A, \overline{B}) \to 0\ .$$

(i) Show that this induced sequence is exact at $\mathrm{Hom}(A, B')$ and at $\mathrm{Hom}(A, B)$.
(ii) Give examples of abelian groups that show that the induced sequence is in general not exact at $\mathrm{Hom}(A, \overline{B})$.

* Please return Wednesday October 4, 2017. Exercises marked with † count for the bonus.

Exercise† 3.3. Let X be a topological space, let $X' \subset X$ be a subspace, let A be an abelian group, and let $n \geq 0$ be an integer. Then we can view $C_n(X'; A)$ as a subgroup of $C_n(X; A)$, and we define the following two subsets of $C_n(X; A)$:

$$Z_n(X, X'; A) = \{c \in C_n(X; A) \mid \partial(c) \in C_{n-1}(X'; A)\}$$

$$B_n(X, X'; A) = \{c \in C_n(X; A) \mid \text{There exist } c' \in C_n(X'; A), e \in C_{n+1}(X; A) \text{ with } c - c' = \partial(e)\}$$

Show the following statements:

(i) The sets $Z_n(X, X'; A)$ and $B_n(X, X'; A)$ are subgroups of $C_n(X; A)$, and $B_n(X, X'; A)$ is contained in $Z_n(X, X'; A)$.

(ii) There is a natural isomorphism $H_n(X, X'; A) \cong Z_n(X, X'; A)/B_n(X, X'; A)$.

Exercise† 3.4. (i) Let

$$0 \longrightarrow A' \xrightarrow{j} A \xrightarrow{q} \overline{A} \longrightarrow 0$$

be a short exact sequence of abelian groups. Show that the following three statements are equivalent:

(a) The map q admits a section, that is, there is a homomorphism $s\colon \overline{A} \to A$ such that $q \circ s = \mathrm{id}_{\overline{A}}$.

(b) The map j admits a retraction, that is, there is a homomorphism $r\colon A \to A'$ such that $r \circ j = \mathrm{id}_{A'}$.

(c) There is a commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{j} & A & \xrightarrow{q} & \overline{A} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A' & \xrightarrow{i} & A' \oplus \overline{A} & \xrightarrow{p} & \overline{A} & \longrightarrow & 0
\end{array}$$

where f is an isomorphism and i and p are the homomorphisms given by $i(a') = (a', 0)$ and $p(a', \overline{a}) = \overline{a}$.

A short exact sequence satisfying the equivalent conditions (a) - (c) is called *split*.

(ii) Give an example of a short exact sequence that is not split! (Hint: Use the group $\mathbb{Z}/4$.)

(iii) Now let X be a space, let $X' \subset X$ be a subspace and suppose that there exists a *retraction* $r\colon X \to X'$, that is, a continuous map such that $r(x') = x'$ for all $x' \in X'$.

Use part (i) and the long exact sequence of the pair (X, X') to construct an isomorphism

$$H_n(X; A) \to H_n(X'; A) \oplus H_n(X, X'; A)\ .$$

for all abelian coefficient groups A and all $n \geq 0$.

Exercises for "Algebraic Topology"

Sheet 4*, October 4, 2017

In the lecture, we have formulated (but not yet proved) the Excision Theorem for singular homology. On this exercise sheet, you may use the Excision Theorem and its consequences established in the lecture.

Exercise† 4.1. (The topological invariance of the dimension) Let m and n be positive integers, let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be non-empty open subsets and let $f\colon U \to V$ be a homeomorphism. Use our computation of the homology of disks relative to their boundary spheres to show that $m = n$.

Exercise† 4.2. (The long exact sequence of a tripel.) Let X be a topological space, let $X'' \subseteq X' \subseteq X$ be subspaces of X, and let A be an abelian group. Let

$$H_n(X', X''; A) \to H_n(X, X''; A)$$

be the map induced by the inclusion $X' \to X$, let

$$H_n(X, X''; A) \to H_n(X, X'; A)$$

be the map induced by the identity of X, and let

$$H_n(X, X'; A) \to H_{n-1}(X', X''; A)$$

be the composite of the map $H_n(X, X'; A) \to H_{n-1}(X'; A)$ from the long exact sequence of (X, X') and the map $H_{n-1}(X'; A) \to H_{n-1}(X', X''; A)$ from the long exact sequence of (X', X'').

Show that the sequence

$$\cdots \to H_n(X', X''; A) \to H_n(X, X''; A) \to H_n(X, X'; A) \to H_{n-1}(X', X''; A) \to \ldots$$

obtained from these maps is exact.

Exercise† 4.3. Let $n \geq 1$ be an integer, let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the n-dimensional disk, let $\partial D^n = \{x \in D^n \mid \|x\| = 1\}$ be its boundary sphere, let

$$\partial\Delta^n = \{(t_0, \ldots, t_n) \in \Delta^n \mid t_i = 0 \text{ for some } i\}$$

be the boundary of the standard n-simplex Δ^n, and let $f\colon \Delta^n \to D^n$ be a homeomorphism. Show that f restricts to a homeomorphism $\partial\Delta^n \to \partial D^n$.

Definition. If A, A', B are abelian groups, a *bilinear map* is a map of sets $f\colon A \times A' \to B$ such that $f(a, -)\colon A' \to B$ and $f(-, a') \to B$ are group homomorphisms for all $a \in A$ and $a' \in A'$. The *tensor product* of A and A' is an abelian group $A \otimes A'$ together with a bilinear map $f\colon A \times A' \to A \otimes A'$ such that the following property is satisfied: For every bilinear map $f'\colon A \times A' \to B'$ to an abelian group B', there is a unique group homomorphism $g\colon A \otimes A' \to B'$ such that $f' = g \circ f$.

The universal property characterizes the tensor product up to isomorphism. We follow the usual convention of writing $x \otimes y$ for $f(x, y)$.

* Please return Wednesday October 11, 2017. Exercises marked with † count for the bonus.

In the next exercise, you may use without proof that the tensor product of abelian groups exists. (It can be constructed as an explicit quotient.)

Exercise 4.4. Let C and D be chain complexes of abelian groups. Let I be the chain complex of abelian groups with $I_0 = \mathbb{Z}[\{\underline{0}, \underline{1}\}]$ the $\mathbb{Z}$-linearization of the two element set $\{\underline{0}, \underline{1}\}$, with $I_1 = \mathbb{Z}[\{e\}]$ the $\mathbb{Z}$-linearization of the one element set $\{e\}$, with $I_n = 0$ if $n \geq 2$, and with the only non-trivial differential given by

$$\partial_1 \colon I_1 \to I_0, \qquad k \cdot e \mapsto k \cdot \underline{1} - k \cdot \underline{0} \ .$$

(i) The tensor product of C and D is defined to be the chain complex with

$$(C \otimes D)_n = \bigoplus_{p=0}^{n} C_p \otimes D_{n-p}$$

where we use the tensor product of abelian groups on the right hand side. The differential $\partial^{C \otimes D}$ of $C \otimes D$ is the group homomorphism induced by the bilinear maps

$$C_p \times D_{n-p} \to (C \otimes D)_{n-1}, \qquad (x, y) \mapsto \partial^C_p(x) \otimes y + (-1)^p \ x \otimes \partial^D_{n-p}(y)$$

(where $\partial^C_0(x)$ and $\partial^D_0(y)$ are understood to be zero). Show that $\partial^{C\otimes D} \circ \partial^{C \otimes D} = 0$!

(ii) Now let $H \colon I \otimes C \to D$ be a chain map. Show that setting

$$H^0 \colon C \to D,\ (H^0)_n(x) = H_n(\underline{0} \otimes x) \qquad \text{and} \quad H^1 \colon C \to D,\ (H^1)_n(x) = H_n(\underline{1} \otimes x)$$

defines chain maps H^0 and H^1.

(iii) Let $H \colon I \otimes C \to D$ be a chain map. Show that maps

$$P_n \colon C_n \to D_{n+1},\ x \mapsto H_{n+1}(e \otimes x)$$

define a chain homotopy from H^0 to H^1.

(iv) Show that for any chain homotopy between two chain maps $C \to D$, there is a chain map $H \colon I \otimes C \to D$ that gives back the original chain homotopy via the formula in (iii).

Remark. The chain complex I in the previous exercise arises from the canonical triangulation of the 1-simplex Δ^1 (compare Exercise 2.4). Therefore, I may be viewed as the chain complex version of the interval. Thinking of the $\otimes$-product as "the" product of chain complexes, the present exercise shows that the notion of chain homotopy is analogous to the notion of homotopy of continuous maps.

Exercises for "Algebraic Topology"

Sheet 5*, October 11, 2017

In the lecture, we have formulated (but not yet proved) the Excision Theorem and the "Small Simplices" Theorem. On this exercise sheet, you may use these results and their consequences established in the lecture.

Exercise† 5.1. Let X be a topological space and let A be an abelian group. Let $U, V \subseteq X$ be two subsets with $X = \text{interior}(U) \cup \text{interior}(V)$ and let

$$i^U : U \cap V \to U, \qquad i^V : U \cap V \to V, \qquad j^U : U \to X, \text{ and } \qquad j^V : V \to X$$

be the inclusion maps. Define group homomorphisms $\partial_n \colon H_n(X; A) \to H_{n-1}(U \cap V; A)$ such that the sequence of homology groups

$$\ldots \longrightarrow H_n(U \cap V; A) \xrightarrow{(i^U_*, i^V_*)} H_n(U; A) \oplus H_n(V; A) \xrightarrow{\binom{j^U_*}{-j^V_*}} H_n(X; A) \xrightarrow{\partial_n} H_{n-1}(U \cap V; A) \longrightarrow \ldots$$

is exact. This sequence is called the *Mayer–Vietoris-sequence* of the cover $\{U, V\}$.

(Hint: Set $\mathcal{O} = \{U, V\}$, replace $H_n(X; A)$ by $H_n(C(\mathcal{S}_{\mathcal{O}}(X); A))$ and look for a short exact sequence of chain complexes inducing the Mayer–Vietoris-sequence.)

Exercise 5.2. In the lecture we calculated $H_n(S^0; A)$ for $n \geq 0$ and $H_0(S^m; A)$ for $m \geq 0$. Use these calculations and the Mayer–Vietoris sequence from the previous exercise to give an alternative proof of the calculation of $H_n(S^m; A)$ for $m \geq 0$ and $n \geq 0$ that was carried out in the lecture.

Definition. Let X be a topological space and let $X' \subseteq X$ be a subspace. Then X' is called a *deformation retract* of X if there exists a continuous map $H \colon X \times [0, 1] \to X$ such that $H(x, 0) = x$ for all $x \in X$, $H(x, t) = x$ for all $x \in X'$ and all $t \in [0, 1]$, and $H(x, 1) \in X'$ for all $x \in X$. A *neighborhood* of X' in X is a subset $U \subseteq X$ with the property that there is an open subset $O \subseteq X$ with $X' \subseteq O \subseteq U$. The subspace X' of X is called a *neighborhood deformation retract* if there exists a neighborhood $U \subseteq X$ of X' such that X' is a deformation retract of U.

Exercise† 5.3. Let X be a topological space, let A be an abelian group, and let $n \geq 0$ be an integer. Let $X' \subseteq X$ be a non-empty closed subspace of X that is a neighborhood deformation retract. As usual, we write X/X' for the quotient of X obtained by identifying all points in X'.

(i) Show that the quotient map $X \to X/X'$ induces an isomorphism of relative homology groups
$$H_n(X, X'; A) \ \to \ H_n(X/X', X'/X'; A) \ .$$

(ii) Show that the relative homology groups $H_n(X, X'; A)$ of (X, X') are isomorphic to the reduced homology groups $\widetilde{H}_n(X/X'; A)$ of the quotient space X/X'.

* Please return Wednesday October 18, 2017. Exercises marked with † count for the bonus.

Exercise† 5.4. (Brouwer's fixed point theorem)
Let $n \geq 1$ be an integer, let $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the n-dimensional disk, and let $\partial D^n = \{x \in D^n \mid \|x\| = 1\}$ be its boundary.

(i) Show that there does not exist a continuous map $f\colon D^n \to \partial D^n$ with the property that $f(x) = x$ for all $x \in \partial D^n$. (Hint: Use our computation of the relative homology groups of (D^n, S^{n-1}).)

(ii) Show that every continuous map $f\colon D^n \to D^n$ has a fixed point, i.e., a point x with $f(x) = x$. (Hint: Use Part (i).)

Exercise 5.5. In the lecture, we used an inductive argument to show that the homology groups $H_n(D_n, S^{n-1}; A)$ and $H_n(S^n; A)$ are isomorphic to A. In this exercise, we will construct explicit isomorphisms relating these groups to A. For this we choose a homeomorphism $f\colon \Delta^n \to D^n$ and use that it restricts to a homeomorphism $\partial\Delta^n \to \partial D^n = S^{n-1}$ by Exercise 4.3.

Show that

$$A \to H_n(D^n, S^{n-1}; A), \qquad a \mapsto [a \cdot f]$$

is a well defined isomorphism of groups and that

$$A \to H_{n-1}(S^{n-1}; A), \qquad a \mapsto \textstyle\sum_{i=0}^{n} (-1)^i [a \cdot (f \circ \delta_i)]$$

is a well defined isomorphism of groups if $n \geq 2$.

Exercises for "Algebraic Topology"

Sheet 6*, October 18, 2017

Exercise 6.1. Let (X, d) be a compact metric space and let $(O_i)_{i\in I}$ be an open cover of X. Show that there is a real number $\varepsilon > 0$ such that for all $x \in X$, there exists an $i \in I$ such that

$$\{y \in X \mid d(x, y) < \varepsilon\} \subseteq O_i \ .$$

The number ε is called the *Lebesgue number* of the cover.

(Hint: Use the functions $d_i\colon X \to \mathbb{R},\ x \mapsto \inf_{y\in X\setminus O_i} d(x, y)$ and $d\colon X \to \mathbb{R},\ x \mapsto \sup_{i\in I} d_i(x)$.)

Exercise† 6.2. Let $m, n \geq 0$ be non-negative integers and let A be an abelian group. Choose basepoints $x_0 \in S^m$ and $y_0 \in S^n$ and let $S^m \vee S^n$ be the one-point union formed with respect to these basepoints. Compute $H_k(S^m \vee S^n; A)$ for all $k \geq 0$!

Definition. Let X be a topological space and let $\sim$ be an equivalence relation on the underlying set of X. Let $q\colon X \to X/{\sim}$ be the canonical map of sets from X to the set of equivalence classes of the equivalence relation $\sim$. Since q is surjective, we can endow $X/{\sim}$ with a topology where a set $O \subseteq X/{\sim}$ is open if and only if $q^{-1}(O) \subseteq X$ is open. The resulting topological space $X/{\sim}$ is called the *quotient of X by the equivalence relation* $\sim$.

Definition. Let X be a topological space. The *suspension* ΣX of X is the quotient of the product $X\times[0, 1]$ by the equivalence relation that is generated by the identifications $(x, 0) \sim (y, 0)$ for all $x, y \in X$ and the identifications $(x, 1) \sim (y, 1)$ for all $x, y \in X$. In other words, ΣX is the quotient of $X \times [0, 1]$ that is obtained by collapsing each of the subspaces $X \times \{0\}$ and $X \times \{1\}$ to a point.

Exercise† 6.3. Let X be a topological space, let A be an abelian group, and let $n \geq 0$ be an integer.

(i) Construct a natural isomorphism $\widetilde{H}_n(X; A) \cong \widetilde{H}_{n+1}(\Sigma X; A)$ relating the n-th reduced homology group of X to the $n + 1$-st reduced homology group of its suspension. (Hint: One approach is to cover ΣX by two cones and to consider the corresponding Mayer–Vietoris sequence.)

(ii) What is ΣS^n?

Definition. Let $f\colon X \to Y$ be a continuous map of topological spaces.

(i) Consider the equivalence relation on the disjoint union of $X \times [0, 1]$ and Y that is generated by the identifications $(x, 1) \sim f(x)$ for all $x \in X$. The quotient space

$$M_f = (X \times [0, 1] \textstyle\coprod Y)/{\sim}$$

is called the *mapping cylinder* of f.

(ii) Now consider the equivalence relation on $X \times [0, 1] \mathbin{\dot\cup} Y$ that is generated by the identifications $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (y, 0)$ for all $x, y \in X$. The quotient space $C_f = (X \times [0, 1] \coprod Y)/{\sim}$ is called the *mapping cone* of f.

* Please return Wednesday October 25, 2017. Exercises marked with † count for the bonus.

Exercise† 6.4. Let $f\colon X \to Y$ be a continuous map of topological spaces and let A be an abelian group.

(i) Let $\iota\colon X \to M_f$ be the continuous map defined by $\iota(x) = (x, 0)$. Construct a homotopy equivalence $p\colon M_f \to Y$ satisfying $p \circ \iota = f$.

(ii) Construct a long exact sequence of reduced homology groups

$$\cdots \to \widetilde{H}_n(X; A) \xrightarrow{f_*} \widetilde{H}_n(Y; A) \to \widetilde{H}_n(C_f; A) \to \widetilde{H}_{n-1}(X; A) \to \dots$$

(Hint: Apply Exercise 5.3 to the pair $(M_f, \iota(X))$.)

Exercise 6.5. Let $X = [0, 1] \times [0, 1]$ the product of two copies of the interval, and let $\sim$ be the equivalence relation on X that is generated by the identifications $(s, 0) \sim (s, 1)$ for all $s \in [0, 1]$ and $(0, t) \sim (1, t)$ for all $t \in [0, 1]$. We write $T = X/{\sim}$ for the resulting quotient space and note that it is homeomorphic to a torus.

Calculate $H_n(T; A)$ for all $n \geq 0$ and all abelian groups A! (Hint: Use the Mayer–Vietoris sequence.)

Exercises for "Algebraic Topology"

Sheet 7*, October 25, 2017

Exercise† 7.1. Show the following two statements about compact spaces:

(i) Any compact space is locally compact.

(ii) Let X and Y be topological spaces, let $x \in X$ be a point and let $L \subseteq Y$ be a compact subspace. If $O \subseteq X \times Y$ is an open subset containing $\{x\} \times L$, then there exists a neighborhood V of x in X with $V \times L \subseteq O$.

Exercise† 7.2. Let X be a topological space and let $f, g \colon \partial D^n \to X$ be two continuous maps. Let $X_f \cup_{\partial D^n} D^n$ be the space obtained by attaching an n-cell to X with attaching map f, and let $X_g \cup_{\partial D^n} D^n$ be the space obtained by attaching an n-cell to X with attaching map g.

Show that $X_f \cup_{\partial D^n} D^n$ and $X_g \cup_{\partial D^n} D^n$ are homotopy equivalent if f and g are homotopic.

Exercise 7.3. Let X be a topological space, let J be a set, let $f \colon J \times \partial D^n \to X$ be a continuous map, and let $Y = X \cup_{J \times \partial D^n} J \times D^n$ be the space obtained by attaching n-cells to X with attaching map f. As explained in the lecture, we can view X as a closed subspace of Y.

Show that X is a neighborhood deformation retract of Y in the sense defined on Exercise sheet 5.

Exercise† 7.4. Let $n \geq 1$ be an integer and let $\sim$ be the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ defined by

$$x \sim x' \text{ if and only if there exists a } \lambda \in \mathbb{R} \setminus \{0\} \text{ with } \lambda \cdot x = x' \ .$$

Let $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\!\sim$ be the resulting quotient space (which is called the *real projective space* of dimension n).

Show that $\mathbb{R}P^{n+1}$ can be obtained from $\mathbb{R}P^n$ by attaching an $n+1$-cell.

* Please return Wednesday November 1, 2017. Exercises marked with † count for the bonus.

Exercises for "Algebraic Topology"

Sheet 8*, November 1, 2017

Definition. Let (X, A) be a relative CW-complex. Then $X_n \setminus X_{n-1}$ is homeomorphic to $J_n \times \mathring{D}^n$ for a suitable indexing set J_n. This justifies to call a path component of $X_n \setminus X_{n-1}$ an *open n-cell of X*. The *closure of an n-cell* is defined to be the closure of the respective path component of $X_n \setminus X_{n-1}$ as a subset of the topological space X.

Exercise† 8.1. Let A be a Hausdorff space and let (X, A) be a relative CW-complex. Show the following two statements.

(i) The closure of an n-cell is compact and contained in X_n.
(Hint: Choose a characteristic map and argue with its image.)

(ii) Let $U \subseteq X$ be a subset with $A \subseteq U$. Then U is a closed subset of X if and only if the intersection of U with the closure of every cell of X is a closed subspace of X.

Exercise 8.2. Let A be a Hausdorff space, let (X, A) be a relative CW-complex, and let (Y, A) be a subcomplex of (X, A). Show that $Y \subseteq X$ is a closed subset.

Exercise† 8.3. Let A be a Hausdorff space, let (X, A) be a relative CW-complex, and let $Y \subseteq X$ be a closed subspace with $A \subseteq Y$. Suppose in addition that for every $n \geq 0$, the intersection $Y \cap (X_n \setminus X_{n-1})$ is a union of open n-cells of X. Show that the filtration of subspaces $Y_n = Y \cap X_n$ defines a CW-complex (Y, A).

Exercise† 8.4. Draw a picture of a CW-structure and compute the Euler characteristic for the following spaces:

(i) Möbius strip
(ii) Solid torus
(iii) Prezel surface
(iv) Klein bottle

* Please return Wednesday November 8, 2017. Exercises marked with † count for the bonus.

Exercises for "Algebraic Topology"

Sheet 9*, November 8, 2017

Definition. A topological space X is *contractible* if the identity of X is homotopic to a constant map. It is *locally contractible* if every neighborhood of a point $x \in X$ contains a contractible neighborhood of x.

Exercise 9.1. Show that any finite-dimensional absolute CW-complex X is locally contractible. (The statement also holds if X is not-necessarily finite-dimensional, but this general case requires a more difficult argument.)

Exercise† 9.2. Let X be an absolute CW-complex and let $Y \subseteq X$ be a subcomplex.

(i) Show that the pair (X, Y) inherits a CW-complex structure from X.
(ii) Show that the quotient space X/Y inherits a CW-complex structure from X.

Definition. A *cell decomposition* of a topological space X is a collection $\mathcal{E}$ of subspaces of X with the following two properties:

(i) Every $e \in \mathcal{E}$ is homeomorphic to the interior $\mathring{D}^n$ of D^n for some $n \geq 0$ (where $\mathring{D}^0 = D^0$ by convention).
(ii) As a set, X is the disjoint union of the subspace $e \in \mathcal{E}$.

We call $e \in \mathcal{E}$ an n-cell if e is homeomorphic to $\mathring{D}^n$. This is well defined since by the topological invariance of dimension, $\mathring{D}^m$ and $\mathring{D}^n$ can only be homeomorphic if $m = n$.

Exercise† 9.3. Let $\mathcal{E}$ be a cell decomposition of a Hausdorff space X with the following properties:

(i) For every n-cell $e \in \mathcal{E}$, there is a continuous map $\chi_e \colon D^n \to X$ which restricts to a homeomorphism $\chi_e \colon \mathring{D}^n \to e$ and which maps ∂D^n to the subspace of X given by the union of all cells of dimension strictly less than n.
(ii) For every cell $e \in \mathcal{E}$, its closure $\overline{e}$ as a subspace of X intersects only with finitely many other cells.
(iii) A subset $U \subseteq X$ is closed if and only if $U \cap \overline{e}$ is closed in X for every $e \in \mathcal{E}$.

Show that X admits a CW-structure. (The results in the lecture and Exercise 8.1 show that every CW-complex gives rise to a cell decomposition satisfying (i)-(iii). The name *CW-complex* is motivated by calling (ii) the *closure finiteness* and (iii) the *weak topology* property.)

* Please return Wednesday November 15, 2017. Exercises marked with † count for the bonus.

Definition. Let $\mathbb{R}^\infty = \mathbb{R}[\mathbb{N}_0]$ be the $\mathbb{R}$-linearization of the set of non-negative integers $\mathbb{N}_0$. The inclusion $\{0, \ldots, n\} \to \mathbb{N}_0$ induces an injection

$$\mathbb{R}^{n+1} = \mathbb{R}[\{0, \ldots, n\}] \to \mathbb{R}[\mathbb{N}_0] = \mathbb{R}^\infty$$

that allows us to view $\mathbb{R}^{n+1}$ as a subset of $\mathbb{R}^\infty$. We topologize $\mathbb{R}^\infty$ by declaring $O \subseteq \mathbb{R}^\infty$ to be open if and only if $O \cap \mathbb{R}^{n+1} \subseteq \mathbb{R}^{n+1}$ is open for all $n \geq 0$.

Via the inclusion $\mathbb{R}^{n+1} \to \mathbb{R}^\infty$, we also view the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ as a subset of $\mathbb{R}^\infty$. The *infinite dimensional sphere* $S^\infty \subset \mathbb{R}^\infty$ is defined to be the union

$$S^\infty = \bigcup_{n \geq 0} S^n .$$

We view S^∞ as a topological space equipped with subspace topology induced from $\mathbb{R}^\infty$.

Exercise† 9.4. (i) Construct a CW-structure on S^∞.

(ii) Show that S^∞ is contractible.

(Hint: One approach is to argue with the self-map $(x_0, x_1, \ldots) \mapsto (0, x_0, x_1, \ldots)$ of S^∞.)

Exercises for "Algebraic Topology"

Sheet 10*, November 15, 2017

Exercise† 10.1. Let $S^\infty \subset \mathbb{R}^\infty$ be the infinite dimensional sphere considered on exercise sheet 9. Let $\mathbb{R}P^\infty$ be the quotient of S^∞ modulo the equivalence relation $\sim$ given by $x \sim -x$ for $x \in S^\infty$. Construct a CW-structure on $\mathbb{R}P^\infty$ and compute $H_n(\mathbb{R}P^\infty;\mathbb{Z})$ for all $n \geq 0$.

(Hint: Use the corresponding computation for $\mathbb{R}P^n$ from the lecture.)

Exercise† 10.2. The Klein Bottle can be defined as the quotient space K of the square $[0,1]\times[0,1]$ modulo the equivalence relation $\sim$ that is generated by the identifications

$$(s,0) \sim (1-s,1) \text{ for all } x \in [0,1] \quad \text{and} \quad (0,t) \sim (1,t) \text{ for all } t \in [0,1]\ .$$

Compute $H_n(K;\mathbb{Z})$ for all $n \geq 0$ using cellular homology.

Exercise† 10.3. Let X and Y be finite CW-complexes and let I_n and J_n be the sets of n-cells of X and Y.

(i) Construct a CW-structure on the product space $X \times Y$ with set of n-cells $\bigcup_{p+q=n} I_p \times J_q$. (Hint: The bookkeeping simplifies if you describe the characteristic maps of the n-cells using $[0,1]^n$ instead of D^n.)

(ii) Show that the Euler characteristics of these CW-complexes satisfy $\chi(X\times Y) = \chi(X)\cdot\chi(Y)$.

Exercise 10.4. Let p be a prime number, let $\zeta = e^{\frac{2\pi i}{p}} \in \mathbb{C}$ be a primitive pth root of unity, let q be an integer that is not divisible by p, and view the 3-sphere

$$S^3 = \{(z_1,z_2)\in\mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$$

as a subspace of $\mathbb{C}^2$ in the indicated way. The *lens space* $L(p,q)$ the quotient of S^3 modulo the equivalence relation $\sim$ that is generated by $(z_1,z_2) \sim (z_1\zeta, z_2\zeta^q)$.

(i) Find a CW-structure on the space $L(p,q)$ with one n-cell for $0 \leq n \leq 3$.

(ii) Compute $H_n(L(p,q);\mathbb{Z})$ for all $n \geq 0$.

(Hint: Generalize the case $L(2,1) \cong \mathbb{R}P^3$ that was treated in the lecture.)

* Please return Wednesday November 22, 2017. Exercises marked with † count for the bonus.

Exercises for "Algebraic Topology"

Sheet 11*, November 22, 2017

Exercise† 11.1. Let (X, A) be a pair of spaces with the homotopy extension property and assume that the inclusion map $i\colon A \to X$ is homotopic to a constant map. Let $p\colon X \to X/A$ be the quotient map to the space obtained from X by collapsing A to a point. Show that there exists a continuous map $r\colon X/A \to X$ such that $r \circ p$ is homotopic to the identity on X.

Exercise† 11.2. Let (X, A) be a pair of spaces with the homotopy extension property and assume that A is a contractible space.

(i) Show that the quotient map $p\colon X \to X/A$ is a homotopy equivalence.

(ii) Give an example of a pair of spaces (X, A) that satisfies the assumptions of Exercise 11.1, but not the assumptions of the present exercise.

Exercise 11.3. Let X and Y be CW-complexes, let $A \subseteq X$ be a subcomplex of X, and let $A \to Y$ be a cellular map of CW-complexes. Show that the pushout $Y \cup_A X$ inherits a CW-structure whose cells correspond to the cells of Y and those cells of X which are not contained in A.

Exercise† 11.4. Let

$$K_n = \{x \in \mathbb{R}^2 \mid \|x - (\frac{1}{n}, 0)\| = \frac{1}{n}\}$$

be the circle with radius $\frac{1}{n}$ around the point $(\frac{1}{n}, 0)$ in $\mathbb{R}^2$. As in lecture 7, we let the *Hawaiian earring* $H = \bigcup_{n\in\mathbb{N}} K_n$ be the union of the circles K_n and view H as a topological space equipped with the subspace topology induced from $\mathbb{R}^2$. We write 0 for the point $(0, 0) \in H$.

(i) Show that there is no n such that the inclusion $K_n \to H$ is homotopic to a constant map. (Hint: Construct a retraction and use $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$.)

(ii) Let $f\colon H \to H$ be a continuous map that is homotopic to the identity. Show that it satisfies $f(0) = 0$. (Hint: Use (i) to show that $f(0) \neq 0$ leads to a contradiction.)

(iii) Show that the pair $(H, \{0\})$ does not have the homotopy extension property.

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* Please return Wednesday November 29, 2017. Exercises marked with † count for the bonus.

Exercises for "Algebraic Topology"

Sheet 12*, November 29, 2017

Definition. (i) Let X and Y be topological spaces. We write $[X, Y]$ for the set of homotopy classes of continuous maps from X to Y. Hence every continuous map $f\colon X \to Y$ represents an element $[f] \in [X, Y]$, and two maps $f, g\colon X \to Y$ represent the same element in $[X, Y]$ if there is a continuous map $H\colon X \times [0,1] \to Y$ with $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$.

(ii) Let X and Y be topological spaces with preferred basepoints $x_0 \in X$ and $y_0 \in Y$. We write $[(X, x_0), (Y, y_0)]_*$ for the set of basepoint preserving homotopy classes of basepoint preserving continuous maps from X to Y. Hence every continuous map $f\colon X \to Y$ with $f(x_0) = y_0$ represents an element $[f] \in [(X, x_0), (Y, y_0)]_*$, and two basepoint preserving maps $f, g\colon X \to Y$ represent the same element in $[(X, x_0), (Y, y_0)]_*$ if there is a continuous map $H\colon X \times [0,1] \to Y$ with $H|_{X\times\{0\}} = f$, $H|_{X\times\{1\}} = g$, and $H(x_0, t) = y_0$ for all $t \in [0,1]$.

(iii) Let $n \geq 0$ be an integer, let S^n be the n-dimensional sphere, and let $s_0 \in S^n$ be a basepoint. If Y is a topological space with basepoint $y_0 \in Y$, then we define $\pi_n(Y, y_0)$ to be the set $[(S^n, s_0), (Y, y_0)]_*$.

(Unraveling this definition, $\pi_0(Y, y_0)$ is the set of path components of Y, and $\pi_1(Y, y_0)$ is the (underlying set of) the fundamental group of Y with basepoint y_0.)

Exercise† 12.1. Let (X, A) be a relative CW-complex and let $x_0 \in A = X_{-1} \subseteq X$ be a basepoint. Show that the inclusion of the m-skeleton induces a map

$$\pi_n(X_m, x_0) \to \pi_n(X, x_0)$$

which is surjective if $m \geq n$ and bijective if $m \geq n + 1$.

(Hint: Use the Cellular Approximation Theorem.)

Exercise† 12.2. Let X and Y be topological spaces with basepoints $x_0 \in X$ and $y_0 \in Y$. Forgetting basepoints defines a map

$$\Phi\colon [(X, x_0), (Y, y_0)]_* \to [X, Y] \ .$$

(i) Suppose in addition that X is an absolute CW-complex and that x_0 is a 0-cell of X, that the set $\pi_0(Y, y_0)$ has only one element, and that for every $y \in Y$ the set $\pi_1(Y, y)$ has only one element. Show that in this situation, Φ is a bijective map.

(Hint: As a first step, show that $(X, \{x_0\})$ and $(X \times [0,1], X \times \{0,1\} \cup \{x_0\} \times [0,1])$ have the homotopy extension property.)

(ii) Show that in general, Φ is neither injective nor surjective.

(Hint: You may use without proof that any group arises as the fundamental group of a topological space.)

* Please return Wednesday December 6, 2017. Exercises marked with † count for the bonus.

Definition. Let A_1, A_2 and A_3 be sets with preferred elements $a_i \in A_i$. Let

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

be a sequence of maps of sets with $f_1(a_1) = a_2$ and $f_2(a_2) = a_3$. We say that this sequence is *exact* if

(i) the composite $f_2 \circ f_1$ is the constant map $x \mapsto a_3$ and
(ii) for any $y \in A_2$ with $f_2(y) = a_3$ there exists an $x \in A_1$ with $f_1(x) = y$.

Exercise† 12.3. Let (X, A) be a pair of spaces with the homotopy extension property, let $a_0 \in A$ be a basepoint, and let Y be a space with basepoint y_0. We write $i\colon A \to X$ for the inclusion map and $q\colon X \to X/A$ for the quotient map.

Consider the sequence

$$[(X/A, q(a_0)), (Y, y_0)]_* \xrightarrow{q^*} [(X, a_0), (Y, y_0)]_* \xrightarrow{i^*} [(A, a_0), (Y, y_0)]_*$$

where $q^*([f]) = [f \circ q]$ and $i^*([g]) = [g \circ i]$.

(i) Show that q^* and i^* are well-defined.
(ii) Show that above sequence is exact. Here we take the homotopy classes of the constant maps as the preferred elements in the sets of homotopy classes involved.

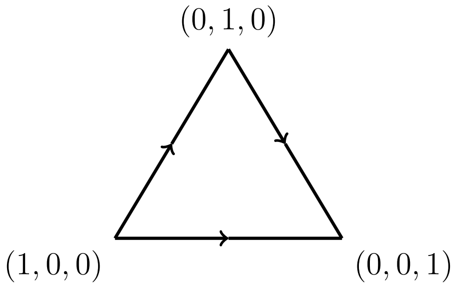
Exercise 12.4. Let X be a finite CW-complex that is path-connected. Show that X is homotopy equivalent to a finite CW-complex with a single 0-cell.

(Hint: Use Exercises 11.2(i) and 11.3.)

Exercises for "Algebraic Topology"

Sheet 13*, December 6, 2017

Exercise† 13.1. Let $\Delta^2 = \{(x_0, \ldots, x_2) \in \mathbb{R}^3 \mid x_i \geq 0, x_0 + x_1 + x_2 = 1\}$ be the 2-simplex. The *topological dunce hat* is the quotient space X obtained from Δ^2 by identifying the points on its 3 edges in the way indicated by the following picture:



More formally, X is the quotient of Δ^2 by the equivalence relation generated by

$$(1-t, t, 0) \sim (0, 1-t, t) \sim (1-t, 0, t) \text{ for each } t \in [0,1] \ .$$

Show that the topological dunce hat X is contractible.
(Hint: Identify X with a cell attachment.)

Definition. An *H-space* is a topological space X with a base point $e \in X$ and a continuous map $\mu\colon X \times X \to X$ such that $\mu(e,e) = e$ and such that the two maps $X \to X$ given by $x \mapsto \mu(x,e)$ and $x \mapsto \mu(e,x)$ are both homotopic relative to $\{e\}$ to the identity on X.

Exercise† 13.2. Let X be an H-space with multiplication $\mu\colon X \times X \to X$ and base point e.

(i) Show that that for all $n \geq 1$, the map

$$\pi_n(X,e) \times \pi_n(X,e) \xrightarrow{\cong} \pi_n(X \times X, (e,e)) \xrightarrow{\mu_*} \pi_n(X,e)$$

coincides with the group structure of $\pi_n(X,e)$. (Here the first map is the canonical bijection obtained from viewing a pair of maps $(f\colon S^n \to X, g\colon S^n \to X)$ as a map $S^n \to X \times X$.)

(ii) Show that the fundamental group $\pi_1(X,e)$ is abelian.

Exercise† 13.3. Let X be a CW-complex and suppose that $\mu\colon X \times X \to X$ defines an H-space structure on X whose base point $e \in X$ is a 0-cell for the CW-structure. Show that there exists a continuous map $\mu'\colon X \times X \to X$ such that $\mu'(e,x) = x = \mu'(x,e)$ for all $x \in X$.
(Hint: Use the homotopy extension property.)

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* Please return Wednesday December 13, 2017. Exercises marked with † count for the bonus.

Definition. Let
$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \dots$$
be a sequence of abelian groups and group homomorphisms. We define $\mathrm{colim}_{i\geq 0} A_i$ to be the cokernel of the group homomorphism
$$\bigoplus_{i\geq 0} A_i \xrightarrow{\mathrm{id} - \oplus\alpha_i} \bigoplus_{i\geq 0} A_i, \qquad (a_0, a_1, \dots) \mapsto (a_0, a_1 - \alpha_0(a_0), a_2 - \alpha_1(a_1), \dots)\ .$$
(One can show that $\mathrm{colim}_{i\geq 0} A_i$ has indeed the universal property of the colimit of the above sequence.)

Exercise 13.4. Let
$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$
be a sequence of topological spaces and continuous maps indexed by the natural numbers. The *mapping telescope* $\mathrm{tel}(X_i)$ of this sequence is the quotient space of the disjoint union
$$\textstyle\coprod_{i\geq 0} X_i \times [0,1]$$
by the equivalence relation $\sim$ that is generated by the identifications $(x,1) \sim (f_i(x), 0)$ for all $i \geq 0$ and all $x \in X_i$.

(i) Show that there is an isomorphism
$$\mathrm{colim}_{i\geq 0} H_n(X_i; A) \to H_n(\mathrm{tel}(X_i); A)$$
for all $n \geq 0$ and all abelian coefficient groups A.

(Hint: Let U and V be the images of
$$\textstyle\coprod_{i\geq 0,\ i \text{ even}} X_i \times [0,1] \quad \text{and} \quad \coprod_{i\geq 0,\ i \text{ odd}} X_i \times [0,1]$$
in $\mathrm{tel}(X_i)$ and argue with the Mayer–Vietoris sequence for the resulting cover.)

(ii) Let p be a prime number, view S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$ and let $f\colon S^1 \to S^1$ be the map $z \mapsto z^p$. Let X be the mapping telescope of the sequence
$$S^1 \xrightarrow{f} S^1 \xrightarrow{f} S^1 \xrightarrow{f} \dots$$
in which each space is S^1 and each map is f. Show that $H_1(X; \mathbb{Z})$ is isomorphic to the subgroup
$$\mathbb{Z}[\frac{1}{p}] = \{\frac{k}{p^n} \mid k, n \in \mathbb{Z}, n \geq 0\}$$
of the underlying additive group of the rationals $\mathbb{Q}$.

Exercises for “Algebraic Topology”

Sheet 14, December 13, 2017

Remark. The exercises on this last sheet are meant to give you an idea about how typical exam questions will look like. They differ from the homework exercises in that I will also ask you to reproduce definitions or the statements of important theorems. (Consequently, you are not allowed to use any notes or books during the exam.) The written exam will most likely consist of 4 or 5 exercises consisting of various subitems that can be answered individually. Given that the exam is scheduled to be 180 minutes long, it will probably be a little bit longer than the present exercise sheet.

The exercises on this final sheet do not need to be handed in and they do not count for the bonus.

Exercise 14.1. Let A be an abelian group and $n \geq 2$ be an integer. We let $\mu_n\colon A \to A$ be the group homomorphism sending $a \in A$ to its n-fold sum $a + \cdots + a$. The *n-torsion* of A is the subgroup $\mathrm{tor}_n(A) \subseteq A$ given by the kernel of μ_n, and nA denotes the image of μ_n.

Now let X be a topological space and let $C(X; A)$ be the singular chain complex of X with coefficients in A.

(i) Show that the group homomorphisms $\mu_n\colon C_k(X;\mathbb{Z}) \to C_k(X;\mathbb{Z})$ for $k \geq 0$ form a chain map $C(X;\mathbb{Z}) \to C(X;\mathbb{Z})$. We also denote it by $\mu_n\colon C(X;\mathbb{Z}) \to C(X;\mathbb{Z})$.

(ii) Show that there is a short exact sequence of chain complexes

$$0 \to C(X;\mathbb{Z}) \xrightarrow{\mu_n} C(X;\mathbb{Z}) \to C(X;\mathbb{Z}/n\mathbb{Z}) \to 0 \ .$$

(iii) Show that for every $k \geq 1$, there is a short exact sequence

$$0 \to H_k(X;\mathbb{Z})/(n\,H_k(X;\mathbb{Z})) \to H_k(X;\mathbb{Z}/n\mathbb{Z}) \to \mathrm{tor}_n(H_{k-1}(X;\mathbb{Z})) \to 0 \ .$$

(iv) Suppose in addition that for all $k \geq 0$, there exists a $\mathbb{Q}$-vector space structure on $H_k(X;\mathbb{Z})$ whose underlying additive group structure is the abelian group structure on $H_k(X;\mathbb{Z})$ resulting from the definition of singular homology. Show that $H_k(X;\mathbb{Z}/n\mathbb{Z})$ is trivial for all $n \geq 2$ and all $k \geq 1$.

Exercise 14.2. Let A be an abelian group.

(i) State the Excision Theorem for singular homology with coefficients in A.

(ii) Give an example of a space X, subspaces $Y \subseteq X' \subseteq X$ and a positive integer n such that the map $H_n(X \setminus Y, X' \setminus Y; A) \to H_n(X, X'; A)$ induced by the inclusion $X \setminus Y \subseteq X$ is not an isomorphism.

(iii) Let $m \geq 1$ be an integer and let X be a topological space such that every point of X admits an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^m$. Show that for every $x \in X$ and every $n \geq 0$, there are isomorphisms

$$H_n(X, X \setminus \{x\}; A) \cong \begin{cases} A & \text{if } n = m \\ 0 & \text{if } n \neq m \ . \end{cases}$$

Exercise 14.3. (i) Let X be a CW-complex and let A be an abelian group. Reproduce the definition of the cellular chain groups $\widetilde{C}_n(X;A)$ and the differential of the cellular chain complex considered in the lecture.

(ii) The torus T can be viewed as the quotient space of $\mathbb{R}^2$ by the equivalence relation $\sim$ generated by the identifications $(x, y) \sim (x + k, y + l)$ where $(x, y) \in \mathbb{R}^2$ and $k, l \in \mathbb{Z}$. Let $p\colon \mathbb{R}^2 \to T$ be the quotient map. For integers $a, b \in \mathbb{Z}$, the path

$$w_{a,b}\colon [0,1] \to \mathbb{R}^2, \qquad t \mapsto t \cdot (a, b)$$

has the property $p(w_{a,b}(0)) = p(w_{a,b}(1))$ and thus induces a continuous map $f_{a,b}\colon S^1 \to T$ from the circle $S^1 = [0,1]/0 \sim 1$ to the torus T.

Show that for every generator $[e^1]$ of $H_1(S^1, \mathbb{Z})$, the induced map on homology groups satisfy $(f_{a,b})_*([e^1]) = a \cdot (f_{0,1})_*([e^1]) + b \cdot (f_{1,0})_*([e^1])$ in $H_1(T, \mathbb{Z})$.

Exercise 14.4. (i) Reproduce the definition of the homotopy extension property for a pair of spaces (X, A) and the retract criterion for the homotopy extension property that applies if $A \subseteq X$ is a closed subspace.

(ii) Now let $f\colon X \to Y$ be a continuous map between topological spaces. As in the lecture, we let the mapping cylinder $M(f)$ be the pushout of

$$X \times [0,1] \xleftarrow{\mathrm{incl}_1} X \xrightarrow{f} Y \ ,$$

and write $j_X\colon X \to M(f)$ for the map induced by $\mathrm{incl}_0\colon X \to X \times [0,1]$. Via j_X, we view X as a subspace of $M(f)$. Show that the pair $(M(f), X)$ has the homotopy extension property.