Uitwerkingen huiswerk week 5

Opgave 17.
Bepaal primitieven $F(x)$ voor de volgende functies:

(i) $f(x) := \dfrac{1}{1+x}$, (ii) $f(x) := \dfrac{x}{1+x}$,

(iii) $f(x) := \dfrac{a^x}{b^x}$ met $a, b > 0,\ a, b \neq 1$, (iv) $f(x) := \dfrac{1}{\sqrt{a^2 - x^2}}$,

(v) $f(x) := \dfrac{1}{\sqrt{x-1} + \sqrt{x+1}}$, (vi) $f(x) := \dfrac{1}{1+\sin(x)}$,

Oplossing.

(i) $F(x) = \ln(1+x)$;

(ii) $f(x) = \frac{x+1}{1+x} - \frac{1}{1+x} = 1 - \frac{1}{1+x} \Rightarrow F(x) = x - \ln(1+x)$;

(iii) $f(x) = (\frac{a}{b})^x = \exp(\ln(\frac{a}{b})x) \Rightarrow F(x) = \frac{1}{\ln(\frac{a}{b})} \exp(\ln(\frac{a}{b})x) = \frac{1}{\ln(a)-\ln(b)}(\frac{a}{b})^x$;

(iv) $\arcsin(x)' = \frac{1}{\sqrt{1-x^2}}$ en $f(x) = \frac{1}{a}\frac{1}{\sqrt{1-(\frac{x}{a})^2}} \Rightarrow F(x) = \arcsin(\frac{x}{a})$;

(v) $f(x) = \frac{1}{\sqrt{x-1}+\sqrt{x+1}} \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = \frac{\sqrt{x+1}-\sqrt{x-1}}{(x+1)-(x-1)} = \frac{1}{2}(\sqrt{x+1} - \sqrt{(}x-1)) \Rightarrow$ $F(x) = \frac{1}{3}((x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}})$;

(vi) $f(x) = \frac{1}{1+\sin(x)} \frac{1-\sin(x)}{1-\sin(x)} = \frac{1-\sin(x)}{1-\sin^2(x)} = \frac{1}{\cos^2(x)} - \frac{\sin(x)}{\cos^2(x)}$. Verder geldt $\tan(x)' = (\frac{\sin(x)}{\cos(x)})' = \frac{1}{\cos^2(x)}$ en $(\frac{1}{\cos(x)})' = \frac{\sin(x)}{\cos^2(x)} \Rightarrow F(x) = \tan(x) - \frac{1}{\cos(x)}$.

Opgave 18.
Bereken de volgende integralen:

(i) $\displaystyle\int_0^1 (1-x)^n \, dx$ voor $n \in \mathbb{N}$ (ii) $\displaystyle\int_0^\pi \sin(mx) \, dx$ voor $m \in \mathbb{Z}$.

Oplossing.

(i) $f(x) = (1-x)^n \Rightarrow F(x) = \frac{-1}{n+1}(1-x)^{n+1} \Rightarrow \int_0^1 (1-x)^n \, dx = F(1) - F(0) = \frac{1}{n+1}$.

(ii) $f(x) = \sin(mx) \Rightarrow F(x) = \frac{-1}{m}\cos(mx) \Rightarrow \int_0^\pi \sin(mx) \, dx = F(\pi) - F(0) = \begin{cases} \frac{2}{m} & \text{als } m \text{ oneven} \\ 0 & \text{als } m \text{ even.} \end{cases}$

Opgave 19.
Bepaal de volgende integralen door partiële integratie:

$$\text{(i)} \int x^2 e^x \, dx, \qquad \text{(ii)} \int \sqrt{x}\, \ln(x)\, dx, \qquad \text{(iii)} \int \ln^2(x)\, dx,$$
$$\text{(iv)} \int \ln^3(x)\, dx, \qquad \text{(v)} \int \cos(\ln(x))\, dx, \qquad \text{(vi)} \int x\, \arctan(x)\, dx.$$

Oplossing.

(i) $\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2x e^x + \int 2 e^x \, dx = (x^2 - 2x + 2) e^x$;

(ii) $\int \sqrt{x}\, \ln(x)\, dx = \frac{2}{3} x^{\frac{3}{2}} \ln(x) - \frac{2}{3} \int \sqrt{x}\, dx = \frac{2}{3} x^{\frac{3}{2}} \ln(x) - \frac{4}{9} x^{\frac{3}{2}}$;

(iii) $\int \ln^2(x)\, dx = (\ln(x)x - x)\ln(x) - \int (\ln(x) - 1)\, dx = (\ln(x)x - x)\ln(x) - (\ln(x)x - x) + x = \ln^2(x)x - 2\ln(x)x + 2x$,

(iv) $\int \ln^3(x)\, dx = (\ln(x)x - x)\ln^2(x) - \int (\ln(x) - 1) 2\ln(x)\, dx = (\ln(x)x - x)\ln^2(x) - 2(\ln^2(x)x - 2\ln(x)x + 2x) + 2(\ln(x)x - x) = \ln^3(x)x - 3\ln^2(x)x + 6\ln(x)x - 6x$;

(v) $\int \cos(\ln(x))\, dx = x\cos(\ln(x)) + \int \sin(\ln(x)) = x\cos(\ln(x)) + x\sin(\ln(x)) - \int \cos(\ln(x)) = \frac{1}{2} x(\cos(\ln(x)) + \sin(\ln(x)))$;

(vi) $\int x\, \arctan(x)\, dx = \frac{1}{2} x^2 \arctan(x) - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx$
$= \frac{1}{2} x^2 \arctan(x) - \frac{1}{2} \int \frac{x^2+1}{1+x^2}\, dx + \frac{1}{2} \int \frac{1}{1+x^2}\, dx = \frac{1}{2} x^2 \arctan(x) - \frac{1}{2} x + \frac{1}{2} \arctan(x)$.

Opgave 20.
Bewijs de volgende reductie formules (m.b.v. partiële integratie):

$$\text{(i)} \int \sin^n(x)\, dx = -\frac{1}{n} \sin^{n-1}(x)\cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x)\, dx;$$
$$\text{(ii)} \int \cos^n(x)\, dx = \frac{1}{n} \cos^{n-1}(x)\sin(x) + \frac{n-1}{n} \int \cos^{n-2}(x)\, dx;$$
$$\text{(iii)} \int \frac{1}{(x^2+1)^n}\, dx = \frac{1}{2n-2} \frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{1}{(x^2+1)^{n-1}}\, dx.$$

Hint: Schrijf $\frac{1}{(x^2+1)^n} = \frac{1 + x^2 - x^2}{(x^2+1)^n} = \frac{1}{(x^2+1)^{n-1}} - \frac{x^2}{(x^2+1)^n}$.

Oplossing.

(i) $\int \sin^n(x)\, dx = \int \sin^{n-1}(x)\sin(x)\, dx$
$= -\sin^{n-1}(x)\cos(x) + \int (n-1)\sin^{n-2}(x)\cos(x)\cos(x)\, dx$
$= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\, dx - (n-1)\int \sin^{n-2}(x)\sin^2(x)\, dx$
$= -\frac{1}{n} \sin^{n-1}(x)\cos(x) + \frac{n-1}{n} \int \sin^{n-2}(x)\, dx$;

(ii) $\int \cos^n(x)\,dx = \int \cos^{n-1}(x)\cos(x)\,dx$
$= \cos^{n-1}(x)\sin(x) + \int (n-1)\cos^{n-2}(x)\sin(x)\sin(x)\,dx$
$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,dx - (n-1)\int \cos^{n-2}(x)\cos^2(x)\,dx$
$= \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$;

(iii) $\int \frac{1}{(x^2+1)^n}\,dx = \int \frac{1}{(x^2+1)^{n-1}}\,dx - \int \frac{x^2}{(x^2+1)^n}\,dx.$
Er geldt $(\frac{1}{(x^2+1)^{n-1}})' = -(n-1)\frac{2x}{(x^2+1)^n}$, dus is
$\int \frac{x^2}{(x^2+1)^n}\,dx = \int x\frac{x}{(x^2+1)^n}\,dx = x\frac{-1}{2n-2}\frac{1}{(x^2+1)^{n-1}} + \frac{1}{2n-2}\int \frac{1}{(x^2+1)^{n-1}}\,dx$ en dus $\int \frac{1}{(x^2+1)^n}\,dx = \int \frac{1}{(x^2+1)^{n-1}}\,dx + \frac{1}{2n-2}\frac{x}{(x^2+1)^n} - \frac{1}{2n-2}\int \frac{1}{(x^2+1)^{n-1}}\,dx = \frac{1}{2n-2}\frac{x}{(x^2+1)^n} + \frac{2n-3}{2n-2}\int \frac{1}{(x^2+1)^{n-1}}\,dx.$

Webpagina: http://www.math.ru.nl/∼souvi/calcanalyse/calcanalyse.html