Oefenopgaven

ontleend aan:

> Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence, *Linear Algebra*
>
> Charles W. Curtis, *Linear Algebra: An Introductory Approach*

Exercise 1.
Label the following statements as true or false. Justify your answers.

(i) Every vector space contains a zero vector.

(ii) A vector space may have more than one zero vector.

(iii) In any vector space, $ax = bx$ implies that $a = b$ (for scalars a and b and a vector x).

(iv) In any vector space, $ax = ay$ implies that $x = y$ (for vectors x and y and a scalar a).

Exercise 2.
In any $\mathbb{F}$-vector space V, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in \mathbb{F}$.

Exercise 3.
A real-valued function f defined on the real line is called an *even function* if $f(-x) = f(x)$ for each real number x. Prove that the set of even functions defined on the real line with the usual operations of addition and scalar multiplication for functions is a vector space.

Exercise 4.
Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) := (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) := (ca_1, a_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Exercise 5.
Let $V := \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{F}\}$, where $\mathbb{F}$ is a field. Define addition of elements of V coordinate wise, and for $c \in \mathbb{F}$ and $(a_1, a_2) \in V$, define

$$c(a_1, a_2) := (a_1, 0).$$

Is V a vector space over $\mathbb{F}$ with these operations? Justify your answer.

Exercise 6.
Let $V := \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) := (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) := (ca_1, ca_2).$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Exercise 7.
Let $V := \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of V coordinate wise, and for $(a_1, a_2) \in V$ and $c \in \mathbb{R}$, define

$$c(a_1, a_2) := \begin{cases} (0, 0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Exercise 8.
Let V and W be vector spaces over a field $\mathbb{F}$. Let

$$Z := \{(v, w) \mid v \in V \text{ and } w \in W\}.$$

Prove that Z is a vector space over $\mathbb{F}$ with the operations

$$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) := (cv_1, cw_1).$$

Exercise 9.
Label the following statements as true or false. Justify your answers.

(i) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace (= lineaire deelruimte) of V.

(ii) The empty set is a subspace of every vector space.

(iii) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

(iv) The intersection of any two subsets of V is a subspace of V.

(v) Let W be the xy-plane of $\mathbb{R}^3$; that is $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$.

Exercise 10.
Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

(i) $W_1 := \{(x, y, z) \in \mathbb{R}^3 \mid x = 3y \text{ and } z = -y\}$

(ii) $W_2 := \{(x, y, z) \in \mathbb{R}^3 \mid x = z + 2\}$

(iii) $W_3 := \{(x, y, z) \in \mathbb{R}^3 \mid 2x - 7y + z = 0\}$

(iv) $W_4 := \{(x, y, z) \in \mathbb{R}^3 \mid x - 4y - z = 0\}$

(v) $W_5 := \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y - 3z = 1\}$

(vi) $W_6 := \{(x, y, z) \in \mathbb{R}^3 \mid 5x^2 - 3y^2 + 6z^2 = 1\}$

Exercise 11.
Prove that $U_1 := \{(x_1, \dots, x_n) \in \mathbb{F}^n \mid x_1 + \ldots + x_n = 0\}$ is a subspace of $\mathbb{F}^n$, but $U_2 := \{(x_1, \dots, x_n) \in \mathbb{F}^n \mid x_1 + \ldots + x_n = 1\}$ is not.

Exercise 12.
Let S be a nonempty set and $\mathbb{F}$ a field. Let $\mathbb{F}_0^S$ denote the set of all functions $f : S \to \mathbb{F}$ such that $f(x) \neq 0$ only for finitely many elements of S. Prove that $\mathbb{F}_0^S$ is a subspace of the vector space of all functions from S to $\mathbb{F}$.

Exercise 13.
Prove that a subset W of a vector space V is a subspace of V if and only if $0 \in W$ and $ax + y \in W$ whenever $a \in \mathbb{F}$ and $x, y \in W$.

Exercise 14.
In each part, determine whether the given vector v is contained in the span (= opspansel) $L(v_1, v_2)$)of the vectors v_1 and v_2.

(i) $v = (2, -1, 1)$, $v_1 = (1, 0, 2)$, $v_2 = (-1, 1, 1)$

(ii) $v = (-1, 2, 1)$, $v_1 = (1, 0, 2)$, $v_2 = (-1, 1, 1)$

(iii) $v = (-1, 1, 1, 2)$, $v_1 = (1, 0, 1, -1)$, $v_2 = (0, 1, 1, 1)$

(iv) $v = (2, -1, 1, -3)$, $v_1 = (1, 0, 1, -1)$, $v_2 = (0, 1, 1, 1)$

Exercise 15.
Show that the vectors $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ generate $\mathbb{F}^3$.

Exercise 16.
Label the following statements as true or false. Justify your answers.

(i) If $S := (v_1, \dots, v_n)$ is linearly dependent, then each vector in S is a linear combination of other vectors in S.

(ii) Subsets of linearly dependent sets are linearly dependent.

(iii) Subsets of linearly independent sets are linearly independent.

Exercise 17.
Determine whether the following sets are linearly dependent or linearly independent.

(i) $((1, -1, 2), (1, -2, 1), (1, 1, 4))$ in $\mathbb{R}^3$.

(ii) $((1,-1,2),(2,0,1),(-1,2,-1))$ in $\mathbb{R}^3$.

Exercise 18.
Let $S := ((1,1,0),(1,0,1),(0,1,1))$ be a 3-tupel of vectors in the vector space $\mathbb{F}^3$.

(i) Show that S is linearly independent if $\mathbb{F} = \mathbb{R}$.

(ii) Show that S is linearly dependent if $\mathbb{F} = \mathbb{F}_2$.

Exercise 19.
Let u and v be distinct vectors in a vector space V. Show that (u, v) is linearly dependent if and only if u or v is a multiple of the other.

Exercise 20.
Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

Exercise 21.
Let $f, g \in \mathbb{R}^{\mathbb{R}}$ be the functions defined by $f(x) := e^{rx}$ and $g(x) := e^{sx}$, where $r \neq s$. Prove that (f, g) is linearly independent in $\mathbb{R}^{\mathbb{R}}$.

Exercise 22.
Label the following statements as true or false. Justify your answers.

(i) The zero vector space has no basis.

(ii) Every vector space that is generated by a finite set has a basis.

(iii) Every vector space has a finite basis.

(iv) A vector space cannot have more than one basis.

(v) If a vector space has a finite basis, then the number of vectors in every basis is the same.

(vi) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.

(vii) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.

(viii) Every subspace of a finite-dimensional space is finite-dimensional.

(ix) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V (= is een volledig stelsel).

Exercise 23.
Determine which of the following tupels are bases for $\mathbb{R}^3$.

(i) $((1,0,-1),(2,5,1),(0,-4,3))$

(ii) $((2,-4,1),(0,3,-1),(6,0,-1))$

(iii) $((1,2,-1),(1,0,2),(2,1,1))$

(iv) $((-1,3,1),(2,-4,-3),(-3,8,2))$

(v) $((1,-3,-2),(-3,1,3),(-2,-10,-2))$

Exercise 24.
The vectors $u_1 = (2,-3,1)$, $u_2 = (1,4,-2)$, $u_3 = (-8,12,-4)$, $u_4 = (1,37,-17)$ and $u_5 = (-3,-5,8)$ generate $\mathbb{R}^3$. Find a subset of u_1, u_2, u_3, u_4, u_5 that is a basis for $\mathbb{R}^3$.

Exercise 25.
Let W denote the subspace of $\mathbb{R}^5$ consisting of all the vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2,-3,4,-5,2), & u_2 &= (-6,9,-12,15,-6),\\ u_3 &= (3,-2,7,-9,1), & u_4 &= (2,-8,2,-2,6),\\ u_5 &= (-1,1,2,1,-3), & u_6 &= (0,-3,-18,9,12),\\ u_7 &= (1,0,-2,3,-2), & u_8 &= (2,-1,1,-9,7) \end{aligned}$$

generate W. Find a subset of $u_1, u_2, \dots, u_8$ that is a basis for W.

Exercise 26.
The vectors $v_1 = (1,1,1,1)$, $v_2 = (0,1,1,1)$, $v_3 = (0,0,1,1)$ and $v_4 = (0,0,0,1)$ from a basis for $\mathbb{F}^4$. Find the unique representation of an arbitrary vector (x_1,x_2,x_3,x_4) in $\mathbb{F}^4$ as a linear combination of v_1, v_2, v_3 and v_4.

Exercise 27.
Let u and v be distinct vectors of a vector space V. Show that if (u,v) is a basis for V and a and b are nonzero scalars, then both $(u+v, au)$ and (au, bv) are also bases for V.

Exercise 28.
Let u, v and w be distinct vectors of a vector space V. Show that if (u,v,w) is a basis for V, then $(u+v+w, v+w, w)$ is also a basis for V.

Exercise 29.
The set of solutions to the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0\\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

Exercise 30.
Let U and W be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(i) Prove that $\dim(U \cap W) \leq n$.

(ii) Prove that $\dim(U + W) \leq m + n$.

Exercise 31.
Determine which of the following subsets of $\mathbb{R}^n$ are subspaces.

(i) All vectors $(x_1, \ldots, x_n)$ such that $x_1 = 1$.

(ii) All vectors $(x_1, \ldots, x_n)$ such that $x_1 = 0$.

(iii) All vectors $(x_1, \ldots, x_n)$ such that $x_1 + 2x_2 = 0$.

(iv) All vectors $(x_1, \ldots, x_n)$ such that $x_1 + x_2 + \ldots + x_n = 1$.

(v) All vectors $(x_1, \ldots, x_n)$ such that $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$ for fixed $a_1, \ldots, a_n$ in $\mathbb{R}$.

(vi) All vectors $(x_1, \ldots, x_n)$ such that $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ for fixed $a_1, \ldots, a_n, b$ in $\mathbb{R}$.

(vii) All vectors $(x_1, \ldots, x_n)$ such that $x_1^2 = x_2$.

Exercise 32.
Test the following tupels of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ to determine whether or not they are linearly independent.

(i) $((1, 1), (2, 1))$

(ii) $((1, 1), (2, 1), (1, 2))$

(iii) $((0, 1), (1, 0))$

(iv) $((0, 1), (1, 0), (x, y))$

(v) $((1, 1, 2), (3, 1, 2), (-1, 0, 0))$

(vi) $((3, -1, 1), (4, 1, 0), (-2, -2, -2))$

(vii) $((1, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 1))$

Exercise 33.
Find a set of linearly independent generators of the subspace of $\mathbb{R}^3$ consisting of all solutions of the equation

$$x_1 - x_2 + x_3 = 0.$$

Exercise 34.
Let v in $\mathbb{R}^n$ be a linear combination of vectors $u_1, \ldots, u_r$ in $\mathbb{R}^n$, and let each

vector u_i, $1 \le i \le r$, be a linear combination of vectors $w_1, \dots, w_s$. Prove that v is a linear combination of $w_1, \dots, w_s$.

Exercise 35.
Determine whether $(1,1,1)$ belongs to the subspace of $\mathbb{R}^3$ generated by $(1,3,4)$, $(4,0,1)$, $(3,1,2)$. Explain your reasoning.

Exercise 36.
Determine whether $(2,0,-4,-2)$ belongs to the subspace of $\mathbb{R}^4$ generated by $(0,2,1,-1)$, $(1,-1,1,0)$, $(2,1,0,-2)$.

Exercise 37.
Let U and W be two-dimensional subspaces of $\mathbb{R}^3$. Prove that $\dim(U \cap W) \ge 1$.

Exercise 38.
Let

$$\begin{aligned} &v_1 = (2,1,0,-1), \quad v_2 = (1,-3,2,0), \quad v_3 = (-2,0,6,1), \\ &v_4 = (4,8,-4,-3), \quad v_5 = (1,10,-6,-2), \quad v_6 = (3,-1,2,4), \end{aligned}$$

and let

$$U := L(v_1, v_2, v_3, v_4), \qquad W := L(v_4, v_5, v_6).$$

Find $\dim U$, $\dim W$, $\dim(U+W)$ and $\dim(U \cap W)$. Also, determine a basis of $U \cap W$.

Webpagina: http://www.math.ru.nl/~souvi/la1_08/la1.html