Groups and representations

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January 2010

Contents

Literature

Standard references for this course are:

- J.L. Alperin, R.B. Bell: *Groups and Representations.* Springer, 1995.
- W. Burnside: *Theory of Groups of Finite Order.* Dover, 2004
- M. Burrow: *Representation Theory of Finite Groups.* Dover, 1993.
- C.W. Curtis, I. Reiner: *Methods of Representation Theory.* Wiley, 1981.
- L.C. Grove: *Groups and Characters.* Wiley, 1997.
- B. Huppert: *Endliche Gruppen I (Chapter V).* Springer, 1967.
- I.M. Isaacs: *Character Theory of Finite Groups.* AMS, 2006.
- G. James, M. Liebeck: *Representations and Characters of Groups.* Cambridge University Press, 2001.
- J.-P. Serre: *Linear Representations of Finite Groups.* Springer, 1996.

Chapter 1

Modules and representations

General note: During this course, all modules and algebras are assumed to be finite dimensional. Mostly we will be concerned with finite groups, but many general notions are also valid for infinite groups. This will be appropriately indicated.

1.1 Representations

1.1.1 Definition Let K be a field and V an n-dimensional K-vector space. Then $GL(V) := \{\varphi : V \to V \mid \varphi \text{ linear, invertible }\}$ is called the *general linear group* of V.

By fixing a basis $(v_1, \ldots, v_n)$ of V, the group $GL(V)$ is seen to be canonically isomorphic with $GL_n(K)$ via the mapping $GL(V) \to GL_n(K) : \varphi \mapsto (a_{ij})$, where $v_i\varphi = \sum_{j=1}^{n} a_{ij}v_j$.

> Note that we will use row convention in this course. This means that matrices act on vectors from the right (i.e. on row vectors) and that the matrix of a linear mapping contains the coordinate vector of the image of the i-th basis vector in its i-th row. This choice is made partially to be compatible with the computeralgebra system Magma which represents vectors as rows.

1.1.2 Definition Let G be a group, K a field.

(i) For a K-vector space V, a group homomorphism $\delta : G \to GL(V)$ is called a *(K-)representation* of G.

(ii) A group homomorphism $\Delta : G \to GL_n(K)$ is called a *(matrix) representation* of G of degree n over K. If Δ is obtained from a K-representation δ by choosing a basis of V we say that Δ belongs to δ.

(iii) Two matrix representations $\Delta, \Delta' : G \to GL_n(K)$ belong to the same representation δ if and only if there exists a matrix $T \in GL_n(K)$ such that $\Delta'(g) = T\Delta(g)T^{-1}$ for all $g \in G$ (T is the basis transformation from the basis corresponding to Δ to the basis corresponding to Δ'). In this case, Δ and Δ' are called *equivalent* representations.

1.1.3 Examples

(1) The mapping $\Delta : G \to K^* : g \mapsto 1$ is called the *trivial representation* of G (over K).

(2) Let $C_n = \langle g \rangle$ be the cyclic group of order n. Then C_n has the 1-dimensional representations $\Delta_k : C_n \to GL_1(\mathbb{C}) : g \mapsto \exp(\frac{2\pi i}{n}k)$ which are pairwise inequivalent.

(3) Let G act on the set $\{1, \dots, n\}$ (e.g. if G is given as a permutation group of degree n) and let $(v_1, \dots, v_n)$ be a basis of the n-dimensional vector space K^n. Then the homomorphism $\Delta : G \to GL_n(K), g \mapsto \Delta(g)$ with $v_i\Delta(g) := v_{i\cdot g}$ is called a *permutation representation* of G.

(4) The symmetric group S_3 acts on $\{1, 2, 3\}$ and is generated by $g = (1, 2)$ and $h = (2, 3)$. The corresponding permutation representation Δ is given by:

$$\Delta(g) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta(h) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

If we choose $(v_1, v_2, v_1 + v_2 + v_3)$ as basis of K^3, then Δ is transformed into the equivalent representation Δ' with

$$\Delta'(g) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta'(h) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we pick out the upper left 2×2 submatrices from Δ' we obtain a representation Δ_2 of degree 2 of S_3, given by

$$\Delta_2(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Delta_2(h) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

1.2 Group ring

1.2.1 Definition Let G be a group, K a commutative ring, then the set

$$KG := \{\sum_{g \in G} a_g g \mid a_g \in K, a_g \neq 0 \text{ only for finitely many } g\}$$

of finite formal sums is called the *group ring* of G over K. Addition and multiplication in KG are defined by

$$(\sum_{g \in G} a_g g) + (\sum_{g \in G} b_g g) := \sum_{g \in G} (a_g + b_g) g$$

$$(\sum_{g \in G} a_g g)(\sum_{g \in G} b_g g) := \sum_{g,h \in G} (a_g b_h) gh = \sum_{g \in G} (\sum_{h \in G} a_h b_{h^{-1}g} g)$$

If K is a field, KG is also called the *group algebra* of G over K (recall that a K-algebra is a ring which at the same time is a K-vector space).

1.2.2 Remarks

(1) By straightforward computation it can be seen that the group ring KG is an associative K-algebra. Checking associativity is a somewhat tedious calculation which nevertheless everybody should have gone through once.

(2) The group G and the ring K are always regarded as being embedded into KG.

(3) KG is commutative if and only if G is an abelian group.

(4) If G contains an element $g \neq 1$ of finite order, then KG has zero divisors: Let g be of order n and define $a := \sum_{i=1}^{n} g^i$. Then $a^2 = \sum_{i=1}^{n}(g^i a) = n \cdot a$, since $g^i a = a$ for all i. This shows that $0 = a^2 - n \cdot a = a(a - n \cdot 1)$. But $a \neq 0$ and $a \neq n \cdot 1$, since $g \neq 1$. Hence, a and $a - n \cdot 1$ are zero divisors.

1.2.3 Examples

(1) Let $C_3 = \langle g \rangle$ be the cyclic group of order 3. An arbitrary element of KC_3 is given by $a_0 \cdot 1 + a_1 \cdot g + a_2 \cdot g^2$. The product of two such elements gives $(a_0 \cdot 1 + a_1 \cdot g + a_2 \cdot g^2)(b_0 \cdot 1 + b_1 \cdot g + b_2 \cdot g^2) = (c_0 \cdot 1 + c_1 \cdot g + c_2 \cdot g^2)$ with $c_0 = a_0 b_0 + a_1 b_2 + a_2 b_1$, $c_1 = a_0 b_1 + a_1 b_0 + a_2 b_2$, $c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$.

(2) Let Δ be the permutation representation of S_3. Then $\Delta(KS_3)$ is a 5-dimensional K-algebra: It is clear that $\dim \Delta(KS_3) \leq 6$, since KS_3 is a 6-dimensional K-algebra. By choosing the basis $(v_1 - v_2, v_1 - v_3, v_1 + v_2 + v_3)$, the permutation representation is transformed into

$$\Delta'((1,2)) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta'((2,3)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which consists of block diagonal matrices with blocks of sizes 2 and 1. Since the shape of block diagonal matrices is preserved under multiplication, this shows that the dimension is at most 5. It can easily be checked that the 2×2 blocks of the group elements indeed contain a basis of $K^{2\times 2}$, therefore the image has in fact dimension 5.

1.2.4 Remark The group rings of cyclic groups can be described in a uniform manner. Let $G = \langle g \rangle \cong C_n$ be a cyclic group of order n. Then the homomorphism $K[x] \to KG : x \mapsto g$ shows that $KG \cong K[x]/(x^n - 1)$ (note that $x^n - 1$ lies in the kernel and that the dimensions of the algebras are the same). The ideals of KG are now easily seen, since ideals in $K[x]/(x^n - 1)$ are of the form $I/(x^n - 1)$ where $I \trianglelefteq K[x]$ is an ideal containing $(x^n - 1)$. But since $K[x]$ is a principal ideal domain this means that $I = (f)$ with $f \mid x^n - 1$. The ideals of KG therefore are in $1-1$-correspondence with the divisors of $x^n - 1$.

If the characteristic $\mathrm{char}(K)$ of K is not a divisor of n, the polynomial $x^n - 1$ is separable (i.e. it has no multiple roots in its splitting field) and therefore $x^n - 1 = f_1 f_2 \ldots f_r$ with f_i distinct irreducible elements of $K[x]$. By the Chinese remainder theorem we can conclude that $KG \cong K[x]/(x^n - 1) \cong K[x]/(f_1) \oplus$

$\ldots \oplus K[x]/(f_r)$. For example, $\mathbb{Q}C_8 \cong \mathbb{Q}(\zeta_8) \oplus \mathbb{Q}(i) \oplus \mathbb{Q} \oplus \mathbb{Q}$ and $\mathbb{C}C_8 \cong \mathbb{C} \oplus \ldots \oplus \mathbb{C}$ (8 times).

The situation is different for $n = \mathrm{char}(K) = p$. In this case $x^p - 1 = (x-1)^p$, therefore the ideals of KG form a chain $KG \supset (g-1) \supset (g-1)^2 \supset \ldots \supset (g-1)^{p-1} \supset \{0\}$.

1.2.5 Remark A representation Δ of G can be uniquely extended to a K-algebra homomorphism $\tilde{\Delta} : KG \to K^{n \times n}$ by

$$\tilde{\Delta}(\sum_{g \in G} a_g g) := \sum_{g \in G} a_g \Delta(g).$$

1.2.6 Proposition *Let K be a field, G, H finite groups and $\varphi : G \to H$ a group homomorphism. Then there exists a unique extension $\hat{\varphi} : KG \to KH$ of φ. Moreover, $\hat{\varphi}$ is injective (surjective) if and only if φ is injective (surjective). The kernel $\ker(\hat{\varphi})$ is the ideal of KG generated by $\{g - 1 \in KG \mid g \in \ker(\varphi)\}$.*

Proof: The uniqueness of $\hat{\varphi}$ follows from the fact that the elements of G are a basis for the K-algebra KG. Comparing the dimensions of KG and $\hat{\varphi}(KG)$ shows the statement about injectivity (surjectivity). Finally, it is clear that for $g \in \ker(\varphi)$ the element $g - 1$ lies in $\ker(\hat{\varphi})$. By the homomorphism theorem we have $\dim \hat{\varphi}(KG) = |G| - \dim \ker(\hat{\varphi})$ and since $\dim \hat{\varphi}(KG) = |\varphi(G)| = |G|/|\ker(\varphi)|$ it follows that $\dim \ker(\hat{\varphi}) = |G| - |G|/|\ker(\varphi)|$.

We now choose a transveral (set of coset representatives) $t_1, \ldots, t_r$ of $\ker(\varphi)$ in G and write the elements of $\ker(\varphi)$ as $h_1 = 1, h_2, \ldots, h_s$. Then $\{t_i h_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ is the set of elements of G and therefore a basis of KG. By a basis transformation we obtain $\{t_i, t_i(h_j - 1) \mid 1 \leq i \leq r, 2 \leq j \leq s\}$ as a different basis for KG and in this basis we find $|G| - |G|/|\ker(\varphi)|$ elements which are in the kernel of $\hat{\varphi}$.

1.3 Modules

1.3.1 Definition Let $(V, +)$ be an abelian group and A a ring (e.g. a group ring KG). Then V is called a (right) A-module if there is a mapping $V \times A \to V$ with

(i) $(v + w)a = va + wa$ for all $v, w \in V$, $a \in A$,

(ii) $v(a + b) = va + vb$ for all $v \in V$, $a, b \in A$,

(iii) $v(ab) = (va)b$ for all $v \in V$, $a, b \in A$,

(iv) $v1 = v$ for all $v \in V$.

We think of module elements as row vectors, therefore mappings on modules will be written on the right.

1.3.2 Examples

(1) If $\Delta : G \to GL_n(K)$ is a representation of G then K^n can be turned into a KG-module by $v(\sum_{g\in G} a_g g) := \sum_{g\in G} a_g(v\Delta(g))$.

(2) The group ring KG is a KG-module. For a finite group G this is called the *regular module*. The corresponding representation (of degree $|G|$) is called the *regular representation* of G. If the elements of G are taken as basis elements for KG this yields a permutation representation of G.

1.3.3 Definition Let A be a ring and V an A-module.

(i) A subgroup $W \leq V$ is called an A*-submodule* of V (denoted as $W \leq_A V$) if W is closed under the action of A. In that case, the factor module V/W is also an A-module with the action $(v+W)a = va + W$.

(ii) V is called a *simple* A*-module* or *irreducible* if $V \neq \{0\}$ and $\{0\}$ and V are the only A-submodules of V. Otherwise V is called *reducible*.

(iii) V is called a *indecomposable* if $V = W \oplus U$ with $W, U \leq_A V$ implies that $W = \{0\}$ or $U = \{0\}$.

(iv) A sequence $V = V_0 > V_1 > \ldots > V_n = \{0\}$ is called an A*-composition series of* V if $V_i \leq_A V$ for all i and if V_{i-1}/V_i are simple A-modules. The number n is called the *length* of the composition series, the factor modules V_{i-1}/V_i are called its *factors*.

1.3.4 Remark For an A-module V and $U, W \leq_A V$, also $U + W := \{u + w \mid u \in U, w \in W\}$ and $U \cap W$ are A-submodules of V.

1.3.5 Remark Representations are called reducible, irreducible, indecomposable etc. if the underlying modules have this property.
If Δ is a representation of KG then the corresponding KG-module V is reducible if and only if there exists $T \in GL_n(K)$ such that

$$T\Delta(g)T^{-1} = \begin{pmatrix} \Delta_1(g) & 0 \\ (*) & \Delta_2(g) \end{pmatrix} \text{ for all } g \in G$$

where Δ_1, Δ_2 are representations of degrees $1 \leq m, n-m < n$ of G. In that case, Δ_1 is the representation of G on a proper submodule $W \leq_{KG} V$ and Δ_2 is the representation on V/W.
If V is decomposable, then T can be chosen such that $(*)$ is 0.

1.3.6 Example Let Δ be the 2-dimensional representation of S_3 given by

$$\Delta((1,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Delta((2,3)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Then $\Delta((1,2))$ has eigenvectors $v_1 + v_2$ (eigenvalue 1) and $v_1 - v_2$ (eigenvalue -1) and $\Delta((2,3))$ has eigenvectors v_1 (eigenvalue 1) and $v_1 + 2v_2$ (eigenvalue -1).
If $\mathrm{char}(K) \neq 3$, there is no common eigenvector, hence there is no S_3-invariant

1-dimensional subspace and Δ is thus irreducible.
If $\mathrm{char}(K) = 3$, then $v_1 - v_2 = v_1 + 2v_2$ is a common eigenvector and with respect to the basis $(v_1 - v_2, v_1)$ the representation is transformed into Δ' with

$$\Delta'((1,2)) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \Delta'((2,3)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, this representation is reducible, but it is indecomposable, since the subspace generated by $v_1 - v_2$ has no S_3-invariant complement.

1.3.7 Definition Let V, W be A-modules for a ring A. A group homomorphism $\varphi : V \to W$ is called an *A-module homomorphism* if $(v\varphi)a = (va)\varphi$ for all $v \in V, a \in A$. Two A-modules which are isomorphic by an A-module isomorphism are denoted by $V \cong_A W$.

1.3.8 Theorem *(Jordan-Hölder)*
Let A be a ring, V an A-module with two composition series $V = V_0 > V_1 > \ldots > V_n = \{0\}$ and $V = W_0 > W_1 > \ldots > W_m = \{0\}$.
Then the two composition series are equivalent, i.e. the lengths of the two series are equal and there is a permutation $\pi \in S_n$ such that $V_{i-1}/V_i \cong_A W_{i\pi-1}/W_{i\pi}$.

This theorem can be proved using the Schreier-Zassenhaus refinement theorem which asserts that the submodules $V_{ij} := \langle V_i, (V_{i-1} \cap W_j)\rangle$ and $W_{ij} := \langle W_j, (W_{j-1} \cap V_i)\rangle$ form a composition series with $V_{i,j-1}/V_{ij} \cong W_{i-1,j}/W_{ij}$. We give a different proof using the first isomorphism theorem.

1.3.9 Theorem *(First Isomorphism Theorem)*
Let A be a ring, V an A-module and $U, W \leq_A V$. Then $(U + W)/W \cong_A U/(U \cap W)$.

Proof: The mapping $\varphi : U \to (U + W)/W : u \mapsto u + W$ is an A-module homomorphism and has kernel $U \cap W \leq_A V$. □

Proof: (Jordan-Hölder) We use induction on the length n of a composition series of V. For $n = 1$, the module V is simple and there is nothing to prove.
Now assume that $n > 1$ and that $V = V_0 > V_1 > \ldots > V_n = \{0\}$ and $V = W_0 > W_1 > \ldots > W_m = \{0\}$ are A-composition series of V. If $V_1 = W_1$, we are done by induction, since V_1 has a composition series of length $n - 1$.
If $V_1 \neq W_1$ we have $V_1 + W_1 = V$, since $V_1 + W_1$ is a module properly containing V_1 and V/V_1 is simple. Define U to be the intersection $V_1 \cap W_1$, then $U \leq_A V$. We see that U has a composition series by looking at the quotients $(U \cap V_{i-1})/(U \cap V_i)$. By the first isomorphism theorem we have $(U \cap V_{i-1})/(U \cap V_i) \cong_A (V_i + (U \cap V_{i-1}))/V_i \leq_A V_{i-1}/V_i$ and thus the quotients are either trivial or isomorphic to V_{i-1}/V_i.
Let $U = U_0 > U_1 > \ldots > U_r = \{0\}$ be a composition series of U. Then $V_1 > \ldots > V_n$ and $V_1 > U > U_1 > \ldots > U_r$ are two composition series of V_1 which are equivalent by induction. Similarly we see that $W_1 > \ldots > W_m$ and $W_1 > U > U_1 > \ldots > U_r$ are two equivalent composition series of W_1. This

shows that $n-1 = r+1 = m-1$ and thus $n = m$. Finally, we conclude from the first isomorphism theorem that $V/V_1 = (V_1+W_1)/V_1 \cong_A W_1/(V_1 \cap W_1) = W_1/U$ and that $V/W_1 \cong_A V_1/U$. Therefore the factors in the composition series $V > V_1 > \ldots > V_n$ are $\{V/V_1, V_1/U, U_{i-1}/U_i (i = 1 \ldots r)\}$ and by replacing V/V_1 by W_1/U and V_1/U by V/W_1 we see that this composition series is equivalent to $V > W_1 > \ldots > W_n$. □

1.3.10 Remark For an arbitrary ring A it is not guaranteed that an A-module has a composition series. However, this is true if A is noetherian and artinian, i.e. if every ascending or descending chain of A-modules becomes constant. This is always true for finite-dimensional modules, but also for finitely generated modules over groups rings of finite groups.

1.3.11 Corollary As a consequence of the Jordan-Hölder theorem each representation Δ of a group G can be written as

$$\Delta(g) = \begin{pmatrix} \Delta_1(g) & 0 & \cdots & 0 \\ (*) & \Delta_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (*) & (*) & \cdots & \Delta_k(g) \end{pmatrix} \text{ for all } g \in G$$

where the Δ_i are the uniquely determined irreducible components of Δ.

1.3.12 Proposition *Let A be a ring. Then every simple A-module V is of the form $V \cong A/L$ for some maximal right-ideal L of A.*

Proof: The map $A \to V : a \mapsto v \cdot a$ is a homomorphism and its kernel is a right-ideal L of A. Since V is simple, the homomorphism is surjective and the kernel is a maximal ideal. □

1.3.13 Example Let G be a group and K a field. Then the kernel I of the K-algebra homomorphism $\varphi : KG \to K : \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g$ is a maximal ideal in KG and is called the *augmentation ideal* of KG. The corresponding simple factor module KG/I is the KG-module of the trivial representation.

1.3.14 Corollary *A finite group G has up to equivalence at most $|G|$ irreducible representations over a field K.*

Proof: The group ring KG is a K-vector space of dimension $|G|$, hence every composition series of KG has at most $|G|$ factors. On the other hand, an irreducible KG-module V is of the form $V \cong_{KG} KG/L$, hence it must be isomorphic to one of the factors in the composition series of KG. □

1.3.15 Proposition *Let K be a field with $\mathrm{char}(K) = p > 0$ and let G be a p-group. Then the trivial representation is the only irreducible representation of G over K.*

Proof: Let V be a simple KG-module and let M be the orbit of some $0 \neq v \in V$. Then G acts on $\mathbb{F}_p M := \{\sum_{g \in G} a_g v g \mid a_g \in \mathbb{F}_p\}$ and the orbits of G on $\mathbb{F}_p M$ have length a power of p. Since $\{0\}$ is an orbit, there has to be another orbit of length 1, $\{w\}$ say. Then $\langle w \rangle$ is a 1-dimensional KG-submodule of V on which G acts trivially and since V is simple the claim follows. □

1.4 Homomorphisms

1.4.1 Definition Let V, W be A-modules for a ring A.

(i) A group homomorphism $\varphi : V \to W$ is called an *A-module homomorphism* if $(v\varphi)a = (va)\varphi$ for all $v \in V, a \in A$.
Two A-modules which are isomorphic by an A-module isomorphism are denoted by $V \cong_A W$.

(ii) $Hom_A(V, W) := \{\varphi : V \to W \mid \varphi \text{ is } A\text{-module homomorphism } \}$ is an abelian group. In case that K is a field and A is a K-algebra, $Hom_A(V, W)$ is a K-vector space.

(iii) $End_A(V) := Hom_A(V, V)$ is called the *endomorphism ring* of V (as A-module) is a ring. In case that K is a field and A is a K-algebra, $End_A(V)$ is a K-algebra.

1.4.2 Remarks Let G be a group and K a field.

(1) If Δ is a representation of G with associated KG-module V, the ring $End_{KG}(V)$ consists of those linear mappings $\varphi : V \to V$ which commute with the action of $\Delta(G)$, i.e.

$$End_{KG}(V) = \{\varphi \in End(V) \mid \Delta(g)\varphi = \varphi\Delta(g) \text{ for all } g \in G\}.$$

(2) G acts on $End(V)$ via $\varphi \mapsto \varphi^g$ where the endomorphism φ^g is defined by $v\varphi^g := ((vg^{-1})\varphi)g$. The ring $End_{KG}(V)$ is the set of fixed points under this action of G.

1.4.3 Theorem (Schur's lemma)
Let K be a field and A a K-algebra.

(i) If V is a simple A-module then $End_A(V)$ is a skew field.

(ii) If K is algebraically closed and V is simple then $End_A(V) \cong K$.

(iii) If V, W are simple A-modules, then $Hom_A(V, W) = \{0\}$ if $V \not\cong_A W$.

Proof: (i)+(iii): Let V, W be simple A-modules and assume that $\varphi \in \mathrm{Hom}_A(V, W)$. Then $\ker(\varphi) = \{0\}$ or $\ker(\varphi) = V$ and $\mathrm{im}(\varphi) = \{0\}$ or $\mathrm{im}(\varphi) = W$. Thus, φ is either 0 or bijective. This shows that φ has to be 0 if $V \not\cong_A W$ and that all elements of $End_A(V) \setminus \{0\}$ are invertible.
(ii): The map $\iota : K \to End_A(V), a \mapsto a \cdot id_V$ is injective, since K is a field. Assume that $\varphi \in End_A(V)$ then the characteristic polynomial of φ has a zero a in K, hence $\varphi - a \cdot id_V \in End_A(V)$ is not invertible and is thus 0 by (i). Therefore $\varphi = a \cdot id_V$ which shows that ι is surjective. □

1.4.4 Remark The fact that $End_A(V) = K \cdot id_V$ does in general not imply that V is irreducible. Let for example $K = \mathbb{F}_p$ and let G be a Sylow p-subgroup of $SL_n(p)$, e.g. the group

$$G = \left\{ \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ a_{ij} & & 1 \end{pmatrix} \mid a_{ij} \in \mathbb{F}_p, i > j \right\}$$

of lower triangular matrices, then the only matrices which commute with all elements of G are the scalar matrices but the first basis vector clearly generates an $\mathbb{F}_pG$-invariant submodule.

1.4.5 Definition Let K be a field, A a K-algebra and V an A-module.

(i) The dual space $Hom(V, K)$ is made a left A-module by defining

$$v(a\lambda) := (va)\lambda \text{ for all } v \in V, a \in A, \lambda \in Hom(V, K).$$

(ii) If A is a group ring KG, then $V^* = Hom(V, K)$ becomes a (right) KG-module by

$$v(\lambda g) := (vg^{-1})\lambda \text{ for all } v \in V, g \in G, \lambda \in V^*.$$

The module V^* is called the *contragredient* or *dual module* of V. The representation Δ^* of G on V^* obtained from this action is called the *contragredient representation* of Δ. If we choose as basis for V^* the dual basis of the basis underlying Δ, then $\Delta^*(g) = (\Delta(g)^{-1})^{tr}$.

1.4.6 Proposition *Let G be a group, K a field and V a KG-module. If $W \leq_{KG} V$ then $W^\perp := \{\lambda \in V^* \mid w\lambda = 0 \text{ for all } w \in W\} \leq_{KG} V^*$. One has $V^*/W^\perp \cong_{KG} W^*$ and $W^\perp \cong_{KG} (V/W)^*$.*

PROOF: We first have to show that $W^\perp$ is G-invariant. Let $\lambda \in W^\perp$, $g \in G$ and $w \in W$, then $w(\lambda g) = (wg^{-1})\lambda = 0$, since $wg^{-1} \in W$, hence $\lambda g \in W^\perp$.
The first isomorphism follows, since the restriction homomorphism $V^* \to W^* : \lambda \mapsto \lambda_{|W}$ is G-invariant and clearly has $W^\perp$ as its kernel. The second isomorphism is due to the bijection between the homomorphisms having W in their kernel and the homomorphisms of V/W. □

1.4.7 Example For a field K of characteristic 3, the mapping $\Delta : S_3 \to GL_2(K) : (1,2) \mapsto \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$, $(2,3) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ defines a reducible but indecomposable representation of S_3. The KS_3-module $V = K^2$ has a unique composition series with 1-dimensional submodule $V_1 = \langle v_1 \rangle$. The module V_1 belongs to the 1-dimensional representation $g \mapsto sign(g)$, the quotient module is the trivial module.
The action on the dual module V^* is given by $(1,2) \mapsto \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ and $(2,3) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, hence the dual module has a unique composition series with the trivial module as a submodule and the module of the signum representation as quotient module.

1.4.8 Theorem (Norton's irreducibility test)
Let $\Delta : G \to GL_n(K)$ be a representation of G on the KG-module V and denote the extension of Δ to KG again by Δ. Let $a \in KG$ such that $0 < \mathrm{rank}(\Delta(a)) < n$. Then Δ is irreducible if and only if

(i) $vKG = V$ for all $0 \neq v \in \ker(\Delta(a))$

(ii) $\lambda KG = V^*$ for one $0 \neq \lambda \in \ker(\Delta(a)^{tr})$

PROOF: Assume that Δ is reducible and let $W \leq_{KG} V$ be a proper submodule. Since our criteria are basis independent we can assume that the basis underlying Δ is chosen such that it extends a basis of W. We therefore have

$$\Delta(a) = \begin{pmatrix} \Delta_1(a) & 0 \\ (*) & \Delta_2(a) \end{pmatrix}.$$

Since $\Delta(a)$ is singular, either $\Delta_1(a)$ or $\Delta_2(a)$ is singular. If $\Delta_1(a)$ is singular, we have $\ker(\Delta(a)) \cap W \neq \{0\}$ and hence there is a vector $0 \neq v \in \ker(\Delta(a))$ such that $vKG \subseteq W$ is a proper submodule of V. Now assume that $\Delta_1(a)$ is invertible. Then every $0 \neq \lambda \in \ker(\Delta(a)^{tr})$ has to lie in $U^\perp$ and since this is a KG-module, λ generates a proper submodule of V^*. □

1.4.9 Algorithm (Richard Parker's MEATAXE)
Let a representation Δ of degree n of a group $G = \langle g_1, \ldots, g_r \rangle$ be given by the images $\Delta(g_1), \ldots, \Delta(g_r)$. This algorithm either splits Δ into smaller representations or proves the irreducibility of Δ.

(1) Choose a number of random words $a \in KG$ and compute their nullity (corank) $nul(a) := n - \mathrm{rank}(\Delta(a))$.

(2) Select $a \in KG$ with $0 < \mathrm{rank}(\Delta(a)) < n$ such that $nul(a)$ is minimal.

(3) Compute $\ker(\Delta(a))$ and check for each 1-dimensional subspace $\langle v \rangle$ of $\ker(\Delta(a))$ whether vKG is a proper submodule. If this is the case, a proper KG-submodule W is found and the actions of G on W and on V/W are computed.

(4) If all 1-dimensional subspaces of $\ker(\Delta(a))$ generate V, compute $0 \neq \lambda \in \ker(\Delta(a)^{tr})$. Check whether $\lambda KG = V^*$ by applying the matrices $\Delta(g)^{tr}$ to λ. If this is the case, Δ is irreducible, otherwise a proper KG-submodule W^* of V^* is found, the actions of G on W^* and on V^*/W^* are computed and the transposed representations are returned as constituents of Δ.

The modules vKG (and λKG) are computed by a *spinning algorithm*: One starts with $W_0 := \langle v \rangle$ and computes $W_{i+1} := \langle u_k \Delta(g_j) \mid 1 \leq j \leq r, 1 \leq k \leq s \rangle$ for $W_i = \langle v_1, \ldots, v_s \rangle$ until $W_{i+1} = W_i$.
Note that this algorithm requires only finitely many steps if either K is a finite field or $nul(a) = 1$.

1.4.10 Example Let K be a field, $G = S_3$ then the 2-dimensional G-module V of S_3 obtained by splitting off the trivial module from the permutation module of degree 3 is gives the representation Δ with

$$\Delta((1,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Delta((2,3)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

For the element $a = (1,2)+(2,3)+(2,3)\cdot(1,2) \in KG$ we get $\Delta(a) = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$, so $\Delta(a)$ has rank 1.

We have $\ker(\Delta(a)) = \langle(1,1)\rangle$ and spinning the vector $(1,1)$ gives $(1,1)\Delta(KG) = \langle(1,1),(0,-1)\rangle = V$. So the first part of the Norton criterion gives no proper submodule of V.

In the second step we look at the dual module V^* and therefore at the transposed matrices. We have $\ker(\Delta(a)^{tr}) = \langle(2,-1)\rangle$ and applying the two generators to the vector $(2,-1)$ gives $(2,-1)\Delta(KG)^{tr} = \langle(2,-1),(-1,2)\rangle$. In case $\mathrm{char}(K) \neq 3$ this is a basis of the full dual module, hence the representation is irreducible. In case $\mathrm{char}(K) = 3$ we have found that $\langle(1,1)\rangle$ is a proper submodule of V^*, so V^* and therefore also V is reducible.

Some adjustments to the basic idea of the MeatAxe allow to improve its efficiency for large finite fields and to extend its application to fields of characteristic 0 in many cases. The ideas involved will be discussed later.

Exercises

1. Determine all irreducible $\mathbb{Q}$-representations of C_3 (up to equivalence).
2. Let K be a field and let G be a group acting on the set $\{1,\ldots,n\}$. Let V be the KG-module with basis $(v_1,\ldots,v_n)$ on which G acts by $v_i g := v_{i\cdot g}$.
   (i) Show that $V_0 := \langle\sum_{i=1}^n v_i\rangle$ and $V_1 := \langle v_1 - v_2, v_1 - v_3, \ldots, v_1 - v_n\rangle$ are KG-submodules of V.
   (ii) Under which condition is $V_0 \leq V_1$ or $V = V_0 \oplus V_1$.
   (iii) Let G be the alternating group A_5 and $K = \mathbb{F}_2$. Show that the action of A_5 on V_1 gives an irreducible representation of degree 4 of A_5 over $\mathbb{F}_2$. (Hint: A_5 is a simple group.)
3. Let V be a KG-module.
   (i) Show that $V_0 := \{v \in V \mid vg = v \text{ for all } g \in G\} \leq_{KG} V$.
   (ii) Show that the mapping $v \mapsto \sum_{g\in G} vg$ is a KG-homomorphism from V to V_0. Is it necessarily surjective?
4. Let $C_4 = \langle g\rangle$ be the cyclic group of order 4 and let
$$a := \frac{1+i}{2}g + \frac{1-i}{2}g^3, \qquad b := \frac{1-i}{2}g + \frac{1+i}{2}g^3 \in \mathbb{C}C_4.$$
   Show that $\{1, g^2, a, b\}$ is a subgroup of the unit group $\mathbb{C}C_4^*$ of $\mathbb{C}C_4$ which is isomorphic with the Klein group V_4.

5. Show that the group rings $\mathbb{C}V_4$ and $\mathbb{C}C_4$ are isomorphic (as $\mathbb{C}$-algebras).
6. The dihedral group D_8 of order 8 is the symmetry group of a square. The action of D_8 on the corners of the square gives a permutation representation of D_8 which turns $\mathbb{R}^4$ into an $\mathbb{R}D_8$-module V. Determine the $\mathbb{R}D_8$-submodules of V.
7. Let Δ and Ψ be equivalent representations of a group G. Show that Δ is irreducible/reducible/indecomposable/decomposable if and only if Ψ is.
8. Let G be a group and K a field. Show that the 1-dimensional representations of G are in bijection with the homomorphisms of G/G' to K^*. (Note: G' is the derived subgroup of G, i.e. generated by the elements $[g,h] := g^{-1}h^{-1}gh$.)
9. Let G be a non-abelian simple group of even order. Show that every non-trivial irreducible representation of G over $\mathbb{C}$ has at least degree 3.
(Hint: G contains an element of order 2. Consider 2-dimensional representations of such an element.)
10. Let G be a finite group and let $a := \sum_{g\in G} g \in KG$. Show that the 1-dimensional module $\langle a\rangle$ spanned by a is the unique submodule of KG that is isomorphic with the trivial G-module.
11. Let G be a finite group, K a field and Δ a 1-dimensional representation of G over K.
   (i) Show that $I := \{\sum_{g\in G} a_g g \mid \sum_{g\in G} a_g \Delta(g) = 0\}$ is a two-sided ideal of KG.
   (ii) Show that Δ is the representation of KG on the module KG/I.
12. Let A be a ring, $V = \oplus_{i=1}^n V_i$ an A-module where V_i pairwise non-equivalent simple A-modules. Show that every $W \leq_A V$ is of the form $W = \oplus_{j=1}^r V_{i_j}$ with $1 \leq i_1 < \ldots < i_r \leq n$. Determine the number of A-submodules of V.
13. Let K be a field, G a finite group and $V = KG$ the regular G-module. Show that $V^* \cong_{KG} V$.
14. Let S_n be the symmetric group on n points and let $V := \langle v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n\rangle$ be a submodule of the natural permuation module.
   (i) Use the Norton criterion to show that V is an irreducible $\mathbb{Q}S_n$-module.
   (Hint: S_n is generated by $g = (1, 2, \ldots, n)$ and $h = (1, 2)$. An obvious linear combination of the matrices of the action of g and h on V has a 1-dimensional kernel.)
   (ii) Find out under which condition on the field K the module V is an irreducible KS_n-module.
   (Hint: Look at small examples, e.g. $n = 3, 4, 5$ and generalize.)

Chapter 2

Semisimple rings

2.1 Maschke's theorem

2.1.1 Definition Let A be a ring.

(i) An A-module V is called *semisimple* if V is a direct sum of simple A-modules.

(ii) A representation corresponding to a semisimple module is called *completely reducible*.

2.1.2 Theorem *Let A be a ring and V an A-module. Then the following conditions are equivalent:*

(i) V is semisimple.

(ii) V is a sum of simple A-modules.

(iii) Every submodule $W \leq_A V$ has a complement U, i.e. $V = W \oplus U$ with $U \leq_A V$.

Proof: (i) $\Rightarrow$ (ii): This is clear.
(ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (i): By assumption we have $V = \sum_{i \in I} V_i$, where V_i are simple A-modules. Let $W \leq_A V$. We will use Zorn's lemma to construct a complement of W in V. Define

$$\mathcal{M} := \left\{ J \subseteq I \mid \sum_{j \in J} V_j = \bigoplus_{j \in J} V_j \text{ and } (\sum_{j \in J} V_j) \cap W = \{0\} \right\}.$$

Then $\mathcal{M} \neq \emptyset$, since $\emptyset \in \mathcal{M}$ and $\mathcal{M}$ is partially ordered with respect to set inclusion. To apply Zorn's lemma we require that every totally ordered subset (chain) $\mathcal{K} \subset \mathcal{M}$ is bounded in $\mathcal{M}$. But for a chain $\mathcal{K}$ we have $K := \cup_{J \in \mathcal{K}} J \in \mathcal{M}$, since every $j \in K$ lies in some $J \in \mathcal{K}$ and thus $V_j \cap W = \{0\}$. Also, $\sum_{j \in K} V_j = \bigoplus_{j \in K} V_j$ because otherwise there would be $J_1 \subseteq J_2 \in \mathcal{K}$ and $j_1 \in J_1, j_2 \in J_2$ with $V_{j_1} \cap V_{j_2} \neq \{0\}$ which contradicts $J_2 \in \mathcal{M}$. By Zorn's lemma we can now conclude that there exists a maximal element $J_0 \in \mathcal{M}$. We claim that

$U := \oplus_{j\in J_0} V_j$ is a complement of W in V.
Assume that $V' := W \oplus U \lneqq V$, then there exists a simple A-module V_i such that $V_i \not\subseteq V'$. Since V_i is simple, this shows that $V' \cap V_i = \{0\}$. In particular we have $V_i \cap W = \{0\}$ and $\sum_{j\in J_0} V_j + V_i = \oplus_{j\in J_0} V_j \oplus V_i$. Therefore, $J_0 \cup \{i\} \in \mathcal{M}$ which contradicts the maximality of J_0 and thus $V' = V$.
Applying this argument to $W = \{0\}$ implies (i), since the complement V of W is written as a direct sum of simple A-modules.
(iii) $\Rightarrow$ (ii): Define $V' := \sum_{W \leq_A V} W$, where the sum is taken over the simple A-submodules of V. By assumption, V' has a complement U in V. If $U \neq \{0\}$, then U contains a cyclic A-module T, i.e. a submodule of the form $T = \{va | a \in A\}$ for some $v \in U$. We can now apply Zorn's lemma to $\mathcal{M} := \{S \leq_A T \mid S \neq T\}$, since $v \notin S$ for all $S \in \mathcal{M}$ implies that every chain is bounded in $\mathcal{M}$. Thus, there exists a maximal submodule T' of T.
By assumption, T' has a complement V'' in V and we have $T = T' \oplus (V'' \cap T)$. But due to the maximality of T', the factor module T/T' is simple and hence $V'' \cap T \cong T/T'$ is a simple A-module. This contradicts the fact that T lies in the complement of V' and hence $U = \{0\}$ and $V' = V$. □

2.1.3 Proposition *Let A be a ring and V an A-module.*

(i) If $W, U \leq_A V$ are semisimple modules, then $\langle W, U\rangle$ is semisimple, i.e. sums of semisimple modules are semisimple.

(ii) If V is semisimple and $W \leq_A V$, then W is semisimple, i.e. submodules of semisimple modules are semisimple.

(iii) If V is semisimple and $W \leq_A V$, then V/W is semisimple, i.e. factor modules of semisimple modules are semisimple.

Proof: (i): Since W and U are sums of simple modules, the sum $\langle W, U\rangle$ is also a sum of simple modules.
(ii): If $W' \leq_A W$ is a submodule of W, then we require to find a complement W'' of W' in W. By assumption we know that W' has a complement $U \leq_A V$ in V. Define $W'' := W \cap U$, then every $w \in W$ is uniquely written as $w = w' + u$ with $w' \in W'$ and $u \in U$, and therefore $W = W' \oplus W''$.
(iii): Let π be the canonical projection of V onto V/W. Since V is semisimple, it is the direct sum of simple A-modules V_i and the image of V_i under π is either $\{0\}$ or a simple A-module (isomorphic with V_i). Thus, the image $\pi(V)$ is the sum of simple A-modules and is therefore semisimple. □

2.1.4 Corollary *A ring A is semisimple as an A-module if and only if every A-module V is semisimple.*

Proof: We only have to prove that semisimplicity of A as an A-module implies semisimplicity of an arbitrary A-module V. Let $(v_i \mid i \in I)$ be a basis of V. Then the mapping $\varphi : \oplus_{i\in I} A \to V$, $(a_i)_{i\in I} \mapsto \sum_{i\in I} v_i a_i$ is an A-module epimorphism. Thus, V is a factor module of the semisimple A-module $\oplus_{i\in I} A$ and is therefore semisimple itself by the previous proposition. □

2.1.5 Definition A ring A is called *semisimple* if A is semisimple as an A-module. By the above corollary this implies that all A-modules are semisimple.

2.1.6 Theorem (Maschke's theorem)
Let K be a field and G a group. The group ring KG is semisimple if and only if $\mathrm{char}(K) \nmid |G|$.

PROOF: $\Rightarrow$: Let I be the augmentation ideal in KG then by assumption $KG = I \oplus I'$ and $I' \cong_{KG} KG/I$ is the trivial KG-module. We have $I' = \langle v \rangle$ with $v = \sum_{g\in G} a_g g$ and $vg = v$ for all $g \in G$, since I' is the trivial module. This shows that $a_g = a \in K$ for all $g \in G$. But $v \notin I$, therefore $\sum_{g\in G} a = |G|a \neq 0$ and thus $\mathrm{char}(K) \nmid |G|$.
$\Leftarrow$: Let $W \leq_{KG} KG$. As a K-module, W has a complement and we denote the projection of KG onto W by π. By averaging over the group elements we turn π into a KG-module homomorphism $\tilde{\pi} : KG \to KG, a \mapsto |G|^{-1} \sum_{g\in G}(ag^{-1})\pi g$. Then $\tilde{\pi}$ is well defined, since we assume that $\mathrm{char}(K) \nmid |G|$ and one checks that $\mathrm{im}(\tilde{\pi}) = W$ and $\tilde{\pi}^2 = \tilde{\pi}$. Thus $KG = \mathrm{im}(\tilde{\pi}) \oplus \ker(\tilde{\pi})$, in other words, $\ker(\tilde{\pi})$ is a complement of W. $\square$

Maschke's theorem is a branching point for the representation theory of finite groups. The first branch is the *ordinary* representation theory, which is concerned with the semisimple case (i.e. $\mathrm{char}(K) \nmid |G|$) where every representation is completely reducible. The other branch is the *modular* representation theory which uses different methods to analyze the situation where the group ring KG is not semisimple. The extreme case of p-groups in characteristic p shows that the irreducible modules are no longer helpful. Instead the *projective indecomposable modules* are studied in the modular representation theory.

In this course we will restrict ourselves almost exclusively to the semisimple case.

2.2 Wedderburn decomposition

2.2.1 Lemma *Let A be a ring and let $A = \oplus_{i\in I} V_i$ be a decomposition of V into a direct sum of right ideals. Then the sum is finite, i.e. I is a finite set.*

PROOF: The element $1 \in A$ can be written as a finite sum $1 = \sum_{i\in I_0} e_i$ with $e_i \in V_i$. Then $A = 1 \cdot A = \sum_{i\in I_0} e_i A \subseteq \sum_{i\in I_0} V_i$, hence $I_0 = I$ and $e_i A = V_i$ for $i \in I$. $\square$

2.2.2 Proposition/Definition *Let $A = \bigoplus_{i=1}^n V_i$ be a decomposition of A into right ideals. Then $1 \in A$ can be written as $1 = e_1 + \ldots + e_n$ with $e_i \in V_i$.*

(i) The e_i are called (orthogonal) idempotents *and fulfill $e_i^2 = e_i$, $e_i e_j = 0$ for $i \neq j$ and $V_i = e_i A$.*

(ii) The V_i are two-sided ideals $V_i \trianglelefteq A$ if and only if the e_i lie in the centre of A. In this case the e_i are called central idempotents.

(iii) *An idempotent e_i is called* a primitive idempotent *if $e_i = e_i' + e_i''$ with e_i', e_i'' orthogonal idempotents implies $e_i' = 0$ or $e_i'' = 0$. The idempotent e_i is primitive if and only if e_iA is an indecomposable right ideal (projective indecomposable module).*

(iv) *A central idempotent e_i is called* a central primitive idempotent *if $e_i = e_i' + e_i''$ with e_i', e_i'' central orthogonal idempotents implies $e_i' = 0$ or $e_i'' = 0$. The idempotent e_i is central primitive if and only if e_iA is a two-sided ideal that can not be decomposed into a direct sum of non-trivial two-sided ideals (block ideal).*

(v) *If $1 = e_1 + \ldots + e_r$ with e_i central primitive idempotents, then the e_i are called* block idempotents. *The block idempotents are unique (up to permutation).*

Proof: (i): The fact that $V_i = e_iA$ follows as in the previous lemma. Furthermore, we have $e_j = 1 \cdot e_j = e_1e_j + \ldots + e_ne_j$ and since $e_ie_j \in V_i$ it follows that $e_ie_j = 0$ for $i \neq j$ and $e_j^2 = e_j$ for all j.
(ii): If $V_i \trianglelefteq A$ for $i \neq j$ we have $V_iV_j \subseteq V_i \cap V_j = \{0\}$, hence $e_ia = e_i(ae_1) + \ldots + e_i(ae_n) = e_iae_i$, since $e_i(ae_j) \in V_iV_j$. On the other hand $ae_i = (e_1a)e_i + \ldots + (e_na)e_i = e_iae_i$, since $(e_ja)e_j \in V_jV_i$. Thus the e_i lie in the centre $Z(A)$ of A.
The other direction is clear, since $e_i \in Z(A)$ implies $V_i = e_iA = Ae_i$ and this is clearly a two-sided ideal in A.
(iii): Assume that e_iA is decomposable into proper right ideals, i.e. $e_iA = U \oplus V$, then $1 = u + v$ with $u \in U, v \in V$. Now, $u = e_iu = (u+v)u = u^2 + vu$, hence $vu = 0$ and $u^2 = u$, thus u and v are orthogonal idempotents and e_i is not primitive.
Conversely, if e_i is not primitive and $e_i = e_i' + e_i''$ is a proper decomposition, then $e_iA = e_i'A \oplus e_i''A$ and thus e_iA is decomposable.
(iv): This is the combination of the conditions in (ii) and (iii).
(v): Assume that $1 = \sum_{i=1}^n e_i = \sum_{j=1}^m f_j$ are two decompositions into block idempotents. Now, $e_i = e_i \cdot 1 = \sum_{j=1}^m e_if_j$ is a decomposition into orthogonal idempotents, since $(e_if_j)(e_if_k) = e_i\delta_{jk}f_j$. But the e_i are primitive, hence there is precisely one $j = j(i)$ with $e_if_{j(i)} = e_i$ and the other e_if_j are 0. The same argument applied to $1 \cdot f_{j(i)}$ shows that $f_{j(i)} = e_if_{j(i)} = e_i$. Hence we have $\{e_1, \ldots, e_n\} \subseteq \{f_1, \ldots, f_m\}$ and by interchanging the roles of the e_i and f_j we get equality of the sets. □

2.2.3 Example Let $A = K^{2\times 2}$, $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then e_1, e_2 are primitive orthogonal idempotents, $e_1A = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in K\}$, $e_2A = \{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \mid c, d \in K\}$ and $e_iA \cong K^2$ are irreducible A-modules. But $f_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $f_2 = I_2 - f_1 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ are also primitive orthogonal idempotents with $A = f_1A \oplus f_2A$. The only central primitive idempotent is $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2.2.4 Lemma *Semisimple rings contain central primitive idempotents.*

PROOF: This is seen by decomposing 1 iteratively into central idempotents. If a central idempotent is not primitive it can be properly split. This process stops after finitely many steps, since by Lemma 2.2.1 a semisimple ring is a finite direct sum of minimal left ideals. □

2.2.5 Proposition *Let A be a semisimple ring and let $1 = e_1 + \ldots + e_r$ be the decomposition into central primitive idempotents.*

(i) Every simple $V \leq_A A$ (minimal right ideal) is contained in precisely one e_iA.

(ii) If $V_i \leq_A e_iA$ and $V_j \leq_A e_jA$ with $i \neq j$, then $V_i \ncong_A V_j$.

(iii) Every e_iA is the (direct) sum of A-isomorphic minimal right ideals and is therefore called a homogeneous *component.*

(iv) Up to isomorphism there exist precisely r simple A-modules.

PROOF: (i): We have $1 \cdot V = e_1V \oplus \ldots \oplus e_rV$ and since V is simple there is precisely one i with $e_iV \neq \{0\}$. But then $V = e_iV \subseteq e_iA$.
(ii): The idempotent e_i acts as identity on V_i and as 0 on V_j, therefore V_i and V_j are not A-isomorphic.
(iii): Let V be a minimal right ideal in e_iA and let $H_V(A) := \sum_{W \leq_A A} W$, where the sum runs over all $W \leq_A A$ with $W \cong_A V$ ($H_V(A)$ is called the V-homogeneous component of A). By (i) and (ii) it follows that $H_V(A) \subseteq e_iA$. We now observe that $H_V(A)$ is a two-sided ideal of A, since multiplication of V from the left by $a \in A$ maps V to the isomorphic A-module aV which is again contained in $H_V(A)$. But e_iA is indecomposable as a two-sided ideal, therefore $H_V(A) = e_iA$. We have seen that e_iA is a sum of simple A-modules, therefore by Theorem 2.1.2 it is semisimple and thus a direct sum of simple A-modules.
(iv): This follows immediately from (i)-(iii), since every simple module is contained in precisely one homogeneous component. □

2.2.6 Definition For a ring $R = (R, +, \cdot)$ the anti-isomorphic ring $R^{op} = (R, +, *)$ is obtained from R by reversing the arguments in the multiplication, i.e. $a * b = b \cdot a$.

2.2.7 Proposition *Let A be a ring and $e \in A$ an orthogonal idempotent. Then $End_A(eA)^{op} \cong eAe$ as rings.*

PROOF: Note that $\varphi \in End_A(eA)$ is determined by its image on e, since $(ea)\varphi = (e\varphi)a$. Moreover, $e\varphi \in eAe$, since $e\varphi = e^2\varphi = (e\varphi)e$. Therefore, the map

$$\Phi : End_A(eA) \to eAe, \quad \varphi \mapsto e\varphi$$

is a homomorphism of the additive groups. It is injective, since $e\varphi = 0$ implies $\varphi = 0$ and it is surjective, since defining φ_a by $(eb)\varphi_a := eab$ gives $\varphi_a \in End_A(eA)$ with $\Phi(\varphi_a) = eae$.
In $End_A(eA)^{op}$, the multiplication $*$ is defined by reversing the arguments of the multiplication in $End_A(eA)$, therefore $\Phi(\varphi*\psi) = e(\psi\varphi) = (e\psi)\varphi = (e\Phi(\psi))\varphi = (e\varphi)\Phi(\psi) = \Phi(\varphi)\Phi(\psi)$. This shows that Φ respects the (suitably chosen) multiplication. □

2.2.8 Theorem (Wedderburn)
Let A be a semisimple ring. Then A is a direct sum of full matrix rings over skew fields. More precisely: Let $1 = e_1 + \ldots + e_r$ be the decomposition into central primitive idempotents. Then $A = e_1A \oplus \ldots \oplus e_rA$ is a decomposition into two-sided ideals. Each e_iA is the sum of A-isomorphic minimal right ideals, i.e. $e_iA \cong \underbrace{V_i \oplus \ldots \oplus V_i}_{n_i}$ and $e_iA \cong D_i^{n_i \times n_i}$ with $D_i = End_A(V_i)^{op}$.

Proof: First note that for a simple A-module V Schur's lemma implies that $End_A(V) \cong D$ for a skew field D. Next one sees that $End_A(V \oplus \ldots \oplus V) \cong D^{n\times n}$ where n is the number of terms in the direct sum, since the mapping $End_A(V \oplus \ldots \oplus V) \to D^{n\times n}, \varphi \mapsto (A_{ij})$ where the element A_{ij} describes the restriction of φ from the i-th to the j-th component is an A-invariant isomorphism. For a skew field D the transposition map $A \mapsto A^{tr}$ gives an isomorphism from $(D^{n\times n})^{op}$ to $(D^{op})^{n\times n}$ therefore we have $End_A(V \oplus \ldots \oplus V) \cong ((D^{op})^{n\times n})^{op}$. The claim now follows from Proposition 2.2.5, since $e_iA = e_iAe_i$ is a direct sum of isomorphic simple A-modules. □

2.2.9 Definition Let K be a field and A a K-algebra. K is called *splitting field* of A if $End_A(V) \cong K$ for all simple A-modules V. It follows from Schur's lemma that algebraically closed fields are always splitting fields.

2.2.10 Theorem *Let A be a semisimple ring and let $V_1, \ldots, V_r$ be representatives of the isomorphism classes of simple A-modules. Then the following are equivalent:*

(i) *K is a splitting field of A.*

(ii) $A \cong \bigoplus_{i=1}^r K^{n_i \times n_i}$.

(iii) $\dim_K A = \sum_{i=1}^r (\dim_K V_i)^2$.

Proof: (i) ⇒ (ii): This follows from Wedderburn's theorem, since $D_i = K$.
(ii) ⇒ (iii): The simple $K^{n\times n}$-modules are isomorphic to K^n, hence $\dim_K V_i = n_i$.
(iii) ⇒ (i): Note that the simple modules in $e_iA = D_i^{n_i\times n_i}$ have dimension n_i over D_i. Applying Wedderburn's theorem we therefore have

$$\sum_{i=1}^r (\dim_K V_i)^2 = \dim_K A = \sum_{i=1}^r \dim_K D_i^{n_i\times n_i} = \sum_{i=1}^r n_i^2 \dim_K D_i$$

$$= \sum_{i=1}^{r} (\dim_{D_i} V_i)^2 \dim_K D_i = \sum_{i=1}^{r} (\dim_K V_i)^2 (\dim_K D_i)^{-1}.$$

This implies that $\dim_K D_i = 1$ for all i, hence K is a splitting field of A. □

2.2.11 Corollary *Let A be a semisimple ring and let $Z(A) := \{b \in A \mid ab = ba \text{ for all } a \in A\}$ be the centre of A. Then the number of isomorphism classes of simple A-modules is $\leq \dim_K Z(A)$. The number is $= \dim_K Z(A)$ if K is a splitting field of A.*

PROOF: This follows since for a homogeneous component $D^{n\times n}$ we have $Z(D^{n\times n}) = Z(D)$ and $\dim_K Z(D) \geq 1$ for a skew field D over K. □

2.2.12 Example Let Q_8 be the quaternion group of order 8, i.e.

$$G = \{\pm 1, \pm i, \pm j \pm k\} = \langle i, j\rangle$$

with $ij = k$, $jk = i$, $ki = j$ and $i^2 = j^2 = k^2 = -1$. Let K be a field with $\mathrm{char}(K) \neq 2$. Q_8 has four 1-dimensional representations over K, namely $\Delta_1 : i \mapsto 1, j \mapsto 1$, $\Delta_2 : i \mapsto -1, j \mapsto 1$, $\Delta_3 : i \mapsto 1, j \mapsto -1$ and $\Delta_4 : i \mapsto -1, j \mapsto -1$. Since $|Q_8| = 8$, this implies that $KQ_8 \cong K \oplus K \oplus K \oplus K \oplus K^{2\times 2}$ or $KQ_8 \cong K \oplus K \oplus K \oplus K \oplus D$ where D is a skew field with $\dim_K(D) = 4$. One sees that the last irreducible representation Δ_5 has to be faithful, since otherwise i^2 would be in the kernel of all irreducible representations and thus in the kernel of the regular representation.

If K contains a primitive 4-th root of unity ζ, then $\Delta_5 : i \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $j \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix}$ is an irreducible 2-dimensional representation. Hence, in that case K is a splitting field of KQ_8.

On the other hand, if $K \subseteq \mathbb{R}$, we can conclude that Δ_5 is not 2-dimensional, since we can assume that i is mapped to the companion matrix of the 4-th cyclotomic polynomial, i.e. to $\Delta_5(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But there exists no matrix $\Delta_5(j)$ of order 4 which conjugates $\Delta_5(i)$ to $-\Delta_5(i)$. (To draw this conclusion it is in fact enough to assume that $x^2 + y^2 = -1$ has no solution in K.) Hence $End_{KQ_8}(V_5) = D$ in this case, and since Q_8 is not abelian, D is a non-abelian skew field with $\dim_K(D) = 4$, called the skew field of *Hamilton quaternions*.

2.2.13 Theorem *Let K be a field and G a group with $\mathrm{char}(K) \nmid |G|$.*

(i) Let $C_1, \ldots, C_r$ be the conjugacy classes of G and $C_i^+ := \sum_{g\in C_i} g$, then $C_1^+, \ldots, C_r^+$ is a basis of $Z(KG)$.

(ii) The number of irreducible representations of G over K (up to equivalence) is $\leq$ the number of conjugacy classes of G.

(iii) If K is a splitting field of KG, then the number of irreducible representations of G over K equals the number of conjugacy classes and $\sum_{i=1}^{r} n_i^2 = |G|$, where n_i is the degree of the i-th irreducible representation.

(iv) If K is a splitting field of KG, then the regular representation of KG contains an irreducible representation of degree n_i with multiplicity n_i.

Proof: (i): Let $a = \sum_{g\in G} a_g g \in Z(KG)$, then $hah^{-1} = \sum_{g\in G} a_g(hgh^{-1}) = \sum_{g\in G} a_{h^{-1}gh} g = a$ for all $h \in G$. Hence the coefficients a_g have to be constant on the conjugacy classes. It is clear that the C_i^+ are linearly independent.
(ii)-(iv): This now follows immediately from Wedderburn's theorem and Corollary 2.2.11. □

2.2.14 Corollary *Let G be a finite group, K a splitting field with $\mathrm{char}(K) \nmid G$. If Δ is an irreducible representation of G with degree n over K, then $\Delta(G)$ contains n^2 linearly independent elements, i.e. a basis of the full matrix ring $K^{n\times n}$.*

2.2.15 Example Let $\mathrm{char}(K) \neq 2, 3$ and $G = S_3$, then KG is semisimple. Clearly, the trivial representation Δ_1 and the 1-dimensional representation $\Delta_2 : g \mapsto sign(g)$ are irreducible. Let Δ_3 be the 2-dimensional representation with

$$\Delta_3((1,2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Delta_3((2,3)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

We compute the endomorphism ring of the module V_3 corresponding to Δ_3 and obtain $C_{K^{2\times 2}}(\Delta_3(S_3)) = \{A \in K^{2\times 2} \mid A\Delta_3(g) = \Delta_3(g)A \text{ for all } g \in S_3\} = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in K\right\}$. Thus $End_K(V_3) \cong K$, hence V_3 is a simple module and K is a splitting field of KS_3.
We have $KS_3 \cong K \oplus K \oplus K^{2\times 2}$ and the central primitive idempotents are given by

$$e_1 = \frac{1}{|G|}\sum_{g\in G} g, \quad e_2 = \frac{1}{|G|}\sum_{g\in G} sign(g)g,$$

$$e_3 = 1 - e_1 - e_2 = \frac{2}{3}1 - \frac{1}{3}(1,2,3) - \frac{1}{3}(1,3,2).$$

If we denote the conjugacy class of 1 by C_1, the class of $(1,2)$ by C_2 and the class of $(1,2,3)$ by C_3, then

$$e_1 = \frac{1}{6}(C_1^+ + C_2^+ + C_3^+), \quad e_2 = \frac{1}{6}(C_1^+ - C_2^+ + C_3^+), \quad e_3 = \frac{1}{3}(2C_1^+ - C_3^+).$$

Exercises

15. Let K be a field and let $R = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in K\}$ be the ring of lower triangular matrices. How many simple modules does R have? Is R a semisimple ring?

16. Let G be a (not necessarily finite) group, $H \leq G$ a subgroup of finite index $[G : H]$. Let K be a field of characteristic $\mathrm{char}(K)$ with $\mathrm{char}(K) \nmid [G : H]$. Let V be a KG-module with submodule $W \leq_{KG} V$ and let $U_0 \leq_{KH} V$ be a KH-module such that $V = W \oplus U_0$. Show that there exists a KG-submodule $U \leq_{KG} V$ with $V = W \oplus U$. (Hint: Mimic the proof of Maschke's theorem.)

17. Let $G = V_4 \cong C_2 \times C_2$ be the Klein group. Write the group ring $\mathbb{Q}G$ as a direct sum of irreducible $\mathbb{Q}G$-modules.

18. Let G be a finite group and let $\Delta : G \to GL_2(\mathbb{C})$ be a representation of G. Suppose that there are elements $g, h \in G$ such that $\Delta(g)$ and $\Delta(h)$ do not commute. Prove that Δ is irreducible.

19. Let $G = C_\infty = \langle g \rangle$ be the infinite cyclic group. The mapping

$$\Delta : G \to GL_2(\mathbb{C}) : g \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is a 2-dimensional representation of G turning $\mathbb{C}^2$ into a $\mathbb{C}G$-module V. Show that V is not semisimple.

20. Prove that every finite simple group G possesses a faithful simple $\mathbb{C}G$-module (i.e. a module such that only $1 \in G$ acts trivially).

21. A hermitian bilinear form on a complex vector space V is a map $\phi : V \times V \to \mathbb{C}$ with $\phi(v + \lambda w, u) = \phi(v, u) + \lambda\phi(w, u)$, $\phi(v, u + \lambda w) = \phi(v, u) + \overline{\lambda}\phi(v, w)$ and $\phi(v, w) = \overline{\phi(w, v)}$. The space of all hermitian bilinear forms on V is denoted by $Bil(V)$.
Let G be a finite group and V a $\mathbb{C}G$-module.
   (i) Show that G acts on $Bil(V)$ via $(\phi g)(v, w) := \phi(vg^{-1}, wg^{-1})$.
   (ii) There exists a G-invariant, positive definite form $\psi \in Bil(V)$, i.e. $\psi g = \psi$ for all $g \in G$ and $\psi(v, v) \geq 0$ for all $v \in V$ and $\psi(v, v) = 0 \Rightarrow v = 0$.
   (iii) If $W \leq_{\mathbb{C}G} V$, then $W^\perp := \{v \in V \mid \psi(v, w) = 0 \text{ for all } w \in W\}$ is a $\mathbb{C}G$-module and $V = W \oplus W^\perp$.

   Note that this is a constructive method to find complements in semisimple modules.

22. Let $A := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\} \subseteq \mathbb{C}^{2\times 2}$.
   (i) Show that $I := \left\{ \begin{pmatrix} z & iz \\ -iz & z \end{pmatrix} \mid z \in \mathbb{C} \right\} \subseteq A$ is a two-sided ideal in A.
   (ii) Show that A is a semisimple ring and find the primitive central idempotents of A.

23. Let $\Delta_1, \ldots, \Delta_r$ be the non-equivalent irreducible representations of a semisimple ring A and let $a_1, \ldots, a_r$ be arbitrary elements of A. Show that there exists an element $a \in A$ such that $\Delta_i(a) = \Delta_i(a_i)$ for all $1 \leq i \leq r$.

24. Show that A is a simple A-module if and only if A is a skew field.

25. Let $V_1, \ldots, V_r$ be non-equivalent simple A-modules and let $U_i := \bigoplus_{j=1}^{n_i} V_i$ be a direct sum of n_i copies of V_i.
   (i) Show that $End_A(U_i) \cong End_A(V_i)^{n_i \times n_i}$.
   (ii) Show that $End_A(\bigoplus_{i=1}^r V_i) \cong \bigoplus_{i=1}^r End_A(V_i)$.
   (iii) Show that $End_A(\bigoplus_{i=1}^r U_i) \cong \bigoplus_{i=1}^r End_A(U_i) \cong \bigoplus_{i=1}^r End_A(V_i)^{n_i \times n_i}$.

26. Let $G = S_3$ and $K = \mathbb{Q}$.
   (i) Give an explicit isomorphism $\mathbb{Q}S_3 \to \oplus_{i=1}^r D_i^{n_i \times n_i}$ according to Wedderburn's theorem. Give as well the inverse mapping for this isomorphism.
   (ii) Determine the centre $Z(\mathbb{Q}S_3)$ and the central primitive idempotents of $\mathbb{Q}S_3$.

27. Let G be a group of order 12. Use Wedderburn's theorem to deduce the possibilities for the degrees of the irreducible representations of G over $\mathbb{C}$.

    Recalling that the number of 1-dimensional representations equals the order of the commutator factor group G/G', determine the degrees of the irreducible representations of the dihedral group D_{12} and of the alternating group A_3 over $\mathbb{C}$.

Chapter 3

Characters

3.1 Class functions

3.1.1 Definition Let G be a group and $\Delta : G \to GL_n(K)$ a representation of G. Then $\chi = \chi(\Delta) : G \to K, g \mapsto \mathrm{tr}(\Delta(g))$ is called the *character* of G afforded by Δ.
A character is defined to be irreducible if the corresponding representation is irreducible.

3.1.2 Remarks

(1) Characters are constant on conjugacy classes, since
$\mathrm{tr}(\Delta(ghg^{-1})) = \mathrm{tr}(\Delta(g)\Delta(h)\Delta(g)^{-1}) = \mathrm{tr}(\Delta(h))$.

(2) Characters of equivalent representations are equal, since $\mathrm{tr}(T\Delta(g)T^{-1}) = \mathrm{tr}(\Delta(g))$.

(3) The character value $\chi(1)$ gives the degree of the corresponding representation.

(4) If Δ is a reducible representation with constituents Δ_1 and Δ_2, then the character of Δ is the sum of the characters of Δ_1 and Δ_2.

3.1.3 Definition Let K be a splitting field of KG and $\mathrm{char}(K) \nmid |G|$. Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G and $g_1, \ldots, g_r$ representatives of the conjugacy classes. Then the matrix $\mathcal{C} = (\chi_i(g_j))_{i,j=1}^r$ is called the *character table* of G.

3.1.4 Example In the character table of a group usually the matrix with the character values is augmented by some information on the conjugacy classes of elements, e.g. the element orders, the size of the class and the order of the centralizer. The character table of S_3 therefore looks like:

$\lvert C_G(g_i)\rvert$	6	3	2
$\lvert C_i\rvert$	1	2	3
$\lvert\langle g_i\rangle\rvert$	1	3	2
g_i	1	$(1,2,3)$	$(1,2)$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Note that the permutation representation of S_3 of degree 3 has the character $\pi = \chi_1 + \chi_3$.

3.1.5 Definition A map $\varphi : G \to K$ is called a *class function* of G if $\varphi(g) = \varphi(hgh^{-1})$ for all $g, h \in G$. The set of all class functions of G is a K-vector space denoted by $Cl_K(G)$.

3.1.6 Theorem *Let* $\mathrm{char}(K) = 0$ *and* G *be a group.*

(i) The irreducible characters of G are linearly independent in the K-vector space of class functions on G.

(ii) Let V, W be KG-modules. Then $V \cong_{KG} W \Leftrightarrow \chi_V = \chi_W$ where χ_V and χ_W are the characters of the representations of G on V and W, respectively.

PROOF: (i): Let χ_i be the irreducible character corresponding to the central primitive idempotent e_i. We have $\sum_{i=1}^r a_i\chi_i(e_j) = a_j\chi_j(1)$, hence the χ_i are linearly independent, since $\mathrm{char}(K) = 0$.
(ii): Let $V = \bigoplus_{i=1}^r m_iV_i$ and $W = \bigoplus_{i=1}^r m_i'V_i$, then $\chi_V = \sum_{i=1}^r m_i\chi_i$ and $\chi_W = \sum_{i=1}^r m_i'\chi_i$ and it follows from (i) that $\chi_V = \chi_W$ implies $m_i = m_i'$ for all i. □

3.1.7 Remark Note that the above statement is not true if $\mathrm{char}(K) \mid |G|$. Let $G = C_p = \langle g\rangle$ be the cyclic group of order p. Then $\Delta_1(g) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\Delta_2(g) = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ both define representation of G over $\mathbb{F}_p$ with character $\chi(g) = 2$ for all $g \in \mathbb{C}_p$. But Δ_1 and Δ_2 are not equivalent, since Δ_1 is decomposable and Δ_2 is indecomposable.

3.1.8 Theorem *Let* $\mathrm{char}(K) \nmid |G|$*, V_i the simple KG-modules with corresponding central primitive idempotents e_i, characters χ_i and skew fields $D_i := End_{KG}(V_i)^{op}$. The idempotent e_i can be written as*

$$e_i = \frac{n_i}{|G|}\sum_{g\in G}\chi_i(g^{-1})g = \frac{\chi_i(1)}{|G|[D_i : K]}\sum_{g\in G}\chi_i(g^{-1})g$$

where $n_i = \dim_{D_i}(V_i)$.

PROOF: Write e_i as $e_i = \sum_{g\in G} a_g g$. Let ρ be the character of the regular representation of KG, then $\rho(g) = |G|$ if $g = 1$ and 0 otherwise. We therefore have $\rho(g^{-1}e_i) = \sum_{h\in G} a_h \rho(g^{-1}h) = a_g\rho(1) = a_g|G|$. On the other hand we know that $\rho = \sum_{i=1}^r n_i\chi_i$. This yields $\rho(g^{-1}e_i) = \sum_{j=1}^r n_j\chi_j(g^{-1}e_i) = n_i\chi_i(g^{-1}e_i) = n_i\chi_i(g^{-1})$, since e_i acts as the identity on V_i and as 0 on V_j. Thus we have $a_g = n_i|G|^{-1}\chi_i(g^{-1})$ as required. □

3.1.9 Corollary (First orthogonality relation)
Let $\mathrm{char}(K) \nmid |G|$ *and let* χ_i, χ_j *be irreducible characters of* G*. Then*

$$\frac{1}{|G|}\sum_{g\in G}\chi_i(gh)\chi_j(g^{-1}) = \begin{cases} 0 & \text{if } \chi_i \neq \chi_j \\ \frac{\chi_i(h)}{n_i} & \text{if } \chi_i = \chi_j \end{cases}$$

In particular one has $\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\chi_j(g^{-1}) = [D_i : K]$ *if* $\chi_i = \chi_j$ *and 0 otherwise.*

PROOF: This is seen by comparing the coefficients of g in the equation $e_ie_j = \delta_{ij}e_i$ using the expression for e_i from theorem 3.1.8. □

3.1.10 Remarks

(1) We can define a symmetric bilinear form $(\cdot,\cdot)_G$ on the vector space of class functions on G over K by setting $(\varphi,\psi)_G := |G|^{-1}\sum_{g\in G}\varphi(g)\psi(g^{-1})$.

(2) For the irreducible characters χ_i of G we have $(\chi_i,\chi_j)_G = \delta_{ij}[D_i : K]$ where $D_i = End_{KG}(V_i)$. (Note that for a K-algebra A we have $[A : K] = [A^{op} : K]$.) Thus, the χ_i form an orthogonal basis which is even orthonormal in case that K is a splitting field for G.

(3) If K is a splitting field for G then a class function φ can be written as $\varphi = \sum_{i=1}^r(\varphi,\chi_i)_G\chi_i$. The norm of a class function $\varphi = \sum_{i=1}^r a_i\chi_i$ is $(\varphi,\varphi)_G = \sum_{i=1}^r a_i^2$ and a character is irreducible if and only if its norm equals 1.

3.1.11 Theorem (Second orthogonality relation)
Let K *be a splitting field for* G*, let* χ_k *be the irreducible characters of* G*,* g_i *representatives of the conjugacy classes of* G*. Then*

$$\sum_{k=1}^r \chi_k(g_i^{-1})\chi_k(g_j) = \begin{cases} 0 & \text{if } i \neq j \\ C_G(g_i) & \text{if } i = j \end{cases}$$

where $C_G(g_i)$ *denotes the centralizer of* g_i *in* G*.*

PROOF: Let φ_i be the i-th class indicator function, i.e. $\varphi_i(g) = 1$ if g is conjugate with g_i and 0 otherwise. Then φ_i can be written as $\varphi_i = \sum_{k=1}^r(\varphi_i,\chi_k)_G\chi_k$ and we have $\varphi_i(g_j) = \sum_{k=1}^r |G|^{-1}(\sum_{g\in G}\varphi_i(g)\chi_k(g^{-1}))\chi_k(g_j)$
$= \sum_{k=1}^r |C_G(g_i)|^{-1}\chi_k(g_i^{-1})\chi_k(g_j)$. □

3.1.12 Example We determine the character table of the symmetric group S_4. The trivial character, the signum character and the 2-dimensional character of the factor group $S_4/V_4 \cong S_3$ are easily found and seen to be irreducible. Since we know the conjugacy classes of S_4 we know that we have to find 5 irreducible characters. We conclude that the two missing characters both have degree 3 since the sum of the squares of the character degrees has to be 24. This gives the following partial character table, which we augment by the character π of the natural permutation representation of S_4.

$C_G(g_i)$	24	8	3	4	4
$\|C_i\|$	1	3	8	6	6
g_i	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,2)$	$(1,2,3,4)$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3				
χ_5	3				
π	4	0	1	2	0

We compute that $(\pi, \chi_1)_G = 1$ and $(\pi, \pi)_G = 2$, therefore $\pi - \chi_1$ is an irreducible character with values $(3, -1, 0, 1, -1)$. For the values of χ_5 we use the second orthogonality relation with the first column and obtain the values $(3, -1, 0, -1, 1)$.

3.2 Character values

3.2.1 Theorem *Let* $\mathrm{char}(K) \nmid |G|$, *let* χ *be a character of* G *over* K *and let* $g \in G$ *be an element of order* m. *Then the character value* $\chi(g)$ *is a sum of* m*-th roots of unity.*
If $\mathrm{char}(K) = 0$, *then all character values of* χ *lie in the cyclotomic field* $\mathbb{Q}(\zeta_e)$, *where* $e = \exp(G) := \mathrm{lcm}(|\langle g \rangle| \mid g \in G)$ *is the exponent of* G.

Proof: By restricting χ to the cyclic group generated by g, we are reduced to the case $G = \langle g \rangle$. Since we are in the semisimple case, we can assume that χ is afforded by a representation Δ for which $\Delta(g)$ is a diagonal matrix. But then each diagonal entry λ of $\Delta(g)$ satisfies $\lambda^m = 1$ and is thus an m-th root of unity. Therefore $\chi(g)$ is a sum of m-th roots of unity.
If $\mathrm{char}(K) = 0$, then $\mathbb{Q}$ is the prime field of K and all m-th roots of unity lie in $\mathbb{Q}(\zeta_m)$. Since $\mathbb{Q}(\zeta_m) \subseteq \mathbb{Q}(\zeta_l)$ for $m \mid l$ if follows that all character values of G lie in $\mathbb{Q}(\zeta_e)$ for $e = \exp(G)$. □

3.2.2 Definition An element $a \in R$ of a ring R is called an *algebraic integer* if a is the root of a monic polynomial with coefficients in $\mathbb{Z}$.

3.2.3 Lemma *Let* R *be a commutative ring with* 1.

(i) *An element* $a \in R$ *is an algebraic integer if and only if* a *is contained in a subring* $S \subseteq R$ *which is finitely generated as* $\mathbb{Z}$*-module.*

(ii) The set of algebraic integers in R forms a subring of R.

PROOF: (i) $\Rightarrow$: Let a be a root of $\sum_{i=0}^{n} a_i X^i$ with $a_i \in \mathbb{Z}$ and $a_n = 1$. Then the finitely generated $\mathbb{Z}$-module $S := \mathbb{Z}[a] = \langle 1, a, a^2, \ldots, a^{n-1}\rangle_{\mathbb{Z}}$ is a subring of R, since $a^n = -\sum_{i=0}^{n-1} a_i a^i \in S$ and thus S is closed under multiplication.
$\Leftarrow$: Let $a \in S = \langle b_1, \ldots, b_m\rangle_{\mathbb{Z}}$, then multiplication by a is described by a matrix $A = (A_{ij}) \in \mathbb{Z}^{m\times m}$ defined by $b_i a = \sum_{j=1}^{m} A_{ij} b_j$. Let f be the characteristic polynomial $\det(X \cdot id - A)$ of A, then f is a monic polynomial with coefficients in $\mathbb{Z}$ and $f(A) = 0$. On the other hand $f(A)$ gives the action of $f(a)$ on S, hence in particular $1 \cdot f(a) = 0$ and thus $f(a) = 0$.
(ii) If a and b are algebraic integers of R we have $\mathbb{Z}[a] = \langle 1, a, \ldots, a^{n-1}\rangle_{\mathbb{Z}}$ and $\mathbb{Z}[b] = \langle 1, b, \ldots, b^{m-1}\rangle_{\mathbb{Z}}$, hence the ring $\mathbb{Z}[a, b]$ is generated by $\{a^i b^j \mid 0 \le i < n,\ 0 \le j < m\}$ and is thus finitely generated as $\mathbb{Z}$-module. This shows that in particular the elements $a + b$, $a - b$ and ab are algebraic integers and thus the algebraic integers form a subring of R. □

3.2.4 Remark This lemma shows that the definition of algebraic integers agrees with what we usually call the integers in the field $\mathbb{Q}$ of rational numbers. It is clear that $a \in \mathbb{Z}$ is an algebraic integer. On the other hand, if $a = \frac{r}{s} \in \mathbb{Q}$ with $\gcd(r, s) = 1$ and $s > 1$, we have $x, y \in \mathbb{Z}$ with $1 = xr + ys$ and hence $\frac{1}{s} = x\frac{r}{s} + y \in \mathbb{Z}[a]$. But $\mathbb{Z}[s^{-1}]$ is not a finitely generated $\mathbb{Z}$-module, since the powers s^{-n} are independent over $\mathbb{Z}$.

3.2.5 Corollary *Let G be a finite group and χ a character of G over a field K with $\mathrm{char}(K) \nmid |G|$. Then $\chi(g)$ is an algebraic integer in K for all $g \in G$. In particular, $\chi(g) \in \mathbb{Q}$ if and only if $\chi(g) \in \mathbb{Z}$.*

PROOF: For an element g of order m the character value $\chi(g)$ is a sum of m-th roots of unity. An m-th root of unity is a root of the polynomial $X^m - 1$ and thus an algebraic integer, sums of algebraic integers are algebraic integers because these elements form a ring. □

3.2.6 Theorem *Let $\mathrm{char}(K) = 0$ and let χ be an irreducible character of G over K, afforded by the representation Δ.*

(i) $|\chi(g)| \le \chi(1)$ for all $g \in G$ and equality holds if and only if $\Delta(g) = a \cdot I_n$ with $a \in K$.

(ii) $\chi(g) = \chi(1) \Leftrightarrow \Delta(g) = I_n \Leftrightarrow g \in \ker(\Delta)$.

(iii) $\chi(g^{-1}) = \overline{\chi(g)}$.

(iv) For p prime $\chi(g^p) \equiv \chi(g)^p \pmod p$.

(v) For p prime and $\chi(g) \in \mathbb{Q}$ we have $\chi(g^p) \equiv \chi(g) \pmod p$.

PROOF: Note that we can regard all character values as complex numbers, since for $e = \exp(G)$ we have $\chi(g) \in \mathbb{Q}(\zeta_e) \subseteq \mathbb{C}$.
By extending K to $L = K(\zeta_e)$ we can assume that $\Delta(g)$ is a diagonal matrix.

Then for g of order m the diagonal entries of $\Delta(g)$ are m-th roots of unity. The claims (i)-(iii) now follow immediately by considering the unit circle.
For (iv) we additionally require that $\binom{p}{i} \equiv 0 \pmod p$, hence $\chi(g^p) = \sum_{i=1}^n \xi_i^p \equiv (\sum_{i=1}^n \xi_i)^p \pmod p$. Finally, (v) follows from (iv), since $a^p \equiv a \pmod p$ for $a \in \mathbb{Z}$ (by Fermat's little theorem) and since algebraic integers of $\mathbb{Q}$ lie in $\mathbb{Z}$. □

3.2.7 Remark The fact that $\chi(g^{-1}) = \overline{\chi(g)}$ shows that the symmetric bilinear form $(\cdot,\cdot)_G$ is in fact a hermitian inner product given by

$$(\chi, \psi)_G = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

3.2.8 Definition For a character χ of a group G we define $\ker(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}$. By the preceeding theorem, for a field of characteristic 0 this coincides with the kernel of a representation by which χ is afforded.

3.2.9 Theorem *Let $\chi_1, \ldots, \chi_r$ be the irreducible $\mathbb{C}$-characters of G.*

(i) $\chi_i(g) \in \mathbb{R}$ for all $i \Leftrightarrow g$ and g^{-1} are conjugate.

(ii) $\chi_i(g) \in \mathbb{Z}$ for all $i \Leftrightarrow g$ and g^j are conjugate for all j with $\gcd(j, |\langle g \rangle|) = 1$.

Proof: It is clear that all characters have the same values on conjugate elements. Vice versa, if all irreducible characters have the same values on two elements $g, h \in G$, also the class indicator functions have the same values on g and h, since the irreducible characters form a basis of the vector space of class functions. Hence, g and h lie in the same conjugacy class. This could in fact also be concluded from the second orthogonality relation.
(i) We have $\chi_i(g) \in \mathbb{R} \Leftrightarrow \chi_i(g) = \overline{\chi_i(g)} = \chi_i(g^{-1})$, hence $\chi_i(g) \in \mathbb{R}$ for all $1 \le i \le r$ if and only if all the irreducible characters have the same values on g and g^{-1} which is the case if and only if g and g^{-1} are conjugate.
(ii) We know that all character values $\chi_i(g)$ lie in $\mathbb{Q}(\zeta_m)$ where m is the order of g. Now on the one hand the Galois automorphisms of $\mathbb{Q}(\zeta_m)$ are of the form $\sigma_j : \zeta_m \mapsto \zeta_m^j$ with $\gcd(j,m) = 1$ and $\chi_i(g) \in \mathbb{Q} \Leftrightarrow \sigma_j(\chi_i(g)) = \chi_i(g)$ for all j. On the other hand, we can assume that $\Delta(g)$ is a diagonal matrix and we can apply σ_j to each of the diagonal entries, thus obtaining $\Delta(g^j)$. This shows that $\sigma_j(\chi_i(g)) = \chi_i(g^j)$. We therefore conclude that $\chi_i(g) \in \mathbb{Q}$ for all $1 \le i \le r$ if and only if all irreducible characters have the same values on g^j for all j with $\gcd(j,m) = 1$ which in turn is the case if and only if g and g^j are conjugate for all j with $\gcd(j,m) = 1$. Since character values are algebraic integers, it follows that $\chi_i(g) \in \mathbb{Q}$ implies $\chi_i(g) \in \mathbb{Z}$. □

3.2.10 Example We determine the character table of the alternating group A_5. As frame we have

$C_G(g_i)$	60	4	3	5	5
$\lvert C_i\rvert$	1	15	20	12	12
g_i	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4,5)$	$(1,2,3,5,4)$
χ_1	1	1	1	1	1
χ_2					
χ_3					
χ_4					
χ_5					
π	5	1	2	0	0

From $(\pi, \chi_1) = 1$ and $(\pi, \pi) = 2$ we see that $\pi - \chi_1$ is an irreducible character which we call χ_2. Thus $\chi_2 = (4, 0, 1, -1, -1)$. Next we conclude from $60 = 1^2 + 4^2 + \chi_3(1)^2 + \chi_4(1)^2 + \chi_5(1)^2$ that $\chi_3(1) = 5$, $\chi_4(1) = \chi_5(1) = 3$. The values on the class of $(1,2)(3,4)$ are rational and are congruent to the character degrees modulo 2. The only possibilities which agree with the orthogonality with the first and second column are $\chi_3((1,2)(3,4)) = 1$, $\chi_4((1,2)(3,4)) = \chi_5((1,2)(3,4)) = -1$. From the orthogonality of the third column with the first two it now follows that $\chi_3((1,2,3)) = -1$ and orthogonality with itself then implies $\chi_4((1,2,3)) = \chi_5((1,2,3)) = 0$. From $(\chi_3, \chi_3) = 1$ one concludes $\chi_3((1,2,3,4,5)) = \chi_3((1,2,3,5,4)) = 0$. Orthogonality between χ_1 and χ_4 now shows that $\chi_4((1,2,3,4,5)) + \chi_4((1,2,3,5,4)) = 1$, analogously we get $\chi_5((1,2,3,4,5)) + \chi_5((1,2,3,5,4)) = 1$. Orthogonality between the second and fourth/fifth column implies $\chi_4((1,2,3,4,5)) + \chi_5((1,2,3,4,5)) = 1$ and $\chi_4((1,2,3,5,4)) + \chi_5((1,2,3,5,4)) = 1$. Finally, orthogonality for the fourth column implies $\chi_4((1,2,3,4,5))^2 - \chi_4((1,2,3,4,5)) - 1 = 0$, thus we have (w.l.o.g.) $\chi_4((1,2,3,4,5)) = \frac{1+\sqrt{5}}{2} = 1 + \zeta_5 + \zeta_5^4$ and $\chi_4((1,2,3,5,4)) = \frac{1-\sqrt{5}}{2} = 1 + \zeta_5^2 + \zeta_5^3$. The full character table of A_5 therefore is:

$C_G(g_i)$	60	4	3	5	5
$\lvert C_i\rvert$	1	15	20	12	12
g_i	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4,5)$	$(1,2,3,5,4)$
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

3.2.11 Proposition *Let G be a finite group, K a splitting field of G with $\mathrm{char}(K) = 0$ and let Δ be an irreducible representation of degree n with character χ. Define $\omega : Z(KG) \to K$ such that $\Delta(z) = \omega(z) \cdot I_n$ for $z \in Z(KG)$. The mapping ω is called* a central character *of G.*

(i) For a conjugacy class C and the class sum $C^+ := \sum_{g \in C} g$ we have $\omega(C^+) = \frac{|C|}{\chi(1)}\chi(g)$ for $g \in C$.

(ii) $\omega(C^+)$ is an algebraic integer.

Proof: (i): We have $\omega(C^+)\cdot I_n = \Delta(C^+) = \sum_{g\in C}\Delta(g)$, which gives $\omega(C^+)\chi(1) = |C|\chi(g)$.
(ii): $C_1^+,\dots C_r^+$ is a K-basis of $Z(KG)$ and we have $C_i^+C_j^+ = \sum_{k=1}^r a_{ijk}C_k^+$ with $a_{ijk}\in K$. But for $g_k\in C_k$ fixed we have $a_{ijk} = |\{(g,h)\mid g\in C_i, h\in C_j, gh = g_k\}|\in\mathbb{Z}_{\geq 0}$. Applying ω gives $\omega(C_i^+)\omega(C_j^+) = \sum_{k=1}^r a_{ijk}\omega(C_k^+)$, thus the ring generated by the $\omega(C_i^+)$ is a finitely generated $\mathbb{Z}$-module which is spanned by the $\omega(C_i^+)$. This implies that $\omega(C_i^+)$ is an algebraic integer (since all elements in a finitely generated $\mathbb{Z}$-module are algebraic integers). □

3.2.12 Algorithm (Dixon-Schneider)
The a_{ijk} are called the *structure constants* of G. There are two ways in which the structure constants allow to compute the character table of G.

(1) If we denote the matrix for fixed first index by $A^{(i)}$, i.e. $A^{(i)} = (a_{ijk})_{j,k=1}^r$, then the relation $\sum_{k=1}^r a_{ijk}\omega(C_k^+) = \omega(C_i^+)\omega(C_j^+)$ show that

$$A^{(i)}\begin{pmatrix}\omega(C_1^+)\\ \vdots\\ \omega(C_r^+)\end{pmatrix} = \omega(C_i^+)\begin{pmatrix}\omega(C_1^+)\\ \vdots\\ \omega(C_r^+)\end{pmatrix},$$

thus the vector $(\omega(C_1^+),\dots,\omega(C_r^+))^{tr}$ is a column-eigenvector of $A^{(i)}$ with eigenvalue $\omega(C_i^+)$. From the eigenvectors, the characters can only be determined up to a scalar multiple, but this is resolved by the first orthogonality relation, since an irreducible character has to have norm 1. We therefore can compute the irreducible characters as simultaneous eigenvectors of the structure constant matrices $A^{(i)}$.

(2) If we let $g_k\in C_k$ run in the definition of the a_{ijk}, we can define the set $S_{ijk} := \{(g,h)\mid g\in C_i, h\in C_j, gh\in C_k\}$ of pairs with product in C_k. Then $|S_{ijk}| = a_{ijk}|C_k|$, since every conjugate of g_k gives rise to a conjugate pair of (g,h). But we have $(g,h)\in S_{ijk}\Leftrightarrow gh = u\in C_k\Leftrightarrow g^{-1}u = h\in C_j\Leftrightarrow (g^{-1},u)\in S_{i'kj}$, where i' is the index of the class $C_{i'}$ containing the inverses of C_i. We therefore have $|S_{ijk}| = |S_{i'kj}|$ and thus $a_{ijk}|C_k| = a_{i'kj}|C_j|$. From $\omega(C_{i'}^+)\omega(C_k^+) = \sum_{j=1}^r a_{i'kj}\omega(C_j^+)$ we now see that $\frac{|C_{i'}||C_k|}{\chi(1)^2}\chi(g_i^{-1})\chi(g_k) = \sum_{j=1}^r a_{i'kj}\frac{|C_j|}{\chi(1)}\chi(g_j) = \sum_{j=1}^r a_{ijk}\frac{|C_k|}{\chi(1)}\chi(g_j)$ which shows that

$$\sum_{j=1}^r \chi(g_j)a_{ijk} = \frac{|C_i|\chi(g_i^{-1})}{\chi(1)}\chi(g_k),$$

since $|C_i| = |C_{i'}|$. Thus, the vector $(\chi(g_1),\dots,\chi(g_r))$ is a row-eigenvector of $A^{(i)}$ with eigenvalue $\frac{|C_i|\chi(g_i^{-1})}{\chi(1)} = \omega(C_{i'}^+)$.

3.2.13 Theorem *Let K be a splitting field for G with* $\mathrm{char}(K) = 0$ *and let χ be an irreducible character of G. Then $\chi(1)\mid |G|$.*

PROOF: The first orthogonality relation says that $|G| = \sum_{g\in G}\chi(g)\chi(g^{-1}) = \sum_{i=1}^{r}|C_i|\chi(g_i)\chi(g_i^{-1})$ and therefore $|G|/\chi(1) = \sum_{i=1}^{r}\omega(C_i^+)\chi(g_i^{-1})$ is an algebraic integer, since both $\omega(C_i^+)$ and $\chi(g_i^{-1})$ are algebraic integers. But an algebraic integer in $\mathbb{Q}$ is an integer. $\square$

3.2.14 Remark A stronger result by Itô says that $\chi(1) \mid [G : N]$ for every abelian normal subgroup $N \trianglelefteq G$.

3.3 Burnsides's $p^a q^b$ theorem

A famous application of character theory is the proof of Burnside's theorem that a group of order $p^a q^b$ with p, q prime is soluble. The proof involves two other theorems, which are interesting in their own right.

3.3.1 Theorem *Let Δ be an irreducible representation of G with character χ and let C be a conjugacy class of G with $\gcd(\chi(1), |C|) = 1$. Then either $\Delta(g) \in Z(\Delta(G))$ or $\chi(g) = 0$.*

PROOF: Let $a, b \in \mathbb{Z}$ with $a|C| + b\chi(1) = 1$, then multiplying by $\frac{\chi(g)}{\chi(1)}$ gives $a\omega(C^+) + b\chi(g) = \frac{\chi(g)}{\chi(1)}$. The left hand side is an algebraic integer, therefore $\frac{\chi(g)}{\chi(1)}$ is an algebraic integer. We have $\Delta(g) \in Z(\Delta(G)) \Leftrightarrow |\chi(g)| = \chi(1)$, since Δ is irreducible, therefore $\left|\frac{\chi(g)}{\chi(1)}\right| < 1$ if $\Delta(g) \notin Z(\Delta(G))$. Let m be the order of g, then $\chi(g) \in \mathbb{Q}(\zeta_m)$. For every Galois automorphism $\sigma \in Aut(\mathbb{Q}(\zeta_m))$ we have $|\chi(g)^\sigma| \leq \chi(1)$, therefore for $\theta := \prod_{\sigma\in Aut(\mathbb{Q}(\zeta_m))} \frac{\chi(g)}{\chi(1)}^\sigma$ we have $|\theta| < 1$. But with $\frac{\chi(g)}{\chi(1)}$ every Galois conjugate $\frac{\chi(g)}{\chi(1)}^\sigma$ is also an algebraic integer, since it is a root of the same monic polynomial as $\frac{\chi(g)}{\chi(1)}$. This shows that θ is an algebraic integer. On the other hand, θ is fixed under all elements of $Aut(\mathbb{Q}(\zeta_m))$, thus $\theta \in \mathbb{Q}$ and since it is an algebraic integer we have $\theta \in \mathbb{Z}$. From $|\theta| < 1$ we now conclude that $\theta = 0$ and therefore $\chi(g) = 0$. $\square$

3.3.2 Theorem *Let G be a non-abelian simple group. Then G has no conjugacy class of prime power length p^n except $\{1\}$.*

PROOF: Let $1 \neq g \in G$, C the conjugacy class of g and assume that $|C| = p^a$. Let χ be a non-trivial irreducible character of G afforded by the representation Δ. Then $Z(\Delta(G)) = \{1\}$, since G is a non-abelian simple group (which implies $G \cong \Delta(G)$). By theorem 3.3.1 we know that $\chi(g) = 0$ if $p \nmid \chi(1)$. The second orthogonality relation for the classes of g and 1 now reads as:

$$0 = \frac{1}{|G|}\sum_{i=1}^{r}\chi_i(g)\chi_i(1) = 1 + \sum_{p|\chi_i(1)}\chi_i(g)\chi_i(1)$$

and therefore $-\frac{1}{p} = \sum_{p|\chi_i(1)}\chi_i(g)\frac{\chi_i(1)}{p}$. The right hand side of this equation is an algebraic integer, since $\chi_i(g)$ is an algebraic integer and $\frac{\chi_i(1)}{p} \in \mathbb{Z}$. But $-\frac{1}{p}$

is not an algebraic integer, which contradicts the assumption of a conjugacy class of prime power length. □

3.3.3 Theorem (Burnside)
If $|G| = p^a q^b$ with p and q prime, then G is soluble.

PROOF: We use induction on $|G|$. Let $N \trianglelefteq G$ be a maximal normal subgroup of G. If $N \neq \{1\}$ then by induction N and G/N are soluble and thus, G is soluble. We can therefore assume that G is simple. Let $P \neq \{1\}$ be a Sylow subgroup of G and $1 \neq g \in Z(P)$. Then $P \leq C_G(g)$ and thus the length $[G : C_G(g)]$ of the conjugacy class of g is a prime power. By theorem 3.3.2 this implies that G is an abelian group and is in particular soluble. □

3.4 Tensor products

3.4.1 Definition For two vector spaces V and W with bases $(v_1, \dots, v_n)$ and $(w_1, \dots, w_m)$, respectively, the *tensor product* $V \otimes_K W$ of V and W is defined to be the vector space spanned by the $n \cdot m$ linearly independent elements $v_i \otimes w_j$. For two elements $v = \sum_{i=1}^n a_i v_i \in V$ and $w = \sum_{j=1}^m b_j w_j \in W$ the *(pure) tensor* $v \otimes w$ is defined by $v \otimes w := \sum_{i=1}^n \sum_{j=1}^m a_i b_j (v_i \otimes w_j)$.

3.4.2 Remarks

(1) We can regard $\otimes_K$ as a K-bilinear mapping $\otimes_K : V \times W \to V \otimes_K W$, since

  (i) $(v + v') \otimes w = v \otimes w + v' \otimes w$ for all $v, v' \in V$, $w \in W$,

  (ii) $v \otimes (w + w') = v \otimes w + v \otimes w'$ for all $v \in V$, $w, w' \in W$,

  (iii) $\lambda v \otimes w = v \otimes \lambda w = \lambda(v \otimes w)$ for all $v \in V$, $w \in W$ and $\lambda \in K$.

  If there is no confusion about the field K we will usually omit the subscript K and write $V \otimes W$ instead of $V \otimes_K W$.

(2) Note that not every element of $V \otimes W$ is a pure tensor of the form $v \otimes w$ for some $v \in V, w \in W$. For $V = W = \mathbb{F}_p^n$ we have $\dim(V \otimes W) = n^2$, thus $|V \otimes W| = p^{n^2}$, but there are only $(p^n)^2 = p^{2n}$ pure tensors.

(3) The construction of the tensor product $V \otimes W$ is independent on the choice of bases for V and W. If we choose different bases $(v'_1, \dots, v'_n)$ for V and $(w'_1, \dots, w'_m)$ for W, then $(v'_1 \otimes w'_1, v'_1 \otimes w'_2, \dots, v'_n \otimes w'_m)$ is also a basis of $V \otimes W$.

3.4.3 Lemma

(i) For two linear mappings $\varphi \in End_K(V)$ and $\psi \in End_K(W)$ there is a unique linear mapping $\varphi \otimes \psi \in End_K(V \otimes W)$ such that $(v \otimes w)(\varphi \otimes \psi) = v\varphi \otimes w\psi$.

(ii) *If φ has matrix A with respect to a basis $(v_1,\dots,v_n)$ of V and ψ has matrix B with respect to a basis $(w_1,\dots,w_m)$ of W, then the matrix of $\varphi\otimes\psi$ with respect to the basis $(v_1\otimes w_1, v_1\otimes w_2,\dots,v_n\otimes w_m)$ is given by the Kronecker product*

$$A\otimes B = \begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{pmatrix}$$

(iii) $(A\otimes B)(A'\otimes B') = AA'\otimes BB'$.

(iv) $\mathrm{tr}(A\otimes B) = \mathrm{tr}(A)\cdot\mathrm{tr}(B)$.

PROOF: (i): Let $(v_1,\dots,v_n)$ and $(w_1,\dots,w_m)$ be bases of V and W, respectively. Uniqueness of $\varphi\otimes\psi$ is clear, since $v_i\otimes w_j$ form a basis of $V\otimes W$. We now define $(v_i\otimes w_j)(\varphi\otimes\psi) := v_i\varphi\otimes w_j\psi$, then the desired property follows from the bilinearity of $\otimes$.
(ii): This follows from (i) by writing out the matrices.
(iii): This is seen by applying both sides to $v_i\otimes w_j$.
(iv): We have $\mathrm{tr}(A\otimes B) = A_{11}\,\mathrm{tr}(B)+\ldots+A_{nn}\,\mathrm{tr}(B) = \mathrm{tr}(A)\cdot\mathrm{tr}(B)$. □

3.4.4 Corollary *If Δ and Δ' are representations of the group G with modules V and W and characters χ and χ', then $\Delta\otimes\Delta' : G\to V\otimes W, g\mapsto\Delta(g)\otimes\Delta'(g)$ is a representation of G with character $\chi\cdot\chi'$. In particular, products of characters are characters again.*

PROOF: We only have to prove that $\Delta\otimes\Delta'$ is a group homomorphism, the rest follows from Lemma 3.4.3. We have $(\Delta\otimes\Delta')(gh) = \Delta(gh)\otimes\Delta'(gh) = \Delta(g)\Delta(h)\otimes\Delta'(g)\Delta'(h) = (\Delta(g)\otimes\Delta'(g))(\Delta(h)\otimes\Delta'(h)) = (\Delta\otimes\Delta')(g)(\Delta\otimes\Delta')(h)$. □

3.4.5 Remark A word of warning: Note that the representation $\Delta\otimes\Delta'$ is defined only on the group elements and has to be extended from there to the group ring KG by linearity. For arbitrary $a\in KG$ we have $(\Delta\otimes\Delta')(a)\neq\Delta(a)\otimes\Delta'(a)$, for example for $c\neq 0,1$ and $g\in G$ we have $(\Delta\otimes\Delta')(cg) = c(\Delta(g)\otimes\Delta'(g))\neq c^2(\Delta(g)\otimes\Delta'(g)) = \Delta(cg)\otimes\Delta'(cg)$.

3.4.6 Theorem (Burnside-Brauer)
Let χ be a faithful character of G and suppose that $\chi(g)$ takes on precisely m different values on G. Then every irreducible character ψ of G is a constituent of one of the powers $\chi^0,\chi^1,\dots,\chi^{m-1}$, i.e. $(\chi^j,\psi)>0$ for some $0\le j<m$.

PROOF: Let $a_1,\dots,a_m$ be the distinct values of $\chi(g)$ and assume that $a_1=\chi(1)$. Define $G_i := \{g\in G \mid \chi(g)=a_i\}$ and $b_i := \sum_{g\in G_i}\overline{\psi(g)}$. The first orthogonality relation now reads as $(\chi^j,\psi) = \frac{1}{|G|}\sum_{i=1}^m a_i^j b_i$. Now assume that ψ is not a constituent of any of the χ^j for $0\le j<m$, then $(b_1,\dots,b_m)$ is a solution for the m linear equations $\sum_{i=1}^m a_i^j b_i = 0$. But the matrix of this system of equations is a *Vandermonde-matrix*, its determinant is $\pm\prod_{i<j}(a_i-a_j)\neq 0$. Hence all b_i have to be zero which contradicts $b_1=\psi(1)\neq 0$. □

3.4.7 Examples

(1) The character ρ of the regular representation takes on only two different values, hence all irreducible characters occur as constituents of $\rho^0 = \chi_1$ and ρ. Of course we knew this already, since ρ contains every irreducible character χ_i with multiplicity $\chi_i(1)$.

(2) Let $\chi_3 = \pi - \chi_1$ be the character of S_4 obtained by subtracting the trivial character from the natural permutation character. Then χ_3 takes on the values $(3, -1, 0, 1, -1)$ on the conjugacy classes, hence $m = 4$. We have $(\chi_3^2, \chi_1) = 1$, $(\chi_3^2, \chi_3) = 1$ and $(\chi_3^2, \chi_3^2) = 4$. Since $\chi_3^2(1) = 9$ we can conclude that the other two irreducible constituents of χ_3^2 are the second 3-dimensional character and the 2-dimensional character obtained from the factor group S_3. We have not found the signum-character χ_2 yet, but from the above theorem we know that $(\chi_3^3, \chi_2) > 0$. It turns out that $(\chi_3^3, \chi_2) = 1$.

3.4.8 Proposition *Let V and W be KG-modules. Then $V^* \otimes W \cong_{KG} Hom_K(V, W)$.*

PROOF: Note that $Hom_K(V, W)$ is a KG-module by $v(\varphi g) := (vg^{-1})\varphi g$. To define an isomorphism between $V^* \otimes W$ and $Hom_K(V, W)$ it is enough to give the image of pure tensors, since these generate $V^* \otimes W$. We define $\Phi : V^* \otimes W \to Hom_K(V, W)$ by $v\Phi(\lambda \otimes w) := (v\lambda)w$, then it is clear that Φ is a homomorphism of K-vector spaces. We have $v(\Phi(\lambda \otimes w)g) = (vg^{-1})\Phi(\lambda \otimes w)g = (vg^{-1})\lambda wg$. On the other hand $\Phi((\lambda \otimes w)g) = \Phi(\lambda g \otimes wg)$ and we have $v(\lambda g) = (vg^{-1})\lambda$ for the action of G on the dual module V^*. Hence, $v\Phi((\lambda \otimes w)g) = (vg^{-1})\lambda wg$, which shows that Φ is a KG-homomorphism.

To show that Φ is injective let $(v_1, \ldots, v_n)$ be a basis of V and let $(\lambda_1, \ldots, \lambda_n)$ be the dual basis of V^*. Note that an arbitrary element of $V^* \otimes W$ can be written as $\sum_{i=1}^n \lambda_i \otimes w_i$ with $w_i \in W$. Now, $v_j\Phi(\sum_{i=1}^n \lambda_i \otimes w_i) = \sum_{i=1}^n v_j(\lambda_i \otimes w_i) = \sum_{i=1}^n (v_j\lambda_i)w_i = \sum_{i=1}^n \delta_{ij}w_i = w_j$. We therefore have $\Phi(\sum_{i=1}^n \lambda_i \otimes w_i) = 0 \Rightarrow w_j = 0$ for all $j \Rightarrow \lambda_j \otimes w_j = 0$ for all $j \Rightarrow \sum_{j=1}^n \lambda_j \otimes w_j = 0$. Since $\dim(V^* \otimes W) = \dim(V) \cdot \dim(W) = \dim(Hom_K(V, W))$ it follows that Φ is also surjective and hence an isomorphism. $\square$

3.4.9 Theorem *Let $\mathrm{char}(K) \neq 2$ and let V be a KG-module of dimension n with character χ.*

(i) $V \otimes V = V^{[2]} \oplus V^{[1,1]}$, where $V^{[2]} := \langle \{v \otimes w \mid v \otimes w = w \otimes v\} \rangle$ is the subspace of dimension $\binom{n+1}{2}$ spanned by the symmetric tensors and $V^{[1,1]} := \langle \{v \otimes w \mid v \otimes w = -w \otimes v\} \rangle$ is the subspace of dimension $\binom{n}{2}$ spanned by the alternating tensors. The modules $V^{[2]}$ and $V^{[1,1]}$ are called the symmetrizations *of $V \otimes V$.*

(ii) The character of G on $V^{[2]}$ is $\chi^{[2]}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$, the character of G on $V^{[1,1]}$ is $\chi^{[1,1]}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$.

PROOF: If $(v_1, \ldots, v_n)$ is a basis of V then $(v_i \otimes v_j + v_j \otimes v_i, v_k \otimes v_k \mid 1 \leq i < j \leq n, 1 \leq k \leq n)$ is a basis of $V^{[2]}$ consisting of $\binom{n+1}{2}$ elements and $(v_i \otimes v_j - v_j \otimes v_i \mid 1 \leq i < j \leq n)$ is a basis of $V^{[1,1]}$ consisting of $\binom{n}{2}$ elements. It is clear that $V^{[2]}$ and $V^{[1,1]}$ are G-invariant submodules, since $(v \otimes w)g = vg \otimes wg$. Now let $g \in G$ and assume that $\Delta(g)$ is a diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. Then the action of g on $(v_i \otimes v_j - v_j \otimes v_i)$ is given by $(v_i \otimes v_j - v_j \otimes v_i)g = \lambda_i \lambda_j (v_i \otimes v_j - v_j \otimes v_i)$ and hence the character on $V^{[1,1]}$ is $\chi^{[1,1]}(g) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}((\sum_{i=1}^n \lambda_i)^2 - (\sum_{i=1}^n \lambda_i^2))^2 = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$. Since $\chi^2 = \chi^{[2]} + \chi^{[1,1]}$ it follows that $\chi^{[2]}(g) = \chi(g)^2 + \chi(g^2)$. □

3.4.10 Definition For a finite group G with irreducible characters $\chi_1, \ldots, \chi_r$ define $\theta_k(g) := |\{h \in G \mid h^k = g\}|$ for $g \in G$. Then θ_k is a class function and we can write it as $\theta_k = \sum_{i=1}^r \nu_k(\chi_i)\chi_i$. The coefficient $\nu_k(\chi_i)$ is called the k-th *Frobenius-Schur indicator* of χ_i.

3.4.11 Corollary *Let G have precisely t involutions (elements of order 2), then $1 + t = \sum_{i=1}^r \nu_2(\chi_i)\chi_i(1)$.*

PROOF: This is clear since $\theta_2(1) = 1 + t$. □

3.4.12 Lemma *For an irreducible character χ of G one has*

$$\nu_k(\chi) := \frac{1}{|G|} \sum_{g \in G} \chi(g^k).$$

PROOF: Since θ_k has integer values we have $\nu_k(\chi) = (\chi, \theta_k) = (\chi, \overline{\theta_k}) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\theta(g) = \frac{1}{|G|} \sum_{g \in G} \sum_{h^k = g} \chi(g) = \frac{1}{|G|} \sum_{h \in G} \chi(h^k)$. □

3.4.13 Corollary *Let K be a splitting field for G with $\mathrm{char}(K) = 0$ and let χ be an irreducible character of G. Then $\nu_2(\chi) \in \{0, 1, -1\}$. More precisely:*

$$\nu_2(\chi) = 0 \Leftrightarrow \chi \neq \overline{\chi}$$

$$\nu_2(\chi) = 1 \Leftrightarrow \chi = \overline{\chi} \text{ and } (\chi^{[2]}, \chi_1)_G = 1$$

$$\nu_2(\chi) = -1 \Leftrightarrow \chi = \overline{\chi} \text{ and } (\chi^{[1,1]}, \chi_1)_G = 1$$

PROOF: Let V be the irreducible module corresponding to the character χ then χ^2 is the character on $V \otimes V$. We have $(\chi^2, \chi_1) = (\chi, \overline{\chi})$ and in case $\chi \neq \overline{\chi}$ the trivial module is not a constituent of $V \otimes V$ and a fortiori not of $V^{[2]}$. From the relation $(\chi^{[2]}, \chi_1) = \frac{1}{2}(\chi^2, \chi_1) + \frac{1}{2}\nu_2(\chi)$ the claim now follows. □

3.4.14 Remark Let G be a finite group and V a simple KG-module. The action of G on $V \otimes V$ can be interpreted as an action of G on the bilinear forms on V with values in K:

Let $(v_1, \ldots, v_n)$ be a basis of V, denote the corresponding representation by Δ, then $v_i g = \sum_{k=1}^n \Delta(g)_{ik}$. A general element $w \in V \otimes V$ is given by $w = \sum_{i,j=1}^n a_{ij}(v_i \otimes v_j)$. Then the action of g on w is given by $wg = \sum_{i,j}^n \sum_{k,l}^n a_{ij}\Delta(g)_{ik}\Delta(g)_{jl}$. If we identify w with the matrix $A = (a_{ij})$, then wg

is identified with the matrix $\Delta(g)A\Delta(g)^{tr}$. Thus, if we interpret A as the Gram matrix of a bilinear form then the action of g becomes the usual action on Gram matrices. Moreover, we see that w is a symmetric tensor (i.e. $w \in V^{[2]}$) if and only if the matrix A is symmetric (i.e. $A = A^{tr}$) and w is an antisymmetric tensor (i.e. $w \in V^{[1,1]}$) if and only if A is antisymmetric (i.e. $A = -A^{tr}$).

3.4.15 Corollary *Let K be a splitting field for G with* $\mathrm{char}(K) = 0$ *and let χ be an irreducible character of G with corresponding module V. If $\chi = \overline{\chi}$ there exists (up to scalar multiples) a unique non-degenerate G-invariant bilinear form Φ on V. This form Φ is symmetric if and only if $\nu_2(\chi) = 1$, it is symplectic (antisymmetric) if and only if $\nu_2(\chi) = -1$.*

Proof: We already know that $(\chi^{[2]}, \chi_1) = 1$ if $\nu_2(\chi) = 1$ and $(\chi^{[1,1]}, \chi_1) = 1$ if $\nu_2(\chi) = -1$. From the above remark we see that in the first case G fixes a symmetric bilinear form and in the latter case an antisymmetric bilinear form. Finally, it is clear that the invariant form Φ has to be non-degenerate, since otherwise the radical $\{v \in V \mid \Phi(v, v') = 0 \text{ for all } v' \in V\}$ of Φ would be a proper G-invariant submodule. □

We have seen that a real-valued character indicates that the corresponding representation fixes a bilinear form and the Frobenius-Schur indicator $\nu_2(\chi)$ distinguishes whether the form is symmetric or antisymmetric. We will now show that $\nu_2(\chi)$ actually tells us much more, namely whether a real-valued character is the character of a representation that can be written over $\mathbb{R}$. For that we have to discuss how we can interpret a $\mathbb{C}G$-module as an $\mathbb{R}G$-module.

3.4.16 Definition Let V be a $\mathbb{C}G$-module with basis $(v_1, \ldots, v_n)$ and representation Δ. Define $V_{\mathbb{R}}$ to be the vector space with basis $(v_1, \ldots, v_n, iv_1, \ldots, iv_n)$. We can turn $V_{\mathbb{R}}$ into an $\mathbb{R}G$-module by

$$v_j g := \sum_{k=1}^{n} Re(\Delta(g)_{jk})v_k + Im(\Delta(g)_{jk})iv_k$$

$$(iv_j)g := \sum_{k=1}^{n} -Im(\Delta(g)_{jk})v_k + Re(\Delta(g)_{jk})iv_k.$$

The representation on $V_{\mathbb{R}}$ has degree $2n$ and can be written as

$$\begin{pmatrix} Re(\Delta(g)) & Im(\Delta(g)) \\ -Im(\Delta(g)) & Re(\Delta(g)) \end{pmatrix}.$$

If χ is the character on V, then the character on $V_{\mathbb{R}}$ is $2Re(\chi) = \chi + \overline{\chi}$.

3.4.17 Lemma *If V is an irreducible $\mathbb{C}G$-module and $V_{\mathbb{R}}$ is a reducible $\mathbb{R}G$-module, then χ is afforded by a real representation.*

Proof: If $V_{\mathbb{R}}$ is a reducible $\mathbb{R}G$-module we have $V_{\mathbb{R}} = U \oplus W$ where U is an $\mathbb{R}G$-module with character χ and W is an $\mathbb{R}G$-module with character $\overline{\chi}$. Since U is an $\mathbb{R}G$-module, the character χ is afforded by a real representation. □

3.4.18 Lemma *Let V be an $\mathbb{R}G$-module and let Φ be a G-invariant symmetric bilinear form on V. Suppose there exist $v, w \in V$ with $\Phi(v, v) > 0$ and $\Phi(w, w) < 0$. Then V is reducible.*

PROOF: From $\sum_{g \in G} \Delta(g)\Delta(g)^{tr}$ we obtain a positive definite bilinear form Ψ on V. We now choose a basis $(v_1, \ldots, v_n)$ of V which is orthonormal with respect to Ψ, i.e. $\Psi(v_i, v_j) = \delta_{ij}$. The Gram matrix F of Φ with respect to this basis is a symmetric matrix. By the spectral theorem we can find an orthogonal matrix Q (i.e. $Q^{-1} = Q^{tr}$) such that $QFQ^{-1} = D$ is a diagonal matrix. Without loss of generality we can assume that $D_{11} > 0$ and $D_{22} < 0$. We now define a new G-invariant bilinear form Φ' on V by $\Phi'(v, w) := \Phi(v, w) - D_{11}\Psi(v, w)$. Then $\Phi'(v_1, w) = 0$ for all $w \in V$, hence v_1 lies in the radical of Φ'. On the other hand $\Phi'(v_2, v_2) = D_{22} - D_{11} < 0$, hence Φ' is not the zero-form and therefore the radical of Φ' is a proper $\mathbb{R}G$-submodule of V. □

3.4.19 Theorem *Let K be a splitting field of the finite group G and let χ be an irreducible character of G. Then χ is afforded by a real representation if and only if $\nu_2(\chi) = 1$.*

PROOF: $\Rightarrow$: Let χ be afforded by the real representation Δ. Then $F := \sum_{g \in G} \Delta(g)\Delta(g)^{tr}$ is the Gram matrix of a symmetric bilinear form fixed by $\Delta(G)$. Moreover, F is positive definite and in particular $\neq 0$. Therefore, $\nu_2(\chi) = 1$.
$\Leftarrow$: Let V be a simple $\mathbb{C}G$-module with character χ. Since $\nu_2(\chi) = 1$, there exists a G-invariant symmetric bilinear form Φ on V. We define a mapping $\iota : V_{\mathbb{R}} \to V$, $\sum_{j=1}^{n} a_j v_j + b_j(iv_j) \mapsto (a_j + ib_j)v_j$ for $a_j, b_j \in \mathbb{R}$, then ι is clearly a bijection. The mapping $\Psi : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{C}$ defined by $\Psi(v, w) := \Phi(v\iota, w\iota)$ is easily seen to be a G-invariant symmetric bilinear form on $V_{\mathbb{R}}$. Since Φ is not the zero-form, there are $v, w \in V$ with $\Phi(v, w) \neq 0$. Therefore we can choose $v_1 \in \{v, w, v + w\}$ such that $\Phi(v_1, v_1) \neq 0$ and by dividing v_1 by a square root of $\Phi(v_1, v_1)$ we can assume that $\Phi(v_1, v_1) = 1$. But then $\Psi(v_1, v_1) = 1$ and $\Psi(iv_1, iv_1) = -1$, hence $V_{\mathbb{R}}$ is reducible by Lemma 3.4.18. By Lemma 3.4.17 this shows that χ is afforded by a real representation. □

3.4.20 Remark The Frobenius-Schur indicator distinguishes over $\mathbb{C}$ the following three cases:

(1) If $\nu_2(\chi) = 0$ then $\chi \neq \overline{\chi}$ and the representation necessarily involves elements from $\mathbb{C} \setminus \mathbb{R}$.

(2) If $\nu_2(\chi) = 1$ then the representation can be realized over $\mathbb{R}$ and fixes a positive definite G-invariant bilinear form.

(3) If $\nu_2(\chi) = -1$ then although the character values are real the representation can not be realized over $\mathbb{R}$ and it fixes a symplectic G-invariant bilinear form.

3.4.21 Example The character table of the quaternion group Q_8 is

$C_G(g_i)$	8	8	4	4	4
$\lvert C_i\rvert$	1	1	2	2	2
g_i	1	i^2	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

For $i = 1, 2, 3, 4$ we have $\chi_i^{[2]}(g) = \frac{1}{2}(\chi_i(g)^2 + \chi_i(g^2)) = 1$, thus $\chi_i^{[2]} = \chi_1$. Furthermore, $\chi_5^{[2]} = \chi_2 + \chi_3 + \chi_4$ and $\chi_5^{[1,1]} = \chi_1$. This can also be seen by computing $\nu_2(\chi_5) = \frac{1}{8}\sum_{g\in Q_8} \chi_5(g^2) = \frac{1}{8}(2 + 2 + 6 \cdot (-2)) = -1$.

Exercises

28. Let K be an arbitrary field and let Δ be an irreducible representation of G over K. Show that $\sum_{g\in G} \Delta(g) = 0$ unless Δ is the trivial representation.

29. Let G be a finite group and K a field with $\mathrm{char}(K) = 0$. Let χ be a character of G over K and let Δ be a representation of G affording χ. Prove that $\det_\chi : G \to K, g \mapsto \det(\Delta(g))$ is a 1-dimensional character of G. Show that $\det_\chi$ is well-defined, i.e. independent of the choice of Δ amongst the equivalent representations.

30. Let G be a non-abelian group of order 8.

    (i) Show that G has a unique nonlinear irreducible character χ over $\mathbb{C}$.

    (ii) Show that $\chi(1) = 2$, $\chi(g) = -2$ for $g \in G' \setminus \{1\}$ and $\chi(h) = 0$ for $h \in G \setminus G'$.

    (iii) Show that in case $G \cong Q_8$ the character $\det_\chi$ from the previous exercise is the trivial character of G whereas in case $G \cong D_8$ it is not.

    Note that D_8 and Q_8 are examples of nonisomorphic groups with identical character tables.

31. Let G and H be groups. Determine the character table of the direct product $G \times H$ in terms of the character tables of G and H.

32. Let G be a finite group and K a field with $\mathrm{char}(K) = 0$.

    (i) Let G act on a finite set Ω. Show that the character π of the permutation representation of G on $V = \langle v_\omega \mid \omega \in \Omega\rangle$ is given by $\pi(g) = |Fix_\Omega(g)|$ where $Fix_\Omega(g) = \{\omega \in \Omega \mid \omega g = \omega\}$.

    (ii) G acts on itself by conjugation. Determine the character of the corresponding permutation representation in purely group theoretic terms.

    (iii) Show that each sum over a row in the character table of G is a non-negative integer. (Hint: Use the bilinear form on class functions and part (ii).)

33. Let χ be a non-trivial character of a group G and suppose that all character values $\chi(g)$ are non-negative real numbers. Show that χ is reducible.

34. Let $g_1, \ldots, g_r \in G$ be representatives of the conjugacy classes of G and let $\mathcal{C} = (\chi_i(g_j))_{i,j=1}^r$ be the character table of G. Show that $|\det(\mathcal{C})|^2 = \prod_{j=1}^r |C_G(g_j)|$. (Hint: The orthogonality relations say that $\mathcal{C}$ is almost a unitary matrix.)

35. Determine the conjugacy classes, the irreducible representations over $\mathbb{C}$ and the character table of the dihedral groups D_{2n}. (Hint: Consider the action of D_{2n} on a regular n-gon.)

36. Let χ be a character of G over a field K with $\mathrm{char}(K) = 0$ and let $\chi = \sum_{i=1}^r a_i\chi_i$ be the decomposition of χ into irreducible characters.

    (i) Show that $\ker(\chi) = \cap\{\ker(\chi_i) \mid a_i > 0\}$.

    (ii) Show that $\cap_{i=1}^r \ker(\chi_i) = \{1\}$.

    (iii) Prove that every normal subgroup $N \trianglelefteq G$ can be 'read off' the character table as the intersection of some of the $\ker(\chi_i)$. (Note: Every normal subgroup is a union of conjugacy classes and a normal subgroup is assumed to be 'known' when the classes it consists of are known.)

37. A character χ is called *faithful* if $\ker(\chi) = \{1\}$.

    (i) Show that the centre $Z(G)$ is cyclic if G has a faithful irreducible representation over $\mathbb{C}$.

    (ii) Assume that G is a p-group and that $Z(G)$ is cyclic. Prove that G has a faithful irreducible representation over $\mathbb{C}$. (Hint: Every normal subgroup of G intersects $Z(G)$ and if $Z(G)$ is cyclic it contains a unique subgroup of order p.)

38. Let G be a finite group, K a splitting field with $\mathrm{char}(K) = 0$ and let $\chi_i, 1 \le i \le r$ be the irreducible characters of G. Recall that the structure constants a_{ijk} are defined by $C_i^+ C_j^+ = \sum_{k=1}^r a_{ijk} C_k^+$, where $C_1^+, \ldots, C_r^+$ are the class sums.

    (i) Show that
    $$a_{ijk} = \frac{|C_i||C_j|}{|G|} \sum_{l=1}^{r} \frac{\chi_l(g_i)\chi_l(g_j)\chi_l(g_k^{-1})}{\chi_l(1)}$$
    where g_i is a representative of the conjugacy class with class sum C_i^+.

    (ii) Conclude that $a_{ijk}|C_k| = a_{i'kj}|C_j|$, where i' is the index of the conjugacy class of g_i^{-1}.

39. In the Dixon-Schneider algorithm we use the fact that the irreducible characters are row-eigenvectors of the structure constant matrices $A^{(i)} = (a_{ijk})_{1 \le j,k \le r}$ with eigenvalues $\omega(C_{i'}^+) = \frac{|C_i|\chi(g_i^{-1})}{\chi(1)}$. Show that no two irreducible characters have the same eigenvalues for all $A^{(i)}$.
    Give an iterative method which splits the space of class functions into 1-dimensional subspaces each containing one irreducible character.

40. Use the Dixon-Schneider algorithm to determine the central characters ω_i and the character table of the dihedral group D_{10} of order 10.

41. Determine the character table of the simple group $GL_3(2)$ of order 168. (Hint: The conjugacy classes are parametrized by the characteristic polynomials of the matrices. Use symmetrized tensor products and permutation characters.)

42. Let χ, ψ, θ be irreducible characters of the finite group G. Show that $(\chi\psi, \theta) \le \theta(1)$.

43. Let G be a group and let χ be the character of a $\mathbb{C}$-irreducible representation of G. Show that if $\nu_2(\chi) = -1$, then $\chi(1)$ is even. Deduce that $\nu_2(\chi) = 1$ for χ with $\chi = \overline{\chi}$ and $\chi(1)$ odd.

44. Let Δ be an irreducible representation of degree 2 of G and let χ be the character of Δ. Show that $\chi^{[1,1]}(g) = \det(\Delta(g))$ for all $g \in G$. Conclude that $\nu_2(\chi) = -1$ if and only if $\det(\Delta(g)) = 1$ for all $g \in G$.

45. Let $G = \langle g, h \mid g^5 = h^2 = 1, hgh = g^{-1}\rangle$ be the dihedral group D_{10} of order 10.

    (i) Show that
    $$\Delta(g) := \begin{pmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^{-1} \end{pmatrix}, \quad \Delta(h) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
    defines a faithful irreducible representation of D_{10} over $\mathbb{C}$.

    (ii) Compute the Frobenius-Schur indicator $\nu_2(\chi)$ of the character χ of Δ. Is Δ equivalent with a representation that can be realized over $\mathbb{R}$? If so, determine such a real representation.

    (iii) Determine $\chi^{[2]}$ and $\chi^{[1,1]}$. Compute a G-invariant bilinear form for $\Delta(G)$.

    (iv) Decompose χ^2 and χ^3 into irreducible characters.

Chapter 4

Induced representations

It is clear that a KG-module can be regarded as a KH-module for a subgroup $H \leq G$ by restricting the action to H. In this chapter we will deal with the opposite situation, i.e. a module for a subgroup H is known and we want to construct a module for the full group from this.

4.1 Induction

4.1.1 Definition Let $H \leq G$ be a subgroup and let V be a KG-module. Then by restricting the action to KH, V becomes a KH-module which is denoted by $V_{|H}$ or $\mathrm{res}_H^G(V)$.

4.1.2 Theorem *Let $H \leq G$ be a subgroup and let W be a KH-module. Let $I \leq W \otimes KG$ be defined by $I := \langle w \otimes hg - wh \otimes g \mid w \in W, h \in H, g \in G\rangle$. Then the quotient space $(W \otimes KG)/I$ becomes a KG-module via $(w \otimes g)g' := w \otimes (gg')$. This KG-module is denoted by W^G.*
If $g_1, \ldots, g_m$ is a transversal for H in G (i.e. $G = \dot{\bigcup}_{i=1}^m Hg_i$), then $W^G \cong \bigoplus_{i=1}^m W \otimes g_i$ and this decomposition is independent of the choice of the transversal. In particular $\dim_K(W^G) = [G : H] \dim_K(W)$.

Proof: If we forget the action of H on W and regard W as a KG-module with trivial G-action, the given action $(w \otimes g)g' = w \otimes (gg')$ is the usual tensor product action on $W \otimes KG$. Moreover, since $(w \otimes hg)x - (wh \otimes g)x = w \otimes h(gx) - wh \otimes (gx)$, we see that I is a KG-submodule of $W \otimes KG$ and hence W^G is a KG-module.

Now let $g_1, \ldots, g_m$ be a transversal for H in G and define

$$\varphi : w \otimes g = w \otimes hg_i \mapsto wh \otimes g_i.$$

Then linearly extending φ gives a mapping $\varphi : W \otimes KG \to \bigoplus_{i=1}^m W \otimes g_i$ which is clearly a surjective homomorphism of vector spaces. To determine the kernel of φ note that a general element $v \in W \otimes KG$ is of the form $v = \sum_{g \in G} w_g \otimes g = \sum_{i=1}^m \sum_{h \in H} w_{hg_i} \otimes hg_i$. Since $v\varphi = (\ldots, \sum_{h \in H} w_{hg_i} h \otimes g_i, \ldots)$ we have $v \in \ker(\varphi)$ if and only if $\sum_{h \in H} w_{hg_i} h = 0$ for all i. It is therefore clear that $I \subseteq \ker(\varphi)$. On the other hand, an element $\sum_{h \in H} w_{hg_i} \otimes hg_i \in \ker(\varphi)$ can be written as $\sum_{h\in H} w_{hg_i} \otimes hg_i = \sum_{h\in H}(w_{hg_i} \otimes hg_i - w_{hg_i}h \otimes g_i) + \sum_{h\in H} w_{hg_i}h \otimes g_i$ and thus lies in I, since $\sum_{h\in H} w_{hg_i}h = 0$. This isomorphism induced by φ implies that $\dim_K(W^G) = [G : H] \dim_K(W)$.

Finally, it is clear that $w \otimes g \in W \otimes g_i$ if and only if $g \in Hg_i$ and in this case $W \otimes g = W \otimes g_i$. This shows that the decomposition $W^G \cong \bigoplus_{i=1}^m W \otimes g_i$ is independent of the choice of the transversal. □

4.1.3 Remark The above construction of $W^G = (W \otimes KG)/I$ is often denoted by $W \otimes_{KH} KG$, indicating that elements of KH are allowed to commute with the tensor product sign. This is consistent with our earlier notation $V \otimes_K W$ for the tensor product of K-vector spaces.

4.1.4 Definition Let $H \le G$ be a subgroup, let W be a KH-module, and let $g_1, \ldots, g_m$ be a transversal for H in G.

(i) The KG-module $W^G := \mathrm{ind}_H^G(W) := W \otimes_{KH} KG = \{\sum_{i=1}^m w_i \otimes g_i \mid w_i \in W\}$ with action $(w \otimes g)g' = w \otimes (gg')$ is called the *induced module* of W.

(ii) If Δ is the representation of H on W then the representation of G on W^G is denoted by Δ^G or and is called the *induced representation.*

(iii) If χ is the character of Δ then the character of Δ^G is denoted by χ^G and is called the *induced character.*

4.1.5 Theorem *Let $H \le G$ be a subgroup with transversal $g_1, \ldots, g_m$ and let $(w_1, \ldots, w_n)$ be a basis of the KH-module W with representation Δ and character χ. Then $B = (w_1 \otimes g_1, \ldots, w_n \otimes g_1, w_1 \otimes g_2, \ldots w_n \otimes g_m)$ is a basis of W^G.*

(i) The representation Δ^G with respect to B is given by

$$\Delta^G(g) = \begin{pmatrix} \dot\Delta(g_1 g g_1^{-1}) & \cdots & \dot\Delta(g_1 g g_m^{-1}) \\ \vdots & & \vdots \\ \dot\Delta(g_m g g_1^{-1}) & \cdots & \dot\Delta(g_m g g_m^{-1}) \end{pmatrix}$$

where $\dot\Delta(h) = \Delta(h)$ if $h \in H$ and 0 otherwise.

(ii) The character χ^G of Δ^G is given by $\chi^G(g) = \sum_{i=1}^m \dot\chi(g_i g g_i^{-1})$ where $\dot\chi(h) = \chi(h)$ if $h \in H$ and 0 otherwise.

(iii) If $\mathrm{char}(K) \nmid |H|$ then $\chi^G(g) = \frac{1}{|H|} \sum_{g' \in G} \dot\chi(g' g g'^{-1})$.

Proof: (i)+(ii): Multiplication by $g \in G$ induces a permutation of the cosets Hg_i, thus $g_i g = hg_j$ for some $h \in H$. Thus, $(w_k \otimes g_i)g = w_k \otimes (hg_j) = (w_k h) \otimes g_j$ where g_j is the unique element in the transversal such that $g_i g g_j^{-1} \in H$.
(iii): We have $\sum_{g' \in G} \dot\chi(g' g g'^{-1}) = \sum_{i=1}^m \sum_{h\in H} \dot\chi((hg_i)g(hg_i)^{-1})$
$= \sum_{i=1}^m \sum_{h\in H} \dot\chi(g_i g g_i^{-1}) = |H| \sum_{i=1}^m \dot\chi(g_i g g_i^{-1}) = |H|\chi^G(g)$. □

4.1.6 Remark Writing $\chi^G(g)$ as $\chi^G(g) = \frac{1}{|H|}\sum_{g'\in G}\dot\chi(g'gg'^{-1})$ again shows that the definition of the induced module is independent of the choice of the transversal, since the right hand side does not depend on the transversal and the character determines the representation.

4.1.7 Corollary *Let $H \leq G$ be a subgroup.*

(i) If Δ is the trivial representation of H then Δ^G is the permutation representation of G on G/H and the permutation character $1_H^G(g)$ gives the number of fixed points of g on G/H.
More generally, representations induced from 1-dimensional representations of a subgroup are called monomial *representations.*

(ii) If Δ is the regular representation of H, then Δ^G is the regular representation of G.

PROOF: (i): Note that $\Delta^G(g)_{ij} = 1$ if $g_i g g_j^{-1} \in H$ and 0 else. Thus, $\Delta^G(g)_{ij} = 1 \Leftrightarrow Hg_ig = Hg_j$ and therefore Δ^G is the permutation character on the cosets in G/H.
(ii): This follows, since for a transversal $g_1, \ldots, g_m$ of H in G the basis $\{h\otimes g_i \mid h \in H, 1 \leq i \leq m\}$ is the same as $\{1\otimes g \mid g \in G\}$. □

4.1.8 Theorem (Frobenius reciprocity)
Let $H \leq G$ be a subgroup, let χ and φ be a characters of G and H, respectively, over K with $\mathrm{char}(K) \nmid |G|$. *Then*

$$(\chi, \varphi^G)_G = (\chi_{|H}, \varphi)_H.$$

PROOF: We have $(\varphi^G, \chi) = \frac{1}{|G|}\sum_{i=1}^m \sum_{g\in G}\chi(g^{-1})\dot\varphi(g_igg_i^{-1})$
$= \frac{1}{|G|}\sum_{i=1}^m \sum_{g\in g_i^{-1}Hg_i}\chi(g^{-1})\dot\varphi(g_igg_i^{-1}) = \frac{1}{|G|}\sum_{i=1}^m\sum_{h\in H}\chi(g_i^{-1}h^{-1}g_i)\varphi(h)$
$= \frac{[G:H]}{|G|}\sum_{h\in H}\chi(h^{-1})\varphi(h) = (\varphi, \chi_{|H})_H.$ □

4.1.9 Example Let $G = S_4$ and $H = S_3 \leq S_4$. Assume first that we know the character tables of G and H:

$C_G(g_i)$	24	8	3	4	4
$\lvert C_i\rvert$	1	3	8	6	6
g_i	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,2)$	$(1,2,3,4)$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

$\lvert C_G(g_i)\rvert$	6	3	2
$\lvert C_i\rvert$	1	2	3
g_i	1	$(1,2,3)$	$(1,2)$
φ_1	1	1	1
φ_2	1	1	-1
φ_3	2	-1	0

We get the restrictions of the characters of S_4 to S_3 as $\chi_{1|H} = \varphi_1$, $\chi_{2|H} = \varphi_2$, $\chi_{3|H} = \varphi_3$, $\chi_{4|H} = \varphi_1 + \varphi_3$, $\chi_{5|H} = \varphi_2 + \varphi_3$. We can write these restrictions into an *induction-restriction table* as follows: The rows are indexed by the irreducible characters of G, the columns by the irreducible characters of H and the (i,j)-entry is a_j if $\chi_{i|H} = \sum_{j=1}^{r} a_j\varphi_j$. In the example, the table is

	φ_1	φ_2	φ_3
χ_1	1	0	0
χ_2	0	1	0
χ_3	0	0	1
χ_4	1	0	1
χ_5	0	1	1

The rows of the induction-restriction table express the restrictions from G to H as linear combinations of the irreducible characters of H and by Frobenius reciprocity the columns express the characters induced from H to G as linear combinations of the irreducible characters of G. Thus we have $\varphi_1^G = \chi_1 + \chi_4$, $\varphi_2^G = \chi_2 + \chi_5$ and $\varphi_3^G = \chi_3 + \chi_4 + \chi_5$.
The induction-restriction table shows that we can construct the irreducible characters of S_4 just from the characters of S_3 and the linear characters χ_1, χ_2 of S_4: We have $\chi_4 = \varphi_1^G - \chi_1$, $\chi_5 = \varphi_2^G - \chi_2$ and finally $\chi_3 = \varphi_3^G - \chi_4 - \chi_5$.

4.1.10 Example Let G be a non-abelian group of order pq where $p > q$ are different primes. Then the Sylow-p subgroup P of G is normal and every element of order q acts as an automorphism of C_p, thus $p - 1 = kq$. Since $Aut(C_p) \cong C_{p-1}$ is cyclic, there is a unique subgroup of order q of $Aut(C_p)$, hence the isomorphism type of G is determined uniquely.

The commutator group of G is P, therefore G has q linear characters $\chi_1, \dots, \chi_q$, namely the characters of C_q, having P in the kernel. Since the character degrees divide the group order, the nonlinear characters of G have degrees p or q. But from $pq = q \cdot 1^2 + l \cdot p^2 + m \cdot q^2$ we conclude that $l = 0$, hence there are m characters of degree q where $m = \frac{p-1}{q} = k$.

To determine the conjugacy classes of G, let $G = \langle a, b\rangle$ where $\langle a\rangle = P$ and b generates a Sylow-q-subgroup. Then the powers of a fall into k conjugacy classes under the action of b, since every orbit has length q. Moreover, if $bab^{-1} = a^s$ we see that $a^{-1}ba = a^{s-1}b$, hence the powers of b are representatives of different conjugacy classes. We thus have k classes of elements of order p and $q - 1$ classes of elements of order q and of course the class consisting of the identity element.

We will now determine the k characters of degree q by inducing linear characters of P to G. The non-trivial linear characters of P are given by $\lambda_i(a) = \zeta_p^i$, where ζ_p is a primitive p-th root of unity and $1 \le i < p$. Since $1, b, b^2, \dots, b^{q-1}$ is a transversal of P in G, we have $\lambda_i^G(b^l) = \sum_{j=0}^{q-1} \dot\lambda_i(b^j b^l b^{-j}) = \sum_{j=0}^{q-1} \dot\lambda_i(b^l) = 0$, since $b^l \notin P$. This could also have been concluded from the fact that the elements of order q have no fixed points on G/P. Since $P \trianglelefteq G$ we furthermore have $\lambda_i^G(a) = \sum_{j=0}^{q-1} \lambda_i(b^j a b^{-j}) = \sum_{j=0}^{q-1} \lambda_i(a^{js}) = \sum_{j=0}^{q-1} (\zeta_p^i)^{js} = \sum_{j=0}^{q-1} \zeta_p^{l_j}$ where $a^{l_0}, \dots, a^{l_{q-1}}$ is a conjugacy class of elements of order p. This shows that the induced characters λ_i^G and $\lambda_{i'}^G$ are different if and only if a^i and $a^{i'}$ represent different conjugacy classes. Thus we obtain k different induced characters $\lambda_{i_j}^G$. To see that these characters are irreducible, we only have to check that they do not have a linear character as a constituent, since the irreducible characters have degrees 1 or q. But by Frobenius reciprocity we have $(\lambda_i^G, \chi_j)_G = (\lambda_i, \chi_{j|P})_P = 0$, since λ_i is a non-trivial linear character and $\chi_{j|H}$ is the trivial character. We therefore conclude that the k induced characters are the distinct irreducible characters of degree q of G.

As an explicit example, let $p = 7$, $q = 3$, then $G = \langle a, b\rangle \cong C_7 \rtimes C_3$. We have $bab^{-1} = a^2$ and the orbits of b on the powers of a are $(a, a^2, a^4), (a^3, a^5, a^6)$. We get the character table of G as

$C_G(g_i)$	21	7	7	3	3
$\lvert C_i\rvert$	1	3	3	7	7
g_i	1	a	a^3	b	b^2
χ_1	1	1	1	1	1
χ_2	1	1	1	ζ_3	ζ_3^2
χ_3	1	1	1	ζ_3^2	ζ_3
λ_1^G	3	$\zeta_7+\zeta_7^2+\zeta_7^4$	$\zeta_7^3+\zeta_7^5+\zeta_7^6$	0	0
λ_3^G	3	$\zeta_7^3+\zeta_7^5+\zeta_7^6$	$\zeta_7+\zeta_7^2+\zeta_7^4$	0	0

4.1.11 Remark *If χ and ψ are characters for representations on KG-modules V and W, then $(\chi, \psi)_G = \dim Hom_{KG}(V, W)$. Thus, Frobenius reciprocity implies $\dim Hom_{KG}(W^G, V) = \dim Hom_{KH}(W, V_{|H})$ and $\dim Hom_{KG}(V, W^G) = \dim Hom_{KH}(V_{|H}, W)$. But a much stronger result holds, namely the Frobenius-Nakayama reciprocity.*

4.1.12 Theorem (Frobenius-Nakayama reciprocity)
Let $H \leq G$ be a subgroup, let V be a KG-module and W a KH-module. Then $Hom_{KG}(W^G, V) \cong Hom_{KH}(W, V_{|H})$ and $Hom_{KG}(V, W^G) \cong Hom_{KH}(V_{|H}, W)$ as K-modules.

PROOF: Let $\varphi \in Hom_{KG}(W^G, V)$, i.e. $\varphi : W \otimes_{KH} KG \to V$. We now define $\varphi' : W \to V$ by $w\varphi' := (w \otimes 1)\varphi$. It is clear that φ' is a KH-module homomorphism, since by the definition of $W \otimes_{KH} KG$ we have $(w\varphi')h = (w \otimes 1)\varphi h = (w \otimes h)\varphi = (wh \otimes 1)\varphi$. We therefore have $\varphi' \in Hom_{KH}(W, V_{|H})$. Moreover, the mapping $\Gamma : \varphi \to \varphi'$ is clearly K-linear and it is injective, since $\varphi' = 0$ implies $(w \otimes 1)\varphi = 0$ for all $w \in W$ and thus $\varphi = 0$.
To show that Γ is surjective, let $\varphi' \in Hom_{KH}(W, V_{|H})$ and let $T = \{g_1, \ldots, g_m\}$ be a transversal for H in G. We now define $(w \otimes hg_i)\varphi := (w\varphi')hg_i = (wh)\varphi' g_i$, then φ is a well-defined K-homomorphism from W^G to V. Moreover, for $g \in G$ and $g_i \in T$ let $g_i g = h' g_j$, then we have $(w \otimes hg_i)\varphi g = (wh)\varphi' g_i g = (wh)\varphi' h' g_j = (wh)h'\varphi' g_j = (w \otimes hh' g_j)\varphi = (w \otimes hg_i)g\varphi$. This shows that φ is G-invariant and thus $\varphi \in Hom_{KG}(W^G, V)$. Since $\Gamma(\varphi) = \varphi'$ this establishes that Γ is bijective.

The second isomorphism is left as an exercise. □

4.2 Permutation characters

4.2.1 Lemma *Let G act transitively on Ω, let $\alpha \in \Omega$ and let $H := G_\alpha := Stab_G(\alpha)$. Then the permutation character π of the action of G on Ω is given as $\pi = (1_H)^G$.*

Proof: Let $T = \{g_1, \dots, g_n\}$ be a transversal of H in G, then $n = |\Omega|$ and $\Omega = \{\alpha g_i \mid 1 \le i \le n\}$. For $g \in G$ we have $\pi(g) = |\{\omega \in \Omega \mid \omega g = g\}| = |\{g_i \in G \mid \alpha g_i g = \alpha g_i\}| = |\{g_i \in G \mid g_i g g_i^{-1} \in H\}| = (1_H)^G(g)$ by the definition of the induced character. □

4.2.2 Proposition *Let G act on Ω and let π be the permutation character of this action. Then $(\pi, 1_G)_G = \frac{1}{|G|}\sum_{g\in G} |fix_\Omega(g)|$ equals the number of orbits of G on Ω.*

Proof: By the definition of the inner product of class functions we have $(\pi, 1_G)_G = \frac{1}{|G|}\sum_{g\in G} \pi(g)$ and since π is the permutation character on Ω we have $\pi(g) = |fix_\Omega(g)|$. Now let $\Omega = \dot\bigcup_{i=1}^r \Omega_i$ be the decomposition of Ω into orbits under the action of G and denote by π_i the permutation character of the action of G on Ω_i, then $\pi = \sum_{i=1}^r \pi_i$. If we select a point $\alpha_i \in \Omega_i$ from each orbit and define $H_i := Stab_G(\alpha_i)$, we have $\pi_i = (1_{H_i})^G$ and by Frobenius reciprocity we get $(\pi_i, 1_G)_G = (1_{H_i}, 1_{H_i})_{H_i} = 1$. From this we conclude that $(\pi, 1_G)_G = \sum_{i=1}^r (\pi_1, 1_G)_G = r$. □

Note: This is the famous Cauchy-Frobenius fixed point theorem also known as Burnside fixed point theorem.

4.2.3 Proposition *Let G act on Ω_1 and Ω_2 with permutation characters π_1 and π_2, respectively. Then $(\pi_1, \pi_2)_G$ is the number of orbits of G on $\Omega_1 \times \Omega_2$. (Here, the action of G on $\Omega_1 \times \Omega_2$ is given by $(\alpha, \beta)g := (\alpha g, \beta g)$.*

Proof: Since the number of fixed points of g and g^{-1} coincide we have $(\pi_1, \pi_2) = \frac{1}{|G|}\sum_{g\in G} |fix_{\Omega_1}(g)||fix_{\Omega_2}(g)| = \frac{1}{|G|}\sum_{g\in G} |fix_{\Omega_1\times\Omega_2}(g)|$ and this is the number of orbits of G on $\Omega_1 \times \Omega_2$. □

4.2.4 Corollary *Let G act transitively on Ω with permutation character π, let $\alpha \in \Omega$ and let $H := Stab_G(\alpha)$. Assume that H has r orbits on Ω, then $(\pi, \pi)_G = r$, i.e. the number of orbits of H on Ω equals the number of orbits of G on $\Omega \times \Omega$. The number $r = (\pi, \pi)_G$ is called the* rank *of the transitive action of G.*

Proof: By Frobenius reciprocity we have $r = (\pi_{|H}, 1_H)_H = (\pi, (1_H)^G)_G = (\pi, \pi)_G$. □

4.2.5 Definition A transitive action of a group G on a set Ω is called *doubly transitive* if the stabilizer of a point $\alpha \in \Omega$ acts transitively on $\Omega \setminus \{\alpha\}$. Analogously, the action is called *k-transitive*, if the pointwise stabilizer of $(k-1)$ points acts transitively on the remaining $|\Omega| - (k-1)$ points. For example, the natural actions of S_n and A_n are n-transitive and $(n-2)$-transitive, respectively.

4.2.6 Corollary *Let G act transitively on Ω with permutation character π, The action is doubly transitive if and only if $\pi = 1_G + \chi$ for an irreducible character χ of G.*

Proof: The action of G is doubly transitive if and only if $(\pi, \pi) = 2$ and since $(\pi, 1_G) = 1$ this equivalent with $\pi = 1_G + \chi$ for an irreducible character χ of G. □

4.2.7 Example Let $G = GL_2(q)$ be the group of invertible 2×2-matrices over the field $\mathbb{F}_q$ of q elements. Then G acts doubly transitive on the set Ω of 1-dimensional subspaces of $\mathbb{F}_q^2$:
The 1-dimensional subspaces of $\mathbb{F}_q^2$ are represented by the vectors $v_a = (1, a)$ with $a \in \mathbb{F}_q$ and $v_\infty = (0, 1)$, hence $|\Omega| = q + 1$. The action on Ω is transitive, since any of the vectors v_x can be chosen as the first row of an element of G, hence all $\alpha \in \Omega$ lie in the orbit of $\alpha_0 = \langle (1, 0) \rangle$. Now let $H := Stab_G(\alpha_0)$, then H consists of the matrices of the form $\begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix}$ with $c \neq 0$. In particular, we find any v_x except for v_0 as the second row of an element in H, hence all the elements of $\Omega \setminus \alpha_0$ lie in the orbit of $\alpha_\infty = \langle (0, 1) \rangle$.
We can therefore conclude that the permutation character π of the action of G on Ω is of the form $\pi = 1_G + \chi$ where χ is a q-dimensional irreducible rational character of G.

4.2.8 Theorem *Let $H \leq G$ be a subgroup and let $\pi = (1_H)^G$ be the permutation character of the action of G on G/H. Then the following hold:*

(i) $\pi(1) \mid |G|$,

(ii) $\pi(g) \in \mathbb{Z}_{\geq 0}$,

(iii) $\pi(g^n) \geq \pi(g)$ *for all* $n \in \mathbb{N}$,

(iv) $(\pi, 1_G)_G = 1$,

(v) $(\pi, \chi)_G \leq \chi(1)$ *for every character* χ *of* G,

(vi) $|\langle g \rangle| \nmid \dfrac{|G|}{\pi(1)} \Rightarrow \pi(g) = 0$,

(vii) $\dfrac{\pi(g)}{\pi(1)} |g^G| \in \mathbb{Z}$, *where* g^G *denotes the conjugacy class of* g *in* G.

Proof: Claim (i) holds for any character, (ii) follows from the interpretation of $\pi(g)$ as the number of fixed points of g, (iii) follows, since every fixed point of g is also a fixed point of g^n and (iv) holds, since the action of G on G/H is transitive.
(v): Write $\pi = (1_H)^G$, then by Frobenius reciprocity we have $(\pi, \chi)_G = (1_H, \chi_{|H})_H$ and the multiplicity of any constituent of $\chi_{|H}$ can not exceed the degree $\chi(1)$.
(vi): Let $m = \langle |g| \rangle$ be the order of g, then every conjugate xgx^{-1} of g also has order m and therefore can not be contained in H. This shows that $\pi(g) = \frac{1}{|H|} \sum_{x \in G} (\dot{1_H})(xgx^{-1}) = 0$.
(vii): Let $\Omega := G/H$ and define X to be the set $X := \{(\omega, x) \mid \omega \in \Omega, x \in g^G, \omega x = \omega\}$. We count the number of elements in X in two different manners:

On the one hand, for a fixed $g \in g^G$ the number of pairs $(\omega, g) \in X$ is $\pi(g)$, hence $|X| = |g^G|\pi(g)$. On the other hand, for a fixed element $\omega \in \Omega$ the number of $(\omega, x) \in X$ is $|Stab_G(\omega) \cap g^G|$. But since G acts transitively on Ω, all stabilizers $Stab_G(\omega)$ are conjugate and since g^G is closed under conjugation, the cardinality $c = |Stab_G(\omega) \cap g^G|$ is independent of ω. Hence, $|X| = |\Omega|c = \pi(1)c$ and we conclude that $\frac{\pi(g)}{\pi(1)}|g^G| = c \in \mathbb{Z}$. □

4.2.9 Remark Theorem 4.2.8 gives a number of necessary conditions which a permutation character has to fulfill. However, they are by now means sufficient.

4.2.10 Example We determine the candidates of transitive permutation characters for the symmetric group S_4. The character table of S_4 looks as follows:

$\lvert\langle g_i\rangle\rvert$	1	2	2	3	4
$\lvert g_i^G\rvert$	1	3	6	8	6
g_i	1	$(1,2)(3,4)$	$(1,2)$	$(1,2,3)$	$(1,2,3,4)$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	2	0	-1	0
χ_4	3	-1	1	0	-1
χ_5	3	-1	-1	0	1

Now let $\pi = \sum_{i=1}^{5} a_i\chi_i$, then from condition (v) of Theorem 4.2.8 we know that $a_i \leq \chi_i(1)$ and $a_1 = 1$ due to condition (iv). Furthermore, the fact that $\pi(g) \geq 0$ gives us a linear inequality for each of the conjugacy classes, i.e. we get the system of linear inequalities

$$\begin{array}{ll} 0 \leq a_2 \leq 1, & 1 + a_2 + 2a_3 - a_4 - a_5 \geq 0 \\ 0 \leq a_3 \leq 2, & 1 - a_2 + a_4 - a_5 \geq 0 \\ 0 \leq a_4 \leq 3, & 1 + a_2 - a_3 \geq 0 \\ 0 \leq a_5 \leq 3, & 1 - a_2 - a_4 + a_5 \geq 0 \end{array}.$$

If we further restrict the integral solutions of this system to solutions with $\pi(1) \mid 24$ we get the following 14 candidates for permutation characters:

$$\begin{array}{lcl} \pi_1 = \chi_1 & = & (1,1,1,1,1) \\ \pi_2 = \chi_1 + \chi_2 & = & (2,2,0,2,0) \\ \pi_3 = \chi_1 + \chi_3 & = & (3,3,1,0,1) \\ \pi_4 = \chi_1 + \chi_4 & = & (4,0,2,1,0) \\ \pi_5 = \chi_1 + \chi_5 & = & (4,0,0,1,2) \\ \pi_6 = \chi_1 + \chi_2 + \chi_3 & = & (4,4,0,1,0) \\ \pi_7 = \chi_1 + \chi_3 + \chi_4 & = & (6,2,2,0,0) \\ \pi_8 = \chi_1 + \chi_3 + \chi_5 & = & (6,2,0,0,2) \\ \pi_9 = \chi_1 + \chi_2 + 2\chi_3 & = & (6,6,0,0,0) \\ \pi_{10} = \chi_1 + \chi_2 + \chi_4 + \chi_5 & = & (8,0,0,2,0) \\ \pi_{11} = \chi_1 + \chi_2 + 2\chi_3 + \chi_4 + \chi_5 & = & (12,4,0,0,0) \\ \pi_{12} = \chi_1 + \chi_3 + 2\chi_4 + \chi_5 & = & (12,0,2,0,0) \\ \pi_{13} = \chi_1 + \chi_3 + \chi_4 + 2\chi_5 & = & (12,0,0,0,2) \\ \pi_{14} = \chi_1 + \chi_2 + 2\chi_3 + 3\chi_4 + 3\chi_5 & = & (24,0,0,0,0) \end{array}.$$

So far, we have not used conditions (iii), (vi) and (vii) of Theorem 4.2.8. Since $(1,2,3,4)^2 = (1,3)(2,4)$ we require that the second component of π_i is not smaller than the last component. This rules out the candidates π_5 and π_{13}. The remaining characters now also fulfill the other conditions of the theorem and we therefore proceed to identifying the true permutation characters. We know that conjugate subgroups yield the same permutation character, hence it is sufficient to look at the actions of G on G/H where H runs over representatives of the conjugacy classes of subgroups of G:
π_1 is the action of S_4 on S_4/S_4,
π_2 is the action of S_4 on S_4/A_4,
π_3 is the action of S_4 on S_4/D_8,
π_4 is the action of S_4 on S_4/S_3,
π_7 is the action of S_4 on $S_4/\langle(1,2),(3,4)\rangle$,
π_8 is the action of S_4 on S_4/C_4,
π_9 is the action of S_4 on $S_4/\langle(1,2)(3,4),(1,3)(2,4)\rangle$,
π_{10} is the action of S_4 on S_4/C_3,
π_{11} is the action of S_4 on $S_4/\langle(1,2)(3,4)\rangle$,
π_{12} is the action of S_4 on $S_4/\langle(1,2)\rangle$,
π_{14} is the action of S_4 on $S_4/\{1\}$, i.e. the regular character.
The only character not found is π_6. To rule out this character we need some additional argument: Assume that $\pi_6 = (1_H)^G$, then $(\pi_6,\chi_2) = (1_H,\chi_{2|H})$, and since $\chi_2(1) = 1$ we have $\chi_{2|H} = 1_H$. This means that the signum-character restricted to H gives the trivial character, hence $H \leq A_4$. But A_4 has no subgroup of order 6, hence π_6 is not a permutation character.

4.2.11 Remark Note that we can read off the maximal subgroups from the transitive permutation characters of G: If $H \leq U \leq G$ we have $(1_H)^G = ((1_H)^U)^G$, where $(1_H)^U = 1_U + \psi$ for some character ψ of U, since U acts transitively on U/H. Therefore, $(1_H)^G = (1_U)^G + \psi^G$, which shows that the permutation character on G/U is completely contained in the permutation character on G/H. We therefore can identify the permutation characters corresponding to maximal subgroups as the permutation characters not containing any other permutation character.

We will finish this section by constructing certain irreducible characters of S_n. Note that the conjugacy classes of S_n are characterized by the cycle structures of the elements, hence there is a $1-1$ correspondence between conjugacy classes of S_n and partitions of n. (Recall that a partition $(n_1,\ldots,n_s) \vdash n$ of n is a sequence $(n_1,\ldots,n_s)$ with $n_i \geq 0$, $n_i \geq n_{i+1}$ and $\sum_{i=1}^{s} n_i = n$.) Since we know that there are as many irreducible representations as conjugacy classes it would be most convenient if we could associate an irreducible representation of S_n to each partition of n. This is actually possible by some clever combinatorial constructions and we will demonstrate here the case of 2-partitions, i.e. partitions of the form $(n-k,k)$.

Let $I_k := \{I \subseteq \{1,\ldots,n\} \mid |I| = k\}$ be the set of k-element subsets of $\{1,\ldots,n\}$. Then S_n acts transitively on I_k via $\{i_1,\ldots,i_k\}g := \{i_1g,\ldots,i_kg\}$. The permutation character π_k of this action of S_n on I_k has degree $\pi_k(1) = \binom{n}{k}$.

4.2.12 Proposition *With the above notation let π_l and π_k be the permutation characters of the action of S_n on I_l and I_k, respectively. Assume that $1 \leq l \leq k \leq \frac{n}{2}$. Then $(\pi_k, \pi_l)_{S_n} = l + 1$.*

PROOF: We know that (π_k, π_l) is the number of orbits of S_n on $I_k \times I_l$. We now claim that the orbits of S_n on $I_k \times I_l$ are $J_0, J_1, \dots, J_l$, where $J_s = \{(A, B) \mid A \in I_k, B \in I_k, |A \cap B| = s\}$. It is clear that pairs from different J_s can not lie in one orbit, since the size of the intersection is invariant under the action of G. Moreover, none of the J_s is empty, since $l \leq k \leq \frac{n}{2}$. Now let $(A, B) \in J_s$, let $A \setminus B = \{i_1, \dots, i_{k-s}\}$, $A \cap B = \{i_{k-s+1}, \dots, i_k\}$ and $B \setminus A = \{i_{k+1}, \dots, i_{k+l-s}\}$. Then we can map (A, B) to (A_0, B_0) with $A_0 = \{1, \dots, k\}$, $B_0 = \{k - s, \dots, k + l - s\}$ by $g \in S_n$ mapping i_j to j for $1 \leq j \leq k + l - s$. Here we require the property of S_n that we can choose the images of all points independently. This now shows that J_s is a single orbit under the action of G, hence the claim follows. □

4.2.13 Theorem *Let π_k be the permutation character of the action of S_n on I_k and assume that $k \leq \frac{n}{2}$. Then $\pi_k = \chi^{(n)} + \chi^{(n-1,1)} + \dots + \chi^{(n-k,k)}$, where $\chi^{(n)} = 1_{S_n}$ and the $\chi^{(n-i,i)}$ are distinct irreducible characters of S_n. In particular, one has $\chi^{(n-k,k)} = \pi_k - \pi_{k-1}$ and thus $\chi^{(n-k,k)}(1) = \binom{n}{k} - \binom{n}{k-1}$.*

PROOF: The proof is by induction on k: For $k = 1$ we already know that $\pi_1 = 1_{S_n} + \chi$ with χ irreducible, since S_n acts doubly transitive. We set $\chi^{(n-1,1)} := \chi$. Now let $k > 1$. From the above proposition we know that $(\pi_k, \pi_{k-1}) = k$ and $(\pi_k, \pi_k) = k+1$. By induction, π_{k-1} is a sum of k irreducible characters, hence it is completely contained in π_k, i.e. $\pi_k = \pi_{k-1} + \chi$ for a character χ of S_n. From $(\pi_k, \pi_k) = k+1$ we conclude that $(\pi_{k-1}, \chi) + (\chi, \chi) = 1$, which shows that χ is an irreducible character distinct from the constituents of π_{k-1}. The claim now follows by defining $\chi^{(n-k,k)} := \chi$. □

4.3 Normal subgroups

4.3.1 Definition Let $H \trianglelefteq G$ be a normal subgroup, let W be KH-module with representation Δ and character φ.

(i) The module $W^g := W \otimes g \leq W^G$ is a KH-module which is conjugate to W. The corresponding representation Δ^g of H is given by $\Delta^g(h) = \Delta(ghg^{-1})$, the corresponding character φ^g by $\varphi^g(h) = \varphi(ghg^{-1})$, since $(w \otimes g)h = w \otimes (ghg^{-1})g = w(ghg^{-1}) \otimes g$.

(ii) The group $T := I_G(\varphi) := \{g \in G \mid \varphi^g(h) = \varphi(h) \text{ for all } h \in H\}$ is called the *inertia group* of φ in G. This is the group of elements $g \in G$ such that $W^g \cong_{KH} W$.

4.3.2 Theorem (Clifford's theorem)
Let $H \trianglelefteq G$ be a normal subgroup, let χ be an irreducible character of G and let φ be an irreducible constituent of $\chi_{|H}$. Let $T := I_G(\varphi)$ be the inertia group of φ in G. Then $\chi_{|H} = e(\sum_{i=1}^{m} \varphi_i)$ where $m = [G : T]$ and $\varphi_i = \varphi^{g_i}$ for a

transversal $g_1, \ldots, g_m$ of T in G (with $g_1 = 1$). Thus, the φ_i are the different conjugates of φ under the action of G and $e = (\chi_{|H}, \varphi)$.

PROOF: We first show that $(\chi_{|H}, \varphi_i)$ is independent of i. Note that since χ is a character of G, we have $\chi_{|H}^g = \chi_{|H}$ for all $g \in G$. But $(\theta^g, \varphi^g) = (\theta, \varphi)$ for any characters θ, φ of H, since with h also ghg^{-1} runs over H. This shows that $(\chi_{|H}, \varphi^g) = (\chi_{|H}^g, \varphi^g) = (\chi_{|H}, \varphi)$ for all $g \in G$ and hence $(\chi_{|H}, \varphi_i) = (\chi_{|H}, \varphi) = e$ for all i.
Next, for the induced character φ^G we have $\varphi^G(h) = \frac{1}{|H|}\sum_{g\in G} \dot{\varphi}(ghg^{-1}) = \frac{1}{|H|}\sum_{g\in G} \varphi^g(h)$ for $h \in H$. This shows that the φ_i are all the irreducible constituents of $\varphi^G_{|H}$. Now let θ be an irreducible character of H distinct from the φ_i, then we have $(\varphi^G_{|H}, \theta) = 0$ and by Frobenius reciprocity this means that $(\varphi^G, \theta^G) = 0$. On the other hand we have $(\chi_{|H}, \varphi) \neq 0$, hence $(\chi, \varphi^G) \neq 0$, and since χ is irreducible this shows that $(\chi, \theta^G) = 0$. From this we conclude that $(\chi_{|H}, \theta) = 0$, hence the φ_i are all the irreducible constituents of $\chi_{|H}$. □

4.3.3 Theorem *Let $H \trianglelefteq G$, let φ be an irreducible character of H and let $T := I_G(\varphi)$. Define $\mathcal{A} := \{\psi$ irreducible character of $T \mid (\psi_{|H}, \varphi) \neq 0\}$ and $\mathcal{B} := \{\chi$ irreducible character of $G \mid (\chi_{|H}, \varphi) \neq 0\}$. Then the following hold:*

(i) If $\psi \in \mathcal{A}$, then ψ^G is irreducible.

(ii) If $\psi^G = \chi$ with $\psi \in \mathcal{A}$, then $(\psi_{|H}, \varphi) = (\chi_{|H}, \varphi)$.

(iii) If $\psi^G = \chi$ with $\psi \in \mathcal{A}$, then ψ is the unique irreducible constituent of $\chi_{|T}$ that lies in $\mathcal{A}$.

(iv) The mapping $\psi \to \psi^G$ is a bijection of $\mathcal{A}$ onto $\mathcal{B}$.

PROOF: Let $\varphi_1 = \varphi, \varphi_2, \ldots, \varphi_m$ be the distinct conjugates of φ in G, thus $[G : T] = m$. Let $\psi \in \mathcal{A}$ and let χ be an irreducible constituent of ψ^G. By Frobenius reciprocity we know that ψ is a constituent of $\chi_{|T}$ and φ is a constituent of $\psi_{|H}$, hence φ is also a constituent of $\chi_{|H}$ and thus $\chi \in \mathcal{B}$. Furthermore, by Clifford's theorem we have $\chi_{|H} = e(\sum_{i=1}^m \varphi_i)$ and $\psi_{|H} = f\varphi$, since $T = I_G(\varphi) = I_T(\varphi)$. As ψ is a constituent of $\chi_{|T}$ we know that $f \leq e$.
(i): We have $em\varphi(1) = \chi(1) \leq \psi^G(1) = m\psi(1) = fm\varphi(1) \leq em\varphi(1)$, hence in particular we have $\chi(1) = \psi^G(1)$ and hence $\psi^G = \chi$ is irreducible.
(ii): From the above equation it also follows that $e = f$, hence $(\chi_{|H}, \varphi) = e = f = (\psi_{|H}, \varphi)$.
(iii): If $\psi, \psi_1 \in \mathcal{A}$ are distinct constituents of $\chi_{|T}$, we have $(\chi_{|H}, \varphi) \geq ((\psi + \psi_1)_{|H}, \varphi) = (\psi_{|H}, \varphi) + ((\psi_1)_{|H}, \varphi) > (\psi_{|H}, \varphi)$ which contradicts (ii).
(iv): The mapping $\psi \to \psi^G$ is well-defined by (i), its image lies in $\mathcal{B}$ by (ii) and it is injective by (iii). To show that it is also surjective, let $\chi \in \mathcal{B}$. Then φ is a constituent of $\chi_{|H} = (\chi_{|T})_{|H}$, hence there exists an irreducible constituent ψ of $\chi_{|T}$ with $(\psi_{|H}, \varphi) \neq 0$. We then have $\psi \in \mathcal{A}$ and by Frobenius reciprocity χ is a constituent of ψ^G. But by (i), ψ^G is irreducible, hence $\chi = \psi^G$ as required. □

4.3.4 Corollary *Let $H \trianglelefteq G$ be a normal subgroup, φ an irreducible character of H with inertia group $I_G(\varphi) = H$. Then φ^G is an irreducible character of G.*

4.3.5 Example Let G be a non-abelian group of order pq where $p > q$ are different primes. Then the Sylow-p subgroup of G is normal and the Sylow-q subgroup acts as automorphisms on it, hence G is isomorphic to the semidirect product $C_p \rtimes C_q$. If we take $C_p = \langle a \rangle$ and let $\lambda(a^i) = \zeta_p^i$ be a non-trivial irreducible character of C_p, then the inertia group $I_G(\lambda) = C_p$, since $\lambda^{b^j}(a) = \lambda(a^j) = \zeta_p^j$ for a generator b of the Sylow-q subgroup. This shows that λ^G is an irreducible character of degree q of G. The other irreducible characters of degree q are obtained in the same manner by inducing different linear characters of C_p.

4.3.6 Proposition *Let K be a splitting field of G and let χ be an irreducible character of G. Then $\chi(1) \mid [G : Z(G)]$.*

PROOF: We use induction on $|G|$. If χ is not faithful, let $N := \ker(\chi)$, then χ is a faithful irreducible character of G/N. We have $Z(G/N) \geq (Z(G) \cdot N)/N$ and hence by induction $\chi(1) \mid [G/N : Z(G/N)] \mid [G/N : (Z(G) \cdot N)/N] = [G : (Z(G) \cdot N)] \mid [G : Z(G)]$. We therefore now assume that χ is faithful.
The elements $z \in Z(G)$ act on the set of conjugacy classes by right multiplication, since $(xgx^{-1})z = x(gz)x^{-1}$. Now assume that g and gz are conjugate for $1 \neq z \in Z(G)$, then $\chi(g) = 0$, since $\chi(gz) = \chi(g)\zeta$ where ζ is a non-trivial root of unity such that $\chi(z) = \zeta \cdot \chi(1)$. Therefore, on the conjugacy classes with $\chi(g) \neq 0$ the orbits of $Z(G)$ have length $|Z(G)|$. Let $C_1, \dots, C_k$ be representatives of the orbits of $Z(G)$ on the conjugacy classes with $\chi(g_i) \neq 0$. Then we have: $|G| = \sum_{g \in G} \chi(g)\overline{\chi(g)} = \sum_{i=1}^k \sum_{z \in Z(G)} |C_i| \chi(g_i z)\overline{\chi(g_i z)} = \sum_{i=1}^k |C_i||Z(G)|\chi(g_i)\overline{\chi(g_i)}$. This implies that

$$\frac{|G|}{|Z(G)|\chi(1)} = \sum_{i=1}^{k} \frac{|C_i|\chi(g_i)}{\chi(1)} \overline{\chi(g_i)} \in \mathbb{Z},$$

since both $\frac{|C_i|\chi(g_i)}{\chi(1)} = \omega(C_i^+)$ and $\overline{\chi(g_i)}$ are algebraic integers. □

4.3.7 Theorem (Ito)
Let K be a splitting field of G, assume that $\operatorname{char}(K) \nmid |G|$ and let χ be an irreducible character of G. If $A \trianglelefteq G$ is an abelian normal subgroup, then $\chi(1) \mid [G : A]$.

PROOF: We use induction on $|G|$. Let λ be an irreducible constituent of $\chi_{|A}$ and let $T := I_G(\lambda)$ be the inertia group of λ in G.
If $T \neq G$ there is an irreducible character ψ of T such that $\chi = \psi^G$. By induction we have $\psi(1) \mid [T : A]$ and since $\chi(1) = [G : T]\psi(1)$ we have $\chi(1) \mid [G : T][T : A] = [G : A]$.
Now assume that $T = G$. Since A is abelian, $\lambda(1) = 1$ and hence restricting the representation Δ affording χ to A gives $\Delta(a) = \lambda(a)I_n$ with $n = \chi(1)$. In particular we have $\Delta(A) \leq Z(\Delta(G))$. Now let $N := \ker(\Delta)$, then Δ can be

regarded as a faithful representation of G/N and we have $(A\cdot N)/N \leq Z(G/N)$. By Proposition 4.3.6 we have $\chi(1) \mid [G/N : Z(G/N)] \mid [G/N : (A\cdot N)/N] = [G : (A\cdot N)] \mid [G : A]$. □

4.3.8 Theorem *Let K be algebraically closed, $H \trianglelefteq G$ with G/H cyclic. Let φ be an irreducible character of H which is G-invariant, i.e. $I_G(\varphi) = G$.*

(i) There exists an irreducible character χ of G with $\chi_{|H} = \varphi$.

(ii) If ψ is any irreducible character of G with $(\psi_{|H}, \varphi)_H > 0$, then $(\psi_{|H}, \varphi)_H = 1$ and $\psi = \lambda \cdot \chi$ where λ is an irreducible (and thus linear) character of G/H.

Proof: (i): Let Δ be the representation of H with character φ and let W be the corresponding KH-module. Let $g \in G$ such that Hg is a generator of G/H, then $G = H\langle g\rangle$. By assumption we have $\Delta^g \sim \Delta$, hence there exists $T \in GL(W)$ with $\Delta(ghg^{-1}) = T\Delta(h)T^{-1}$ for all $h \in H$. For $n = [G : H]$ we have $g^n \in H$ and therefore $T^n\Delta(h)T^{-n} = \Delta(g^nhg^{-n}) = \Delta(g^n)\Delta(h)\Delta(g^{-n})$ for all $h \in H$. Thus, $\Delta(g^{-n})T^n \in End_{KH}(W) = K \cdot id_W$ by Schur's lemma. We choose $c \in K$ such that $\Delta(g^{-n})T^n = c^n \cdot id_W$ and define $\Delta(g) := c^{-1}T$.
It now remains to check that this extends Δ to a representation of G, i.e. that $\Delta(hg^i) := \Delta(h)c^{-i}T^i$ defines a homomorphism $G \to GL(W)$: We have

$$\Delta(hg^i \cdot h'g^j) = \Delta((hg^ih'g^{-i})g^{i+j}) = \Delta(h)\Delta(g^ih'g^{-i})c^{-i-j}T^{i+j}$$

$$= \Delta(h)T^i\Delta(h')T^{-i}c^{-i-j}T^{i+j} = (\Delta(h)c^{-i}T^i)(\Delta(h')c^{-j}T^{-i}T^{i+j})$$

$$= \Delta(hg^i)\Delta(h'g^j).$$

(ii): Let $\lambda_1, \ldots, \lambda_n$ be the irreducible (linear) characters of $G/H \cong C_n$, then $\psi_i := \lambda_i \cdot \chi$ are irreducible characters of G. Moreover, $\psi_{i|H} = \chi_{|H}$, hence $(\psi_{i|H}, \varphi) > 0$. We obtain ψ_i as the character of the representation extending Δ by defining $\Delta(g) := c^{-1}\zeta_n^iT$. Since W is an irreducible KH-module, we have $End_{KH}(W) = K$ by Schur's lemma, hence two representations extending Δ can only be equivalent if they are equal and therefore the ψ_i are characters of non-equivalent representations. By Frobenius reciprocity we now see that $(\psi_i, \varphi^G) > 0$ for all i. But $\varphi^G(1) = n \cdot \chi(1)$, hence we have $\varphi^G = \sum_{i=1}^n \psi_i$. □

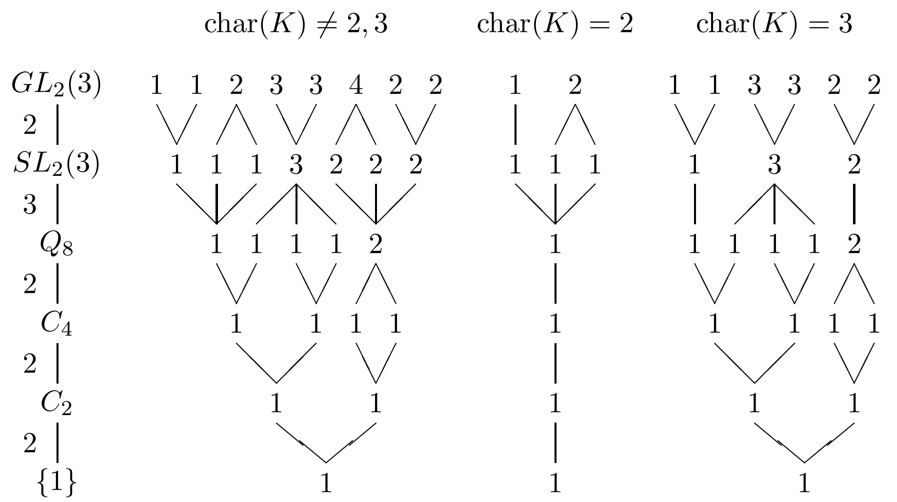
4.3.9 Corollary *If $H \trianglelefteq G$ such that $[G : H] = p$ is a prime number and let φ be an irreducible character of H. Then one of the following holds:*

(i) $I_G(\varphi) = H$, then φ^G is irreducible and H has p characters which are conjugate with φ. The character values for φ^G are $\varphi^G(g) = 0$ if $g \in G\backslash H$.

(ii) $I_G(\varphi) = G$, then φ can be extended to an irreducible character χ of G. If $\mathrm{char}(K) \neq p$ there are p such extensions, if $\mathrm{char}(K) = p$ there is one. Two extensions of φ differ by a linear character of $G/H \cong C_p$.

4.3.10 Remark For a soluble group G the above theorems allow us to explicitly construct all irreducible representations by climbing up a composition series: Let $G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = \{1\}$ with $G_{i-1}/G_i \cong C_{p_i}$ for primes p_i. First assume that $\mathrm{char}(K) \nmid |G|$. If we have already constructed the irreducible representations Δ_j of G_i, then the irreducible representation of G_{i-1} are obtained by either extending Δ_j in p_i ways or by inducing Δ_j to G_{i-1}, thereby joining p_i representations of G_i into one of G_{i-1}.
If $\mathrm{char}(K) = p_i$ in some step, the situation is even simpler: We either induce or we have a unique extension, since C_{p_i} has the trivial module as its only simple module in characteristic p_i.
Note that both ways of moving up in the composition series are constructive: Induction was already described in an earlier section and extending a representation requires solving the system of linear equations $\Delta(ghg^{-1})T = T\Delta(h)$ for T and finding an n-th root of the scalar a with $\Delta(g^{-n})T^n = a \cdot I_n$. The last step is the source of some complications in practice, since it requires to deal with algebraic extensions of growing degrees.

4.3.11 Example Let $G := GL_2(3)$ be the group of invertible 2×2-matrices over $\mathbb{F}_3$, then $|G| = 48$ and G has a composition series $G = GL_2(3) \trianglerighteq SL_2(3) \trianglerighteq Q_8 \trianglerighteq C_4 \trianglerighteq C_2 \trianglerighteq \{1\}$ with cyclic quotients of orders $2, 3, 2, 2, 2$. The following figure shows the character degrees of G over a splitting field of characteristic $\mathrm{char}(K) \neq 2, 3$, $\mathrm{char}(K) = 2$ and $\mathrm{char}(K) = 3$, respectively.



Note that the trivial character can always be extended and that a nonabelian group must have a nonlinear character. If the number of characters of a fixed degree on a certain level is not a multiple of p_i, then at least one of them can be extended.
$\mathrm{char}(K) \neq 2, 3$: It is clear that C_4 has 4 linear characters. Not all of them can extend to Q_8, hence a pair induces to a 2-dimensional character. On the next level, $SL_2(3)/C_2 \cong A_4$ is not abelian, hence three of the 1-dimensional characters induce to an irreducible character of $SL_2(3)$. The only character of degree 2 has to be extendible. Finally, $GL_2(3)/Q_8 \cong S_3$ is not abelian, hence not all 1-dimensional characters can be extendible. The only character of degree 3 and

one of the 2-dimensional characters have to be extendible. The fact that the other two 2-dimensional characters induce to an irreducible character has to be concluded from the action of $GL_2(3)$ on the corresponding characters.
$\mathrm{char}(K) = 2$: It is clear that Q_8 is in the kernel of every irreducible representation and since $GL_2(3)/Q_8 \cong S_3$ is not abelian, not all the characters of $SL_2(3)$ can be extendible to $GL_2(3)$.
$\mathrm{char}(K) = 3$: The same arguments as in the case $\mathrm{char}(K) \neq 2, 3$ hold, with the exception that the trivial character and the 2-dimensional character of Q_8 extend only to a single character of $SL_2(3)$.

4.3.12 Example We compute the character table of S_5 from the character table of A_5. The characters φ_1, φ_2 and φ_3 of A_5 of degrees 1, 4 and 5 are S_5-invariant and therefore have two extensions to S_5, differing by a factor of -1 on the classes outside A_5. From this we obtain the trivial character χ_1 and the signum-character χ_1'. Furthermore, since we know that $\pi - \chi_1$ is an irreducible character, where π is the natural permutation character of S_5, we can also determine the extensions χ_2 and χ_2' of φ_2. The two characters φ_4, φ_5 of degree 3 have A_5 as their inertia group and their induction to S_5 gives the same irreducible character $\chi_{4,5}$ with values 0 outside A_5. We thus obtain the following partial character table:

$C_G(g_i)$	120	8	6	5	12	4	6
$\lvert C_i\rvert$	1	15	20	24	10	30	20
g_i	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4,5)$	$(1,2)$	$(1,2,3,4)$	$(1,2)(3,4,5)$
χ_1	1	1	1	1	1	1	1
χ_1'	1	1	1	1	-1	-1	-1
χ_2	4	0	1	-1	2	0	-1
χ_2'	4	0	1	-1	-2	0	1
χ_3	5	1	-1	0	a	b	c
χ_3'	5	1	-1	0	$-a$	$-b$	$-c$
$\chi_{4,5}$	6	-2	0	1	0	0	0

From the second orthogonality relations it follows that $|a| = |b| = |c| = 1$ and that $ab = -1$ and $ac = 1$, thus if we choose $a = 1$ we conclude that $b = -1$ and $c = 1$.

Exercises

46. Prove the transitivity of induction: Let $H \leq U \leq G$ be subgroups and let W be a KH-module with character φ. Show that $(\varphi^U)^G = \varphi^G$.
47. Let $H \leq G$ be a subgroup and let χ be a character of G and φ a character of H.
    (i) Show that $(\varphi \cdot \chi_{|H})^G = \varphi^G \cdot \chi$.
    (ii) Show that $\ker(\varphi^G) = \bigcap_{g\in G} g \ker(\varphi) g^{-1}$.
    (iii) Let $N \trianglelefteq G$ be a normal subgroup of G and let χ be an irreducible character of G with $(\chi_{|N}, 1_N)_N \neq 0$. Prove that $N \leq \ker(\chi)$.

48. Let $H \leq G$ be a subgroup, let V be a KG-module and W a KH-module. Show that $Hom_{KG}(V, W^G) \cong Hom_{KH}(V_{|H}, W)$ as K-modules. (Hint: For a homomorphism $\varphi \in Hom_{KH}(V_{|H}, W)$ consider the map $\varphi' : V \to W^G : v \mapsto \sum_{i=1}^{m}(vg_i^{-1})\varphi \otimes g_i$.)
49. Let $H \leq G$ be a subgroup, let φ be an irreducible character of H and let $\varphi^G = \sum_{i=1}^{r} a_i\chi_i$ be the decomposition of φ^G into irreducible characters of G. Show that $\sum_{i=1}^{r} a_i^2 \leq [G : H]$.
50. Compute the induction-restriction table between the alternating groups A_5 and A_4.
51. Let $H \leq G$ be a subgroup.
    (i) Let χ be an irreducible character of G with $H \cdot \ker(\chi) = G$. Show that $\chi_{|H}$ is an irreducible character of H.
    (ii) Assume that H is a maximal subgroup of G and let $\chi \neq 1_G$ be a non-trivial constituent of $\pi := (1_H)^G$, i.e. $(\chi, \pi) \neq 0$. Show that $\ker(\chi) = \ker(\pi)$.
52. Let G act transitively on Ω with $|\Omega| > 1$. Show that G contains a *fixed-point free* element g, i.e. an element g such that $|fix_\Omega(g)| = 0$.
53. Let G act doubly transitive on Ω and let $H \leq G$ with $[G : H] < |\Omega|$. Show that H acts transitively on Ω.
54. Determine all candidates of transitive permutation characters of the alternating group A_5 from the character table of A_5. Which of the so obtained characters are in fact permutation characters?
55. Let $H \trianglelefteq G$ and let φ be an irreducible character of H such that φ^G is irreducible. Show that $\varphi^G(g) = 0$ for all $g \in G \setminus H$.
56. Let $H \trianglelefteq G$, let χ be an irreducible character of G and let φ be an irreducible character of H such that $(\chi_{|H}, \varphi) \neq 0$. Show that $\varphi(1) \mid \chi(1)$.
57. Suppose that G has exactly one nonlinear irreducible character. Prove that the derived subgroup G' is an elementary abelian group. (Hint: Consider the action of G on the irreducible characters of G' and use the fact that the restriction of an irreducible character of G to G' is the sum of irreducible characters of G' lying in one orbit.)
58. Let G be a finite group, p a prime and suppose that $\chi(1)$ is a power of p for every irreducible character of G. Show that G has a normal abelian p-complement, i.e. a subgroup $H \leq G$ with $p \nmid |H|$ and $[G : H] = p^a$. (Hint: Show that p divides $[G : G']$ and use induction on $|G|$.)
59. Let $H \trianglelefteq G$, then G acts on the conjugacy classes of H by conjugation and on the irreducible characters of H by $\chi^g(h) = \chi(ghg^{-1})$. Show that the number of fixed points for these two actions coincide and that the number of orbits for these two actions are also the same. (Hint: Regard the two actions as actions on the columns and rows of the character table of H which is an invertible matrix.)
60. Let $A \trianglelefteq G$ be an abelian normal subgroup of G and let φ be an irreducible character of A with $I_G(\varphi) = G$. Show that φ can be extended to G if A has a complement in G, i.e. if there is a subgroup $H \leq G$ with $H \cdot A = G$ and $H \cap A = \{1\}$.
    Give an example that demonstrates that the conclusion is not necessarily true if A does not have a complement in G.