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Ordinary Differential Equations

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Preface

These lecture notes are written for the course “Ordinary Differential Equations” (NWI-WB104) at Radboud University, Nijmegen (second year, 6EC). The course covers the basic theory of ordinary differential equations. Further details, applications and many additional topics can be found in, e.g. the monographs by M. Braun [1], J. J. Duistermaat and W. Eckhaus [2] and W. Walter [6]. To follow the course a solid understanding of analysis, calculus and linear algebra is required.

We discuss several classes of ordinary differential equations that can be solved explicitly. We investigate the existence and uniqueness of solutions of ordinary differential equations and their stability with respect to perturbations. Moreover, we analyze the qualitative behavior of solutions of linear and nonlinear ordinary differential equations, including the stability of equilibria and the large time behavior of solutions. Several concrete applications will be considered, including models in mechanics and population dynamics.

Contents

Chapter 1

Introduction and explicit solutions

1.1 ODEs and initial value problems

An **ordinary differential equation** (ODE) is an equation that relates values of an unknown function $u : I \to \mathbb{R}^n$, $n \in \mathbb{N}$, $I \subset \mathbb{R}$ an interval, with values of its derivatives. The order of the highest derivative in the equation determines the **order of an ODE**.

We use the following notation

$$u'(t) = \frac{du}{dt}(t),\ u''(t) = \frac{d^2u}{dt^2}(t),\ \ldots,\ u^{(m)}(t) = \frac{d^m u}{dt^m}(t), \quad t \in I, \quad m \in \mathbb{N},$$

and when we write $A \subset \mathbb{R}^n$ then either $A \subsetneq \mathbb{R}^n$ or $A = \mathbb{R}^n$. Moreover, we always assume that the interval I is *proper*, i.e. $I \neq \emptyset$ and I is not a singleton set. It can be open, closed, right-open and left-closed or right-closed and left-open.

Example. The equation

$$u'(t) = 2tu(t), \qquad t \in \mathbb{R},$$

is an ODE of first order, where $u : \mathbb{R} \to \mathbb{R}$ is the unknown function. We can easily verify that $u(t) = ce^{t^2}, t \in \mathbb{R}$, satisfies the ODE for any constant $c \in \mathbb{R}$.

Another example is the equation

$$u''(t) = \frac{2}{t^2}u(t) + u'(t), \qquad t > 0.$$

It is an ODE of second order. We can verify that for any $c \in \mathbb{R}$ the function $u(t) = ct^2e^t, t > 0$, satisfies this ODE.

We will first analyze **explicit ODEs of first order**,

$$u'(t) = f(t, u(t)), \tag{1.1}$$

where $f : D \to \mathbb{R}^n$, $n \in \mathbb{N}$, is a function defined on a set $D \subset \mathbb{R} \times \mathbb{R}^n$. Equations of higher order we will address later.

If $n = 1$, the ODE (1.1) is a *scalar equation*; otherwise, if $n > 1$, it is a *system of ODEs*. In the latter case, the ODE (1.1) is a short notation for the system of n ODEs

$$\begin{aligned} u_1'(t) &= f_1(t, u_1(t), \ldots, u_n(t)), \\ u_2'(t) &= f_2(t, u_1(t), \ldots, u_n(t)), \\ &\vdots \\ u_n'(t) &= f_n(t, u_1(t), \ldots, u_n(t)). \end{aligned}$$

In ODE models, the independent variable t often represents the time, but it can also denote a location, distance or displacement. In those cases, it is typically denoted by x.

If the function f in (1.1) does not explicitly depend on time, i.e.

$$u'(t) = f(u(t)),$$

the ODE is called **autonomous**; otherwise, it is called **non-autonomous**. In case of autonomous equations, the rate of change $u'(t)$ only depends on the "state" $u(t)$ itself and not on external factors.

Definition 1.1. Let $I \subset \mathbb{R}$ be an interval. The function $u : I \to \mathbb{R}^n$ is called a **solution of the ODE** (1.1) on I if

- u is differentiable in I,
- $(t, u(t)) \in D \ \ \forall t \in I$,
- $u'(t) = f(t, u(t)) \ \ \forall t \in I$.

If the interval I is closed or half-closed, we consider one-sided derivatives.

A special solution is a **steady state** or **equilibrium solution** which is a constant solution of the ODE, i.e.

$$u(t) = u^* \qquad \forall t \in I,$$

for some $u^* \in \mathbb{R}^n$.

We observe that

$$u(t) = u^* \quad \forall t \in I \qquad \Longleftrightarrow \qquad f(t, u^*) = 0 \quad \forall t \in I.$$

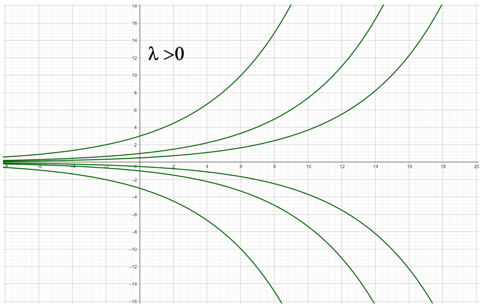
In particular, for autonomous ODEs the equilibria correspond to the zeros of the function f.

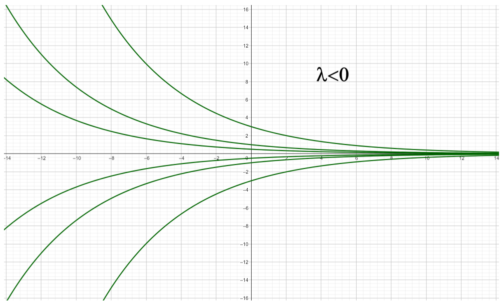
Example 1.2. One of the simplest ODEs is

$$u'(t) = \lambda u(t), \qquad t \in \mathbb{R}, \tag{1.2}$$

for some $\lambda \in \mathbb{R}$. Here, $D = \mathbb{R}^2$ and $f : D \to \mathbb{R}$ is defined as $f(t, u) = f(u) = \lambda u$, i.e. the ODE is autonomous.

The solutions are given by $u(t) = ce^{\lambda t}$, $t \in \mathbb{R}$, where $c \in \mathbb{R}$ is an arbitrary constant. The solutions are plotted below for the cases $\lambda > 0$ and $\lambda < 0$.





If $\lambda \neq 0$ then $u^* = 0$ is the only steady state of the ODE (1.2). Moreover, we observe that for $\lambda < 0$ all solutions converge to the steady state $u^* = 0$ as $t \to \infty$. On the other hand, for $\lambda > 0$ all non-zero solutions tend to $+\infty$ or $-\infty$ as $t \to \infty$.

The ODE (1.2) is used as a simplistic model for the growth of populations. Let $u(t)$ denote the density of a population at time $t \geq 0$, e.g. bacteria in a Petri dish where nutrients are abundantly available. Then, the growth of the population can be described by the ODE

$$u'(t) = \alpha u(t) - \beta u(t),$$

where $\alpha > 0$ denotes the growth rate and $\beta > 0$ the death rate of the population. Hence, this corresponds to the ODE (1.2) with $\lambda = \alpha - \beta$.

In this context, u describes the density of a population and hence, we should only consider non-negative solutions. Then, the only possible scenarios are that the population dies out (if $\beta > \alpha$) or that the population shows exponential growth (if $\alpha > \beta$). Bacterial populations that initially show exponential growth are commonly observed. However, their growth rate tends to decrease as the population size increases since resources eventually become limited. Therefore, this simple growth model can be considered a good approximation for a short time period, but it is unrealistic for longer time periods.

Example 1.3. We aim to improve the growth model in the previous example in the case of growth, i.e. $\alpha > \beta$. We assume that the death rate is not constant, but it increases as the population size increases. Let us suppose it increases linearly in u and it is given by $\beta + \gamma u$, for some constants $\beta, \gamma > 0$. Then, we obtain the ODE

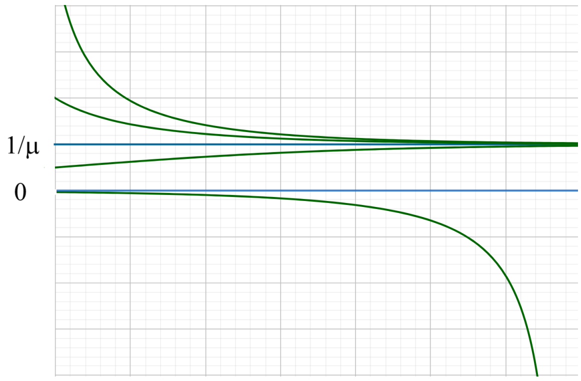
$$u'(t) = (\alpha - \beta - \gamma u(t))u(t) = \lambda u(t)(1 - \mu u(t)),$$

where $\lambda = \alpha - \beta > 0$ and $\mu = \frac{\gamma}{\lambda} > 0$.

The steady states are $u_1^* = 0$ and $u_2^* = \frac{1}{\mu}$. As we will see later, this ODE can be solved explicitly and the non-zero solutions are of the form

$$u(t) = \frac{e^{\lambda t}}{c + \mu(e^{\lambda t} - 1)}, \qquad t \geq 0,$$

where $c \in \mathbb{R}$ is an arbitrary constant. We observe that for $c > 0$ solutions exist for all $t \geq 0$, while for $c < 0$ the solutions only exist for a finite time (as $c + \mu(e^{\lambda t} - 1)$ can become zero). If $c < 0$ the solutions take negative values while they are strictly positive if $c > 0$. Several solutions are plotted below where the blue curves are the steady state solutions.



Since we aim to describe population growth we limit ourselves to non-negative solutions which corresponds to the case $c \geq 0$. We observe that every strictly positive solution tends to the positive steady state $\frac{1}{\mu}$ as $t \to \infty$. Solutions starting from a value larger than $\frac{1}{\mu}$ are decreasing, while solutions starting from a positive value below $\frac{1}{\mu}$ are increasing.

We will later look at more interesting models for population dynamics involving several different species that interact with each other. They are formulated as *systems of ODEs*.

We observed in the examples that solutions are not uniquely determined by the ODE itself. However, if we specify an *initial value* for the solution, the solutions in the considered examples are unique. Moreover, we observed in Example 1.3 that solutions may only exists for a certain finite time interval and this interval of existence can depend on the initial value.

Definition 1.4. An **initial value problem** (IVP) for an explicit first order ODE is of the form

$$u'(t) = f(t, u(t)), \tag{1.3}$$

$$u(t_0) = u_0, \tag{1.4}$$

where $f : D \to \mathbb{R}^n$, $D \subset \mathbb{R} \times \mathbb{R}^n$, is a given function and $(t_0, u_0) \in D$ a given point.

Let $I \subset \mathbb{R}$ be an interval with $t_0 \in I$. A **solution of the IVP** is a function $u : I \to \mathbb{R}^n$ such that

- u is a solution of the ODE (1.3);
- $u(t_0) = u_0$.

The maximal interval on which the solution of an IVP exists is called the **maximal interval of existence**.

Questions we will address in this course include the following:

- **Existence:** Does an IVP

  $$u'(t) = f(t, u(t)), \qquad u(t_0) = u_0$$

  have a solution?
- **Uniqueness:** Is there only one solution of an IVP?
- **Explicit solutions:** Can we find an explicit formula for the solution?

  This is possible for few particular classes of ODEs, but in general, we cannot expect to solve an ODE explicitly.

- **Qualitative behavior of solutions:** If we at least know that a unique solution exists which properties does it have?

  For instance, we will investigate maximal intervals of existence, non-negativity and boundedness of solutions, their asymptotic behavior as $t \to \infty$, steady states and their stability, existence of periodic solutions, ...

As the classes of ODEs that can be solved explicitly are very limited we need to develop a general theory that ensures the existence and uniqueness of solutions of IVPs. This is important as, once we know that a unique solution exists, we can study qualitative properties of solutions (analytically) and we can also apply numerical schemes to approximate and simulate the solutions. This is particularly important in applications if we want to analyze and validate models based on experimental data. In this course we focus on the analysis, numerical methods will not be covered.

Before we develop a general existence and uniqueness theory we discuss several classes of ODEs that can be solved explicitly.

1.2 Separable equations

A scalar ODE of the form

$$u'(t) = g(t)h(u(t)), \tag{1.5}$$

where $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are continuous functions defined on the intervals $I, J \subset \mathbb{R}$, is called **separable**. In this case, the function f is a product of two functions $f(t,u) = g(t)h(u)$, where g only depends on t and h only depends on u.

We can *formally* solve separable ODEs according to the following "recipe":

- Write the ODE as
$$\frac{du}{dt} = g(t)h(u) \qquad \Rightarrow \qquad \frac{du}{h(u)} = g(t)dt.$$
  Now, the variables u and t are separated, i.e. the left hand side only depends on u, the right hand side only on t.

- Integrating both sides we obtain
$$\int^{u(t)} \frac{1}{h(v)} dv = \int^{t} g(s)ds,$$
  which is an equation that implicitly defines the solution u.

- Finally, we try to find an explicit expression for the solution u by solving the integral equation.

Example 1.5. Consider the ODE

$$u'(t) = 2tu(t),$$

so $g(t) = 2t$ and $h(u) = u$.

First, we separate the variables and write

$$\frac{du}{u} = 2t\,dt.$$

Then, we integrate which leads to

$$\int^{u(t)} \frac{1}{v} dv = \int^{t} 2s ds.$$

Hence, we obtain

$$\begin{aligned} & \ln|u(t)| + c_1 = t^2 + c_2, && \text{for some integration constants } c_1, c_2 \in \mathbb{R}, \\ \Rightarrow \quad & \ln|u(t)| = t^2 + c_2 - c_1. && \end{aligned}$$

Finally, we take the exponential and obtain the explicit solution

$$u(t) = e^{t^2} e^{c_2 - c_1} = a e^{t^2}, \qquad \text{for some constant } a \in \mathbb{R}.$$

While this recipe often works in practice and allows us to explicitly compute solutions, the steps are not mathematically rigorous and need to be justified. In fact, we made the following mistakes:

- We cannot separate du and dt in $\frac{du}{dt}$.
- The integral $\int^u \frac{1}{v} dv$ has a singularity for $v = 0$.

We can make this derivation rigorous and prove that the method indeed leads to the unique solution of an IVP for a separable ODE, if the function h has suitable properties.

Theorem 1.6. *Let $I, J \subset \mathbb{R}$ be intervals, $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ be continuous functions and $t_0 \in I, u_0 \in J$.*

We assume that u_0 is an interior point of J and $h(u_0) \neq 0$. Then, there exists a unique local solution of the IVP (1.5) *with $u(t_0) = u_0$ which is determined by*

$$\int_{u_0}^{u(t)} \frac{1}{h(v)} dv = \int_{t_0}^{t} g(s) ds. \tag{1.6}$$

Otherwise, if $h(u_0) = 0$, then $u \equiv u_0$, i.e. $u : I \to \mathbb{R}$, is a steady state solution of the ODE.

Proof. If $h(u_0) = 0$, then $u \equiv u_0$ is a constant solution. In fact, we have

$$0 = \frac{d}{dt} u_0 = u'(t) = g(t) h(u(t)) = g(t) h(u_0) = 0 \qquad \forall t \in I.$$

Otherwise, if $h(u_0) \neq 0$ we define

$$H(u) := \int_{u_0}^{u} \frac{1}{h(v)} dv, \qquad G(t) = \int_{t_0}^{t} g(s) ds.$$

By assumption, $h(u_0) \neq 0$ and hence, $h(u) \neq 0$ for u in an open interval $J_0 \subset J$ around u_0. For $u \in J_0$, the function H is well-defined, continuously differentiable and $H'(u) = \frac{1}{h(u)} \neq 0$.

Therefore, the inverse function theorem implies that the inverse function of H exists which we denote by $H^{-1} : H(J_0) \to J_0$. Moreover, H^{-1} is continuously differentiable and

$$\left(H^{-1}\right)'(u) = \frac{1}{H'(H^{-1}(u))}, \qquad u \in H(J_0).$$

We now define

$$u(t) := H^{-1}(G(t))$$

which corresponds to applying H^{-1} to (1.6). We observe that $H(u_0) = 0 = G(t_0)$, and therefore, $G(t_0) = 0 \in H(J_0)$. Therefore, if we take a sufficiently small interval I_0 such that $t_0 \in I_0$ and $I_0 \subset I$ open, then $G(I_0) \subset H(J_0)$, i.e. $u(t) = H^{-1}(G(t))$ is well-defined for $t \in I_0$.

We aim to show that this function u is a solution of the IVP on I_0. In fact, u is continuously differentiable, and by the chain rule, it follows that

$$\begin{aligned} u'(t) &= \frac{d}{dt}\left(H^{-1}(G(t))\right) = \left(H^{-1}\right)'(G(t))G'(t) = \frac{1}{H'(H^{-1}(G(t)))}g(t) \\ &= h\left(H^{-1}(G(t))\right)g(t) = h(u(t))g(t), \end{aligned}$$

where we used that $H'(u) = \frac{1}{h(u)}$ and $u(t) = H^{-1}(G(t))$. This shows that u satisfies the ODE on I_0. Moreover, we observe that $u(t_0) = H^{-1}(G(t_0)) = H^{-1}(0) = u_0$, i.e. u is a local solution of the IVP.

Finally, we show that the solution is unique. Let us assume that v is another solution of the IVP. Then, as long as $h(v) \neq 0$ (which certainly holds in a sufficiently small interval around u_0) we have

$$\frac{v'(t)}{h(v(t))} = g(t).$$

Integrating the equation from t_0 to t yields

$$\int_{t_0}^{t} g(s)ds = \int_{t_0}^{t} \frac{v'(s)}{h(v(s))}ds = \int_{u_0}^{v(t)} \frac{1}{h(w)}dw,$$

where we used the substitution $w = v(s)$ in the last step. By the definition of G and H, this implies that $G(t) = H(v(t))$, and applying the inverse function H^{-1} on both sides we obtain

$$v(t) = H^{-1}(G(t)) = u(t),$$

i.e. $u \equiv v$ in an interval around t_0. □

Remark. We remark that every scalar autonomous ODE, $u'(t) = f(u(t))$, is separable and hence, Theorem 1.6 applies.

If $h(u_0) \neq 0$, the theorem only guarantees the *local existence and uniqueness* of solutions, i.e. the existence and uniqueness of a solution on a small time interval around t_0. However, the solution may not exists or be unique on the entire interval I where g is defined. For instance, the logistic equation in Example 1.3 is separable and we can calculate the solution of the IVP by solving the integral equation (1.6). However, in the case $c < 0$ the solution only exist on a finite time interval $[0, T), T > 0$, and blows up as $t \to T$, i.e. $\lim_{t \to T} u(t) = \infty$.

In the case that $h(u_0) = 0$ we can immediately conclude that $u \equiv u_0$ is a steady state solution. However, there might exist other solutions, i.e., the solutions of the IVP might not be unique in this case.

Example 1.7. Consider the IVP for a separable ODE

$$u'(t) = 3u^{\frac{2}{3}}(t), \qquad u(0) = 0,$$

i.e. $g \equiv 1$ and $h(u) = 3u^{\frac{2}{3}}$ with $h(u_0) = h(0) = 0$. Then, $u \equiv 0$ is s steady state solution and solves the IVP.

However, following the "recipe" for separable ODEs to compute solutions we obtain

$$\int_0^t 1 ds = t = \frac{1}{3}\int_0^{u(t)} v^{-\frac{2}{3}} dv = u^{\frac{1}{3}}(t),$$

and consequently, $u(t) = t^3$. This is another solution of the IVP.

This IVP has even infinitely many solutions. In fact, for arbitrary $a < 0 < b$, the functions

$$u_{a,b}(t) = \begin{cases} (t-a)^3 & t \leq a \\ 0 & t \in (a,b) \\ (t-b)^3 & t \geq b \end{cases}$$

satisfy the ODE and the initial value $u_{a,b}(0) = 0$.

In Chapter 3 we will formulate general conditions on the function f that ensure the local existence and uniqueness of solutions of IVPs for explicit first order ODEs (1.3)-(1.4).

1.3 First order linear equations

A scalar first order ODE of the form

$$u'(t) = a(t)u(t) + b(t), \tag{1.7}$$

where $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ are continuous functions, $I \subset \mathbb{R}$ an interval, is called **linear**. Moreover, if $b \equiv 0$, the equation is called **homogeneous**, and **inhomogeneous** otherwise.

The ODE is called linear since the right hand side of the equation, $f(t,u) = a(t)u + b(t)$, is a linear function of u (even though the functions a and b might be nonlinear).

To find a solution formula we first consider the homogeneous problem, i.e. $f(t,u) = a(t)u$ and $b \equiv 0$. In this case the ODE is separable. Moreover, we observe that $u \equiv 0$ is a steady state solution. To find nontrivial solutions let $t_0 \in I, u_0 \in \mathbb{R}$ and assume that $u(t_0) = u_0 \neq 0$. Using the notation in (1.5) we have $g(t) = a(t), h(u) = u$. As $h(u_0) = u_0 \neq 0$ we can apply Theorem 1.6. In fact, we conclude that there exists a unique local solution of the IVP and it is determined by the integral equation

$$\int_{u_0}^{u(t)} \frac{1}{v} dv = \int_{t_0}^t a(s) ds.$$

Without loss of generality we assume that $u_0 > 0$, and hence, $u > 0$ in a sufficiently small interval around t_0 (otherwise we consider $w = -u$). Then, we obtain

$$\ln(u(t)) - \ln u_0 = \int_{u_0}^{u(t)} \frac{1}{v} dv = \int_{t_0}^t a(s) ds,$$

and taking the exponential we find the explicit solution formula

$$u(t) = u_0 e^{\int_{t_0}^t a(s)ds}.$$

Moreover, we observe that the solution u exists for all $t \in I$.

Proposition 1.8. Let $I \subset \mathbb{R}$ be an interval, $a : I \to \mathbb{R}$ be continuous, $u_0 \in \mathbb{R}$ and $t_0 \in I$. Then, there exists a unique solution $u : I \to \mathbb{R}$ of the homogeneous IVP $u'(t) = a(t)u(t)$ with $u(t_0) = u_0$, and it is given by

$$u(t) = u_0 e^{\int_{t_0}^t a(s)ds}, \qquad t \in I.$$

Proof. The function u is well-defined for all $t \in I$ and continuously differentiable (as a is continuous). Moreover, by the chain rule it follows that

$$u'(t) = u_0 e^{\int_{t_0}^t a(s)ds} a(t) = a(t)u(t), \qquad t \in I,$$

and $u(t_0) = u_0$. This shows that u is a solution of the IVP.

To prove the uniqueness of solutions let us assume that v is another solution of the IVP. We define $\varphi(t) := v(t)e^{-\int_{t_0}^t a(s)ds}$. Then, we obtain

$$\begin{aligned} \varphi'(t) &= v'(t)e^{-\int_{t_0}^t a(s)ds} + v(t)(-a(t))e^{-\int_{t_0}^t a(s)ds} \\ &= a(t)v(t)e^{-\int_{t_0}^t a(s)ds} - a(t)v(t)e^{-\int_{t_0}^t a(s)ds} = 0, \end{aligned}$$

where we used that $v'(t) = a(t)v(t)$. Consequently, $\varphi \equiv c$, for some $c \in \mathbb{R}$. We observe that $c = \varphi(t_0) = v(t_0) = u_0$, which implies that

$$v(t) = \varphi(t)e^{\int_{t_0}^t a(s)ds} = u_0 e^{\int_{t_0}^t a(s)ds} = u(t),$$

i.e. $u \equiv v$ on I. □

To find a solution formula for the inhomogeneous IVP we use an ansatz called *variation of constants*. Namely, we replace the constant in the solution formula for the homogeneous equation by a function and consider

$$u(t) = c(t)e^{\int_{t_0}^t a(s)ds} = c(t)u_h(t), \qquad t \in I,$$

where $u_h(t) = e^{\int_{t_0}^t a(s)ds}$ is a solution of the homogeneous equation. Then, we obtain

$$u'(t) = c'(t)u_h(t) + c(t)u_h'(t) = c'(t)u_h(t) + c(t)a(t)u_h(t) = c'(t)e^{\int_{t_0}^t a(s)ds} + a(t)u(t).$$

Hence, u solves the ODE (1.7) with the initial value $u(t_0) = u_0$ if and only if c satisfies

$$c'(t) = e^{-\int_{t_0}^t a(s)ds} b(t), \qquad c(t_0) = u_0,$$

which implies that

$$c(t) = \int_{t_0}^t e^{-\int_{t_0}^s a(r)dr} b(s)ds + u_0.$$

Finally, using that $u = cu_h$ we conclude that

$$u(t) = u_0 e^{\int_{t_0}^t a(s)ds} + \int_{t_0}^t b(s) e^{\int_s^t a(r)dr} ds, \qquad t \in I,$$

solves the ODE (1.7) with initial value $u(t_0) = u_0$.

We observe that as in case of homogeneous IVP the solution is defined for all $t \in I$. The following theorem summarizes the results.

Theorem 1.9. *Let $I \subset \mathbb{R}$ be an interval, $a, b : I \to \mathbb{R}$ be continuous, $u_0 \in \mathbb{R}$ and $t_0 \in I$. Then, there exists a unique solution $u : I \to \mathbb{R}$ of the IVP* (1.7) *with $u(t_0) = u_0$, and it is given by*

$$u(t) = u_0 e^{\int_{t_0}^t a(s)ds} + \int_{t_0}^t b(s) e^{\int_s^t a(r)dr} ds, \qquad t \in I.$$

This formula is known as the variation of constants formula.

Proof. The function u is well-defined for $t \in I$ and continuously differentiable (as a and b are continuous). To verify that u satisfies the ODE we apply the Leibniz rule,

$$\begin{aligned} u'(t) &= u_0 e^{\int_{t_0}^t a(s)ds} a(t) + \int_{t_0}^t b(s) e^{\int_s^t a(r)dr} a(t) ds + b(t) \\ &= a(t)u(t) + b(t). \end{aligned}$$

Moreover, $u(t_0) = u_0$ which proves that u is a solution of the IVP.

To show uniqueness of solutions we assume that v is another solution of the IVP. Then, the difference $w := u - v$ satisfies

$$w'(t) = u'(t) - v'(t) = a(t)u(t) + b(t) - (a(t)v(t) + b(t)) = a(t)w(t),$$

and $w(t_0) = u(t_0) - v(t_0) = 0$. Consequently, Proposition 1.8 implies that

$$w(t) = w(t_0) e^{\int_{t_0}^t a(s)ds} = 0,$$

i.e. $u \equiv v$ on I. □

Remark. We recall the *Leibniz integral rule*: Consider open intervals (t_0, t_1) and (α, β) and assume that the functions $a, b : (t_0, t_1) \to (\alpha, \beta)$ and $f : (t_0, t_1) \times (\alpha, \beta) \to \mathbb{R}, (t, x) \mapsto f(t, x)$, are continuously differentiable. Then, the function

$$F(t) = \int_{a(t)}^{b(t)} f(t, x) dx, \qquad t \in (t_0, t_1),$$

is continuously differentiable and

$$F'(t) = f(t, b(t))b'(t) - f(t, a(t))a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t, x) dx.$$

Example 1.10. If $a \equiv a_0 \in \mathbb{R}$ is constant, the variation of constants formula becomes

$$u(t) = e^{a_0(t-t_0)} u_0 + \int_{t_0}^t e^{a_0(t-s)} b(s) ds.$$

If $b \equiv b_0 \in \mathbb{R}$ is also constant and $a_0 \neq 0$ we obtain

$$u(t) = e^{a_0(t-t_0)} u_0 + \frac{b_0}{a_0} \left(e^{a_0(t-t_0)} - 1 \right).$$

1.4 Change of variables

Several classes of ODEs that have a particular structure can be transformed into separable or linear ODEs by a suitable substitution (change of variables). Below we discuss three examples.

Homogeneous equations

Let $J \subset \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ be a continuous function. An ODE of the form

$$u'(t) = f\left(\frac{u(t)}{t}\right), \qquad t \neq 0,$$

is called **homogeneous**. Note that the term homogeneous has a different meaning here than for linear ODEs.

Setting $v(t) = \frac{u(t)}{t}$ leads to the ODE

$$v'(t) = \frac{1}{t}(f(v(t)) - v(t)),$$

which is separable. Hence, the theory in Section 1.2 applies and we can, in principle, solve the ODE for v (if f has suitable properties). Transforming back, $u(t) = tv(t)$, yields the solution of the original equation (see Exercises E1.3 and E1.4).

ODEs of the form $\mathbf{u' = f(at + bu + c)}$

Let $J \subset \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ be a continuous function. We consider an ODE of the form

$$u'(t) = f(at + bu(t) + c), \tag{1.8}$$

where $a, b, c \in \mathbb{R}$ and $b \neq 0$. Let $v(t) = at + bu(t) + c$. Then, if u is a solution of (1.8) then v satisfies

$$v'(t) = a + bu'(t) = a + bf(v(t)),$$

which is separable. Hence, we can apply the method for separable ODEs to compute the solution v (if possible) and transforming back, $u(t) = \frac{1}{b}(v(t) - at - c)$, we obtain the solution of the original ODE (1.8).

Example 1.11. We consider the ODE

$$u'(t) = (t + u(t))^2.$$

Then, $v(t) = t + u(t)$ satisfies the ODE

$$v'(t) = v^2(t) + 1,$$

which we can solve explicitly. In particular, the function $h(v) = v^2 + 1$ satisfies $h(v) \neq 0\ \forall v \in \mathbb{R}$ and hence, by Theorem 1.6 all solutions are given by

$$v(t) = \tan(t + c),$$

for some $c \in \mathbb{R}$. Finally, transforming back we obtain the solutions of the original ODE, $u(t) = \tan(t + c) - t$.

Bernoulli equations

Let $J \subset \mathbb{R}$ be an interval and $a, b : J \to \mathbb{R}$ be continuous functions. *Bernoulli equations* are ODEs of the form

$$u'(t) = a(t)u(t) + b(t)u^{\alpha}(t),$$

where $\alpha \in \mathbb{R} \setminus \{0, 1\}$. Non-negative solutions $u \geq 0$ can be found by introducing the function $v(t) = u^{\beta}(t)$, for some $\beta \in \mathbb{R}$. Choosing a suitable exponent β, the ODE for v becomes linear and we can apply Theorem 1.9. Finally, we transform back and obtain the solution of the original ODE, $u(t) = v^{\frac{1}{\beta}}(t)$ (see Exercise E1.7).

1.5 Exact equations

Let $I \subset \mathbb{R}$ be an interval, $D \subset \mathbb{R}^2$ be open and connected, $\psi : D \to \mathbb{R}$ be continuously differentiable and $u : I \to \mathbb{R}$ be continuously differentiable with $(t, u(t)) \in D$. Assume that u satisfies the implicit equation

$$\psi(t, u(t)) = c, \qquad c \in \mathbb{R}. \tag{1.9}$$

Then, the chain rule implies that

$$0 = \frac{d}{dt}\psi(t, u(t)) = \partial_t \psi(t, u(t)) + \partial_u \psi(t, u(t))u'(t),$$

i.e. u is a solution of the ODE

$$a(t, u(t)) + b(t, u(t))u'(t) = 0,$$

where

$$a(t, u) = \partial_t \psi(t, u), \qquad b(t, u) = \partial_u \psi(t, u), \qquad (t, u) \in D.$$

Conversely, assume that $a, b : D \to \mathbb{R}$ are continuous functions. Then, an ODE of the form

$$a(t, u(t)) + b(t, u(t))u'(t) = 0 \tag{1.10}$$

is called **exact** if there exists a continuously differentiable function $\psi : D \to \mathbb{R}$ such that

$$a(t, u) = \partial_t \psi(t, u), \qquad b(t, u) = \partial_u \psi(t, u), \qquad (t, u) \in D.$$

Then, the solutions are determined by (1.9) and the *contour lines* of ψ, i.e.

$$\psi(t, u) = c, \qquad c \in \mathbb{R},$$

are the graphs of the solution curves of the ODE (1.10).

It is important to be able to recognize whether an ODE is exact and to find the corresponding function ψ. The following theorem yields a criterion that allows us to check whether an ODE is exact.

Theorem 1.12. *Let $D = (\alpha, \beta) \times (\gamma, \delta) \subset \mathbb{R}^2$, where $\alpha < \beta$ and $\gamma < \delta$, and let $a, b : D \to \mathbb{R}$ be continuously differentiable. Then, the following statements are equivalent:*

(i) There exists a continuously differentiable function $\psi : D \to \mathbb{R}$ such that

$$\partial_t \psi(t,u) = a(t,u), \qquad \partial_u \psi(t,u) = b(t,u) \qquad \forall (t,u) \in D.$$

(ii) The integrability condition,

$$\partial_u a(t,u) = \partial_t b(t,u) \qquad \forall (t,u) \in D,$$

holds.

Proof. (i)⇒ (ii): Since the second partial derivatives of ψ exist and are continuous, it follows from Schwarz' Theorem and the relations in (i) that

$$\partial_u a(t,u) = \partial_u \partial_t \psi(t,u) = \partial_t \partial_u \psi(t,u) = \partial_t b(t,u).$$

(ii)⇒ (i): Let $(t,u) \in D$ and let ψ be defined by

$$\psi(t,u) = \int^t a(s,u)ds + \int^u \left(b(t,v) - \int^t \partial_u a(s,v)ds \right) dv.$$

Then, using the integrability condition (ii) we conclude that

$$\begin{aligned} \partial_t \psi(t,u) &= a(t,u) + \int^u (\partial_t b(t,v) - \partial_u a(t,v))\, dv = a(t,u), \\ \partial_u \psi(t,u) &= \int^t \partial_u a(s,u)ds + b(t,u) - \int^t \partial_u a(s,u)ds = b(t,u), \end{aligned}$$

i.e. the ODE is exact. □

Remark. In Theorem 1.12 we assumed that D is rectangular. However, the result remains valid for open subsets $D \subset \mathbb{R}^2$ that are *simply connected*, i.e. they have no holes.

Example 1.13. A separable ODE

$$u'(t) = h(u(t))g(t),$$

is not exact (unless $h \equiv 1$). However, if $h \neq 0$ we can rewrite the equation and obtain the exact ODE

$$\frac{1}{h(u(t))} u'(t) = g(t).$$

Some ODEs that are not exact can be made exact by multiplication with a suitable function. An example is the separable ODE in Example 1.13. A function $\mu : D \to \mathbb{R}$ is called an **integrating factor** for the ODE

$$a(t,u(t)) + b(t,u(t))u'(t) = 0,$$

if the ODE

$$\mu(t,u(t))a(t,u(t)) + \mu(t,u(t))b(t,u(t))u'(t) = 0$$

is exact.

To find such a function μ is typically very difficult. In fact, by Theorem 1.12 μ has to satisfy

$$\frac{\partial}{\partial u}(\mu(t,u)a(t,u)) = \frac{\partial}{\partial t}(\mu(t,u)b(t,u)),$$

i.e.

$$\partial_u\mu(t,u)a(t,u) + \mu(t,u)\partial_u a(t,u) = \partial_t\mu(t,u)b(t,u) + \mu(t,u)\partial_t b(t,u).$$

This is a *partial differential equation* (PDE) for μ, i.e. an equation involving partial derivatives with respect to different independent variables. Solving PDEs is generally much more involved than solving ODEs. However, in a few special cases, e.g. if μ only depends on u, or if μ only depends on t, the equation for μ becomes an ODE that can be solved (see Exercise E1.9).

1.6 Exercises

E1.1 **ODEs and IVPs**

(a) Show that the function $u = (u_1, u_2) : (0,\infty) \to \mathbb{R}^2$,

$$u_1(t) = t^2 - t\ln(t),$$
$$u_2(t) = \ln(t),$$

is a solution of the IVP

$$u_1'(t) = 2t - u_2(t) - 1$$
$$u_2'(t) = \frac{1}{t^2}u_2(t) + \frac{1}{t^3}u_1(t),$$
$$u_1(1) = 1, \quad u_2(1) = 0.$$

(b) Show that the function

$$u(t) = ce^{1+\sin(t)} + e^{1+\sin(t)}\int_0^t e^{-1-\sin(s)}ds, \qquad t \in \mathbb{R},$$

where $c \in \mathbb{R}$ is an arbitrary constant, is a solution of the ODE

$$u'(t) = \cos(t)u(t) + 1.$$

In both cases, specify the function f and the set D in Definitions 1.1 and 1.4.

E1.2 **Logistic differential equation**

The *logistic differential equation* is often used to model the growth of populations. If $u(t)$ denotes the population density at time $t \geq 0$, its time evolution can be described by the ODE

$$u'(t) = \lambda u(t)\left(1 - \frac{u(t)}{\kappa}\right),$$

where $\lambda > 0$ is the *growth rate* and $\kappa > 0$ the *carrying capacity* of the population.

(a) Determine a formula for the solution of the corresponding IVP with initial value $u(0) = u_0 > 0$.

Hint: Note that the ODE is separable and use partial fraction decomposition.

(b) Sketch the graphs of the solutions corresponding to the initial values $u_0 = 1$ and $u_0 = 20$ for $\lambda = 1$ and $\kappa = 10$.

(c) Determine the limit $\lim_{t\to\infty} u(t)$ for the solution u in (a).

E1.3 **Homogeneous equations**

An ODE of the form

$$u'(t) = f\left(\frac{u(t)}{t}\right)$$

is called *homogeneous*. Apply the substitution $v(t) = \frac{u(t)}{t}$ and derive an ODE for v. Of what type is the resulting ODE?

E1.4 **IVP for a homogeneous equation**

Determine the (local) solution of the IVP

$$\begin{aligned} tu^2(t)u'(t) &= u^3(t) - t^3, \\ u(1) &= 1. \end{aligned}$$

Hint: Use Problem E1.3.

E1.5 **Maximal interval of existence**

(a) Solve the IVP

$$\begin{aligned} u'(t) &= e^{u(t)} \sin(t), \\ u(0) &= 0, \end{aligned}$$

and determine the maximal interval of existence of the solution. What is the behavior of the solution as t tends to the endpoints of this interval?

(b) Find all solutions of the ODE

$$u'(t) = \left(2t + \frac{1}{t^2}\right) u(t)$$

and determine their interval of existence.

E1.6 **Comparison of solutions**

Let $u, v : [0, \infty) \to \mathbb{R}$ be continuously differentiable, $a, b, \hat{b} : [0, \infty) \to \mathbb{R}$ be continuous and

$$u'(t) + a(t)u(t) = b(t), \qquad v'(t) + a(t)v(t) = \hat{b}(t) \le b(t).$$

Show that $v(0) \le u(0)$ implies that $v(t) \le u(t)$ for all $t \ge 0$.

E1.7 **Bernoulli equation**

Let $I \subset \mathbb{R}$ be an interval, $\alpha \in \mathbb{R} \setminus \{0, 1\}$ and let $a, b : I \to \mathbb{R}$ be continuous functions. We consider the ODE

$$u'(t) = a(t)u(t) + b(t)u^{\alpha}(t). \tag{1.11}$$

(a) Show that one can transform the ODE (1.11) via the ansatz $w(t) = u^\beta(t)$ with a suitable exponent $\beta \in \mathbb{R}$ into a linear ODE.

(b) Solow's model describes the growth of an economy. It assumes that a single good (output Q) is produced using two factors of production, labor L and capital K. It assumes that

$$K'(t) = \mu Q(L(t), K(t)),$$

where $\mu > 0$, i.e. a constant fraction of Q is saved, and

$$L'(t) = \lambda L(t),$$

where $\lambda > 0$, i.e. the labor grows exponentially. Moreover, the output is given by the Cobb-Douglas production function,

$$Q(L, K) = K^\alpha L^{1-\alpha}, \qquad \alpha \in (0, 1).$$

Show that the capital-labor fraction $\kappa = \frac{K}{L}$ satisfies

$$\kappa'(t) = -\lambda\kappa(t) + \mu\kappa^\alpha(t).$$

Determine the solution corresponding to the initial value $\kappa(0) = \kappa_0 > 0$ and determine the behavior of $\kappa(t)$ as $t \to \infty$.

E1.8 **Exact ODE**

Determine the solution of the IVP

$$3t^2u(t) + 8tu^2(t) + (t^3 + 8t^2u(t) + 12u^2(t))u'(t) = 0, \qquad u(0) = 1.$$

Remark: You obtain an equation that determines the solution implicitly.

E1.9 **Integrating factor**

Consider the ODE

$$(2t^2 + 2tu^2(t) + 1)u(t) + (3u^2(t) + t)u'(t) = 0.$$

Verify that the equation is not exact. Then, assume that there exists an integrating factor μ that depends on t but is independent of u. Compute μ and then, use μ to find an implicit equation that determines the solution u of the original ODE.

E1.10 **Bug on a rubber band**

A rubber band with initial length $L > 0$ has one end tied to a wall in $x = 0$ and the other end is attached to a car. At $t = 0$, the car starts driving away from the wall at a constant speed $v > 0$ (assume that the rubber band stretches uniformly). At the same time, a bug located at $x = 0$ begins to crawl along the rubber band toward the car, with constant speed u relative to the band.

(a) Argue that the following IVP describes the position x of the bug,

$$x'(t) = \frac{v}{L + vt}x(t) + u, \qquad x(0) = 0.$$

(b) Will the bug reach the car and if so, at what time?

Chapter 2

Scalar ODEs: some qualitative properties

2.1 Direction field

First order scalar ODEs can be interpreted geometrically and "solved" graphically.

Definition 2.1. Let $D \subset \mathbb{R}^2$. We consider the first order scalar ODE

$$u'(t) = f(t, u(t)), \tag{2.1}$$

where $f : D \to \mathbb{R}$ is continuous. The **direction field** $v : D \to \mathbb{R}^2$ of the ODE is defined by

$$v(t, u) = \begin{pmatrix} 1 \\ f(t, u) \end{pmatrix}, \qquad (t, u) \in D.$$

An **isocline** of the ODE is a curve along which the direction field has a constant value, i.e. $f(t, u) = c$, for some $c \in \mathbb{R}$.

Assume that $u : I \to \mathbb{R}$ is a solution of (2.1) on an interval $I \subset \mathbb{R}$. Then, the slope $u'(\hat{t})$ of the function u at time $\hat{t} \in I$ is $u'(\hat{t}) = f(\hat{t}, \hat{u})$, where $\hat{u} = u(\hat{t})$. Hence, v is a vector field such that the vector $v(\hat{t}, \hat{u})$ is tangential to the solution curve $u : I \to \mathbb{R}$ in every point $(\hat{t}, \hat{u}) \in D$.

By drawing the direction field we can sketch the graph of the solution by following the curves determined by the vector field. This allows to get an impression of the qualitative behavior of solutions even if we cannot explicitly solve the ODE.

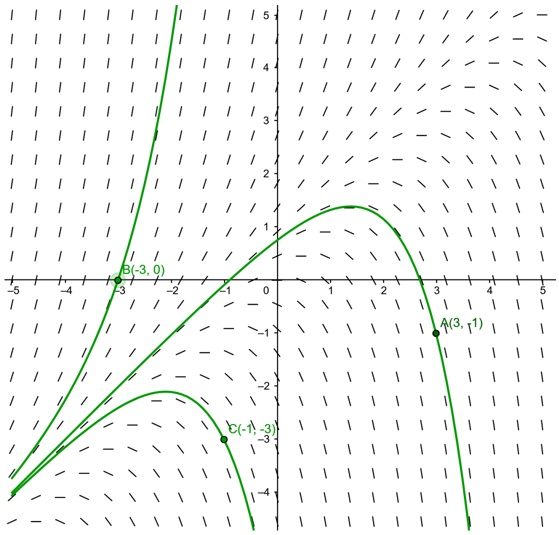
Example 2.2. We consider the ODE

$$u'(t) = u(t) - t.$$

The isoclines correspond to the family of curves

$$u(t) = c + t, \qquad c \in \mathbb{R}, t \in R.$$

Along these lines the derivative of the solution is constant and equals c. Plotting the direction field we can sketch the solution curves as illustrated in the figure below.



2.2 Orthogonal trajectories

In physical applications it is often necessary to find *orthogonal trajectories* for a given family of curves, i.e. curves that intersect each member of a family of curves at right angles. For instance, a charged particle moving through a magnetic field travels along a curve that is perpendicular to the magnetic field lines.

We consider a family of curves in $\mathbb{R}^2$ given by

$$F(t,u) = c, \qquad (t,u) \in D, \ c \in \mathbb{R}, \tag{2.2}$$

where $D \subset \mathbb{R}^2$ is open and $F : D \to \mathbb{R}$ is continuously differentiable.

For instance, the family of parabolas with maximum/minimum in $(0,0)$, including the t-axis, is given by

$$\left\{(t,u) \in \mathbb{R}^2 : u = ct^2\right\}_{c\in\mathbb{R}}.$$

These curves cover $\mathbb{R}^2 \setminus (\{0\}\times\mathbb{R})$ and, in fact, through each point $(t,u) \in \mathbb{R}^2 \setminus (\{0\}\times\mathbb{R})$ runs exactly one curve. The curves can be parametrized by $u_c : \mathbb{R} \to \mathbb{R}$, $u_c(t) = ct^2$. Writing the equation as

$$\frac{u_c(t)}{t^2} = c, \qquad t \neq 0,$$

we observe that u_c satisfies the exact ODE

$$\frac{d}{dt}\left(\frac{u_c(t)}{t^2}\right) = \frac{u_c'(t)}{t^2} - 2\frac{u_c(t)}{t^3} = 0.$$

We aim to find orthogonal trajectories for this family.

To this end we consider the general family of curves in (2.2), i.e.

$$F(t,u) = c, \qquad (t,u) \in D, \ c \in \mathbb{R}.$$

If $t \mapsto u(t), t \in I, I \subset \mathbb{R}$ and interval, is a local parametrization, i.e. $F(t,u(t)) = c$, then

$$0 = \frac{d}{dt}\left(F(t,u(t))\right) = \partial_t F(t,u(t)) + \partial_u F(t,u(t))u'(t). \tag{2.3}$$

Let the orthogonal trajectories be parametrized by $t \mapsto w(t)$. If the curves u and w intersect in a point $(t, u) \in D$, then we have

$$u = u(t) = w(t),$$
$$\begin{pmatrix} 1 \\ u'(t) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ w'(t) \end{pmatrix} = 0.$$

The last equation reflects that the tangential vectors of u and w at time t are perpendicular and we conclude that $u'(t) = -\frac{1}{w'(t)}$. Inserting these two conditions in (2.3) we obtain

$$0 = \partial_t F(t, w(t)) + \partial_u F(t, w(t)) \frac{(-1)}{w'(t)},$$

i.e. w satisfies the ODE

$$\partial_t F(t, w(t)) w'(t) = \partial_u F(t, w(t)).$$

Example 2.3. We consider again the family of parabolas

$$\left\{(t, u) \in \mathbb{R}^2 : u = ct^2\right\}_{c \in \mathbb{R}}.$$

Then, $F(t, u) = \frac{u}{t^2}$, and hence, the orthogonal trajectories satisfy the ODE

$$-2\frac{w(t)}{t^3} w'(t) = \frac{1}{t^2}.$$

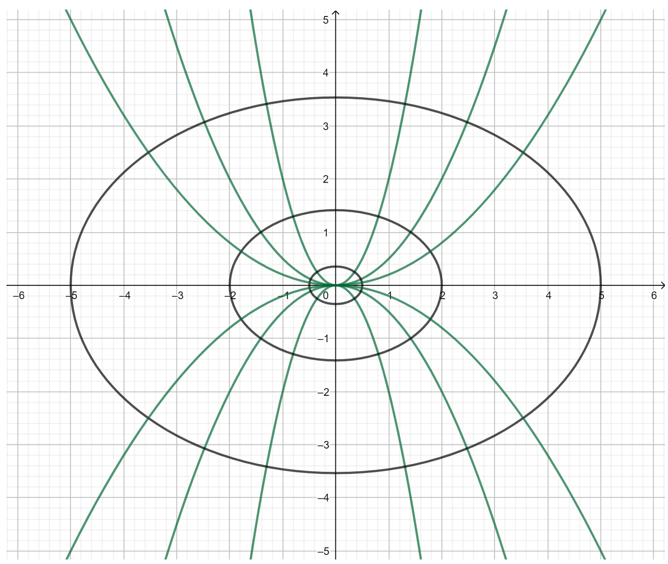
The equation can be rewritten as

$$w'(t) = -\frac{1}{2}\frac{t}{w(t)}.$$

The ODE is separable and can be solved explicitly. In fact, the (local) solutions are determined by

$$w^2(t) = \frac{1}{2}(-t^2 + d^2), \qquad \text{for some } d \in \mathbb{R}.$$

Thus, the family of ellipses are the orthogonal trajectories to the family of parabolas.



2.3 Regularity of solutions

When we speak about the *regularity of solutions* of ODEs we mean how often the solutions are (continuously) differentiable.

We show that the higher the regularity of the function f on the right hand side of an ODE is, the higher is the regularity of the solution.

Proposition 2.4. Consider the ODE (2.1) and assume that a solution u exists. If f is continuous, then u is continuously differentiable.

If f is k times continuously differentiable, then u is $k+1$ times continuously differentiable.

Proof. Let $f : D \to \mathbb{R}$ be continuous and $u : I \to \mathbb{R}$, $I \subset \mathbb{R}$ an interval, be a solution of the ODE. Then, u is continuous (as u is differentiable) and this implies the continuity of $t \mapsto f(t, u(t))$ (as the composition of continuous functions is continuous). Consequently, the function $t \mapsto u'(t) = f(t, u(t))$ is continuous, i.e. u is continuously differentiable.

The second statement can be shown by the same arguments and by induction in $k \in \mathbb{N}$. □

In this section we consider *autonomous* ODEs,

$$u'(t) = f(u(t)), \tag{2.4}$$

where $f : J \to \mathbb{R}$ is continuous on an interval $J \subset \mathbb{R}$.

We show that every solution of an autonomous ODE is monotone. Note that in the following proposition we do not require the uniqueness of solutions.

Proposition 2.5. Let $u : I \to \mathbb{R}$ be a solution of the autonomous ODE (2.4), then u is monotone.

Proof. By Proposition 2.4 we know that the solution u is continuously differentiable.

We argue by contradiction. Let us assume that u is not monotone. Then, there exist $a, b \in I$ such that $u'(a) > 0$ and $u'(b) < 0$. Without loss of generality we assume that $a < b$. Choose $\hat{t} \in [a, b]$ such that

$$u(\hat{t}) = \max_{t \in [a,b]} \{u(t)\}.$$

Then, $\hat{t} \in (a, b)$ and $u(\hat{t}) > \max\{u(a), u(b)\}$, as $u'(a) > 0$ and $u'(b) < 0$.

If $u(b) < u(a)$, let $M := \{t \in (\hat{t}, b] : u(t) = u(a)\}$. By the intermediate value theorem, $M \neq \emptyset$ and we conclude that $b_0 = \min\{M\} > \hat{t}$. Moreover, since $u(t) > u(a)$ for $t \in [\hat{t}, b_0)$, it follows that $u'(b_0) \leq 0 < u'(a)$.

Finally, $u(a) = u(b_0)$ and the ODE now imply that

$$0 < u'(a) = f(u(a)) = f(u(b_0)) = u'(b_0) \leq 0,$$

which is a contradiction.

If $u(a) = u(b)$ we choose $b_0 = b$ and obtain the same contradiction.

Otherwise, if $u(b) > u(a)$ let $M := \{t \in [a, \hat{t}) : u(t) = u(b)\}$. By the intermediate value theorem $M \neq \emptyset$ and we conclude that $a_0 = \max\{M\} < \hat{t}$. Moreover, since $u(t) > u(a_0)$ for $t \in (a_0, \hat{t}]$, it follows that $u'(a_0) \geq 0 > u'(b)$. As before, the ODE leads to a contradiction,

$$0 \leq u'(a_0) = f(u(a_0)) = f(u(b)) = u'(b) < 0.$$

□

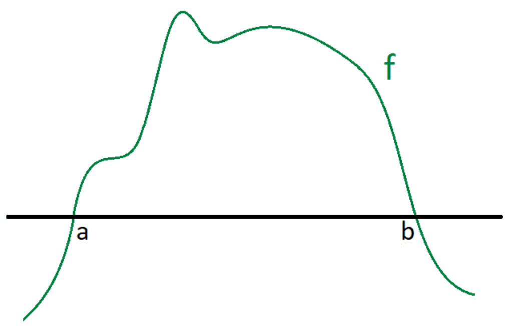
Example 2.6. Consider the IVP

$$u'(t) = f(u(t)), \qquad u(0) = u_0,$$

where $u_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous. We assume that f has the following properties:

$$\begin{aligned} &f(a) = 0, \qquad f(b) = 0, \\ &f(u) > 0 \qquad \text{if } a < u < b, \\ &f(u) < 0 \qquad \text{if } u < a \text{ or } u > b, \end{aligned}$$

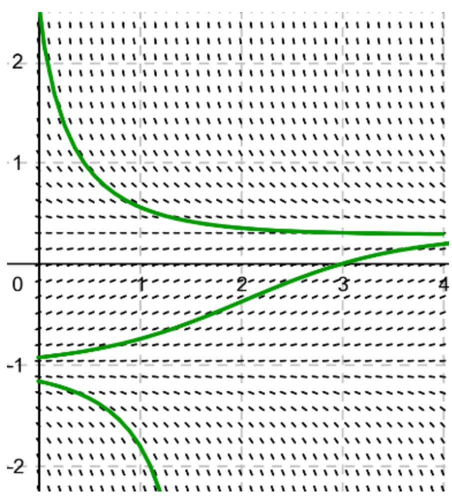
where $a, b \in \mathbb{R}$ with $a < b$.



We observe that $u_1^* = a$ and $u_2^* = b$ are steady states, and Theorem 1.6 implies that the IVP has a unique solution for every $u_0 \in \mathbb{R} \setminus \{a, b\}$. All non-constant solutions are strictly monotone and the sign of f determines whether the solution is increasing or decreasing.

In fact, if $u_0 > b$ the solution is strictly monotone decreasing and converges to b as $t \to \infty$. If $a < u_0 < b$ the solution is strictly monotone increasing and also converges to b as $t \to \infty$. Finally, if $u_0 < a$, the solution is strictly monotone decreasing (depending on the behavior of f the solution may only exist for a finite time). Hence, the qualitative behavior (steady states and monotonicity) of solutions does not depend on the precise form of f.

As an example the direction field and a few solution curves are plotted below for the function $f(u) = (0.3 - u)(u + 1)$.



2.4 Exercises

E2.1 **Asymptotic behavior**

(a) Let $t_0 \in \mathbb{R}$, the function $g : [t_0, \infty) \to \mathbb{R}$ be differentiable and let the limit

$$\lim_{t\to\infty} g'(t) =: a, \qquad a \in \mathbb{R},$$

exist. Show that if g is bounded, then $a = 0$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, $t_0, u_0 \in \mathbb{R}$ and let $u : [t_0, \infty) \to \mathbb{R}$ be a solution of the IVP

$$\begin{aligned} u'(t) &= f(u(t)), \\ u(t_0) &= u_0. \end{aligned} \tag{2.5}$$

Show that if the limit $\lim_{t\to\infty} u(t) =: a$ exists then $u \equiv a$ is a steady state solution of the IVP (2.5) on $I = [t_0, \infty)$.

E2.2 **Direction field**

Sketch the direction field for the ODE

$$u'(t) = t^2 + 4u^2(t),$$

and several isoclines, i.e. curves along which the slope is constant. Without solving the ODE explicitly, sketch the solution of the IVP with initial value $u(0) = 1$.

E2.3 **An autonomous ODE**

Consider the IVP

$$u'(t) = (e^{u(t)} - 1)(u^2(t) - 1), \qquad u(0) = u_0 \in \mathbb{R}.$$

Discuss the qualitative behavior of solutions (steady states and monotonicity) and sketch the graphs of the solutions for the following cases:

$$\begin{array}{llll} u_0 < -1, & u_0 = -1, & u_0 \in (-1, 0), & u_0 = 0, \\ u_0 \in (0, 1), & u_0 = 1, & u_0 > 1. & \end{array}$$

E2.4 **Differential inequality**

Let $b > a$ and $u : [a, b] \to \mathbb{R}$ be differentiable.

(a) Assume that there exists $\lambda \in \mathbb{R}$ such that

$$u'(t) \leq \lambda u(t) \qquad \forall t \in [a, b].$$

Show that the function $g : [a, b] \to \mathbb{R}$, $g(t) = u(t)e^{-\lambda t}$, is monotone decreasing.
Conclude that $u(a) = 0$ implies that $u(t) \leq 0$ and $u(b) = 0$ implies that $u(t) \geq 0$ for all $t \in [a, b]$.

(b) Assume that $u \geq 0$ in $[a, b]$ and that

$$|u'(t)| \leq \lambda u(t) \qquad \forall t \in [a, b].$$

Show that if there exists $t_0 \in [a, b]$ such that $u(t_0) = 0$, then $u \equiv 0$ in $[a, b]$.

E2.5 **Finite time existence**

In the lecture we have seen that solutions may cease to exist as u blows up in finite time. Solutions can also cease to exist because u' tends to plus or minus infinity (without u blowing up). An example is given below.

Determine the solutions of the ODE

$$u'(t) = \frac{\lambda}{u(t)},$$

where $\lambda \in \mathbb{R}$. What is the maximal interval of existence? Sketch the graph of the solutions for the cases $\lambda > 0$ and $\lambda < 0$.

Chapter 3

Existence and uniqueness

Most IVPs cannot be solved explicitly, but we can often study qualitative properties of solutions or approximate solutions and compute them numerically. However, given an IVP we first need to show that a solution exists and that the solution is unique.

We aim to specify conditions that ensure the existence and uniqueness of solutions for IVPs of the form

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(t_0) &= u_0, \end{aligned} \tag{3.1}$$

where $f : D \to \mathbb{R}^n$ is a continuous function defined on $D \subset \mathbb{R} \times \mathbb{R}^n$ and $(t_0, u_0) \in D$. Recall that $u : I \to \mathbb{R}^n$ is a solution of the IVP (3.1) if $I \subset \mathbb{R}$ is an interval containing t_0, the function u is differentiable, $(t, u(t)) \in D$ for all $t \in I$, u satisfies the ODE and $u(t_0) = u_0$.

In previous examples we have seen that solutions may not be unique (Example 1.7), a solution may only exist for a finite time as the solution u blows up (Example 1.3) or because the derivative of the solution u' blows up (see Problem 2.4). It can also happen that a solution does not exist for particular initial values, see the example below.

Example. Consider the IVP

$$u'(t) = \frac{u(t)}{t}, \qquad u(0) = 1.$$

The ODE is separable, and the solutions are of the form $u(t) = ct$, for some constant $c \in \mathbb{R}$. However, there does not exist a solution with $u(0) = 1$. In fact, the function $f(t, u) = \frac{u}{t}$ is not continuous and not even defined in $t = 0$.

We will prove a general existence and uniqueness result for solutions of the IVP (3.1). The strategy to prove existence is to first derive an equivalent integral representation for the solution. Using this integral representation we construct a sequence of approximate solutions. We show that the approximate solutions converge and finally, that the limit is indeed a solution of the IVP.

3.1 Picard iteration

Instead of considering the IVP (3.1) we can also look for solutions of the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds, \qquad t \in I. \tag{3.2}$$

In fact, u satisfies the integral equation (3.2) if and only if u is a solution of the IVP (3.1) on I.

Lemma 3.1. *Let $D \subset \mathbb{R} \times \mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ be a continuous function. Then, $u : I \to \mathbb{R}^n$ is a solution of the IVP* (3.1) *if and only if $u : I \to \mathbb{R}^n$ is continuous and satisfies the integral equation* (3.2).

Proof. If u is a solution of the IVP on $I \subset \mathbb{R}$ and f is continuous, then u is continuously differentiable on I (see Proposition 2.4). Integrating the ODE from t_0 to $t \in I$, it follows that u satisfies (3.2).

On the other hand, if u is continuous and satisfies (3.2), then u is continuously differentiable, $u(t_0) = u_0$ and differentiating the equation it follows that u satisfies the ODE. □

We notice that if u and f are continuous, the right hand side of the integral equation (3.2) defines a continuously differentiable function. We now use this representation to define a sequence $(u_k)_{k \in \mathbb{N}_0}$ of continuous functions by

$$\begin{aligned} u_0(t) &\equiv u_0 \\ u_{k+1}(t) &= u_0 + \int_{t_0}^{t} f(s, u_k(s))ds, \end{aligned} \tag{3.3}$$

for $t \in [t_0, T]$, where $T > t_0$. If f has suitable properties this sequence converges to a function $u : [t_0, T] \to \mathbb{R}^n$ as $k \to \infty$ that satisfies (3.2). Hence, $(u_k)_{k \in \mathbb{N}_0}$ approximates the solution of the IVP (3.1). It is called the *sequence of successive iterations* or the Picard *iteration*. Before we state and prove the main result we introduce a few notions from functional analysis.

3.2 Preliminaries

Let $t_0 \in \mathbb{R}$, $T > t_0$ and let $C^0([t_0, T]; \mathbb{R}^n)$ denote the space of continuous functions $u : [t_0, T] \to \mathbb{R}^n$. We define a mapping (called an *operator*)

$$\mathcal{T} : C^0([t_0, T]; \mathbb{R}^n) \to C^0([t_0, T]; \mathbb{R}^n), \quad u \mapsto \mathcal{T}u = v,$$

by

$$\mathcal{T}u(t) = v(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds, \qquad t \in [t_0, T]. \tag{3.4}$$

Then, by Lemma 3.1 the function u is a solution of (3.1) on $[t_0, T]$ if and only if u satisfies

$$\mathcal{T}u = u,$$

i.e. u is a *fixed point* of the operator $\mathcal{T}$. Hence, showing the existence of a solution of the IVP is equivalent to showing the existence of a fixed point of the operator $\mathcal{T}$.

Definition 3.2. Let V be a real vector space. A **norm** $\|\cdot\|$ is a mapping from V to $[0, \infty)$ with the following properties:

- $\|v\| = 0 \iff v = 0 \quad \forall v \in V$

- $||\alpha v|| = |\alpha|\,||v|| \quad \forall \alpha \in \mathbb{R}, v \in V$
- $||u + v|| \leq ||u|| + ||v|| \quad \forall\, u, v \in V$ (*triangle inequality*)

A **normed space** $(V, ||\cdot||)$ is a vector space V with a norm $||\cdot||$.

Definition 3.3. Let $(V, ||\cdot||)$ be a normed space and $(v_k)_{k\in\mathbb{N}_0}$ be a sequence in V.

- The sequence $(v_k)_{k\in\mathbb{N}_0}$ is **convergent** if there exists $v \in V$ such that

$$\forall \varepsilon > 0\ \exists K \in \mathbb{N}_0 : \qquad ||v_k - v|| < \varepsilon \quad \forall k \geq K.$$

- The sequence $(v_k)_{k\in\mathbb{N}}$ is a **Cauchy sequence** if

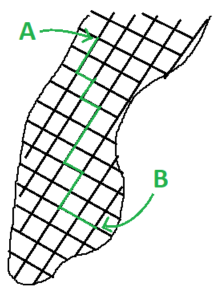
$$\forall \varepsilon > 0\ \exists K \in \mathbb{N}_0 : \qquad ||v_k - v_l|| < \varepsilon \quad \forall k, l \geq K.$$

We recall that every convergent sequence is a Cauchy sequence, but in general, not every Cauchy sequence converges.

Example 3.4. The Euclidean space $\mathbb{R}^n$ is a normed space. For instance, we can consider the following norms in $\mathbb{R}^n$: Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

- Euclidean norm: $||v||_{euc} = \sqrt{\sum_{i=1}^n v_i^2}$
- Maximum norm: $||v||_{max} = \max_{i=1,\ldots,n}\{|v_i|\}$
- Manhatten norm: $||v||_{man} = \sum_{i=1}^n |v_i|$

  The Manhatten norm is also called taxicab norm. The name refers to the grid layout of most streets on the island of Manhattan which causes that the shortest path a taxi can take from one intersection to another equals the distance of the two intersections in the Manhatten norm. The distance between two vectors in this norm is the sum of the absolute values of the differences of the components of the vectors.



Remark 3.5. In $\mathbb{R}^n$ all norms are *equivalent*, i.e. if $||\cdot||_1$ and $||\cdot||_2$ are two norms, then there exist constants $a, b > 0$ such that

$$a||v||_1 \leq ||v||_2 \leq b||v||_1 \qquad \forall v \in V.$$

In particular, this implies that the convergence of a sequence in one norm implies the convergence of the sequence in the other norm. This is generally not the case in other normed spaces.

Example 3.6. Let $t_0 \in \mathbb{R}$, $T > t_0$ and let $\|\cdot\|$ be an arbitrary norm in $\mathbb{R}^n$. The space $C^0([t_0, T]; \mathbb{R}^n)$ of continuous function on $[t_0, T]$ is a normed space. For instance, we can consider the following norms: Let $u \in C^0([t_0, T]; \mathbb{R}^n)$.

- Maximum norm: $\|u\|_{max} = \max_{t\in[t_0,T]}\{\|u(t)\|\}$
- Exponentially weighted norm: $\|u\|_{exp} = \max_{t\in[t_0,T]}\{\|u(t)\|e^{-\lambda t}\}, \quad \lambda > 0$
- Integral norm: $\|u\|_{int} = \int_{t_0}^{T} \|u(t)\|dt$

The maximum norm and the exponentially weighted norm are equivalent, since

$$e^{-\lambda T}\|u\|_{max} \leq \|u\|_{exp} \leq e^{-\lambda t_0}\|u\|_{max} \qquad \forall u \in C^0([t_0, T]; \mathbb{R}^n).$$

However, the integral norm and the maximum norm are *not equivalent* as we only have

$$\|u\|_{int} \leq (T - t_0)\|u\|_{max} \quad \forall u \in C^0([t_0, T]; \mathbb{R}^n),$$

but the second inequality does not hold.

Definition 3.7. A normed space $(V, \|\cdot\|)$ is called a **Banach space** if it is *complete*, i.e. if every Cauchy sequence converges to an element $v \in V$.

The Euclidean space $\mathbb{R}^n$ with an arbitrary norm is a Banach space. Another example is the space of continuous functions $C^0([t_0, T]; \mathbb{R}^n)$ with the maximum norm. Before proving this statement we recall that a sequence of functions $f_k : [t_0, T] \to \mathbb{R}^n, k \in \mathbb{N}_0$, *converges uniformly* to $f : [t_0, T] \to \mathbb{R}^n$ if $\forall \varepsilon > 0\ \exists K \in \mathbb{N}_0$ such that

$$\|f_k(t) - f(t)\| < \varepsilon \qquad \forall t \in [t_0, T],\ k \geq K.$$

Proposition 3.8. Let $t_0 \in \mathbb{R}$ and $T > t_0$. Then, $C^0([t_0, T]; \mathbb{R}^n)$ with the maximum norm $\|\cdot\|_{max}$ is a Banach space.

Proof. Let $(u_k)_{k\in\mathbb{N}_0}$ be a Cauchy sequence in $C^0([t_0, T]; \mathbb{R}^n)$ and $\varepsilon > 0$. Then, $\exists K \in \mathbb{N}_0$ such that

$$\max_{t\in[t_0,T]} \|u_k(t) - u_l(t)\| < \frac{\varepsilon}{2} \qquad \forall k, l \geq K.$$

Consequently, for every $t \in [t_0, T]$ it follows that

$$\|u_k(t) - u_l(t)\| < \frac{\varepsilon}{2} \quad \forall k, l \geq K,$$

i.e. $(u_k(t))_{k\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is complete, the limit $\lim_{k\to\infty} u_k(t) = u(t)$ exists $\forall t \in [t_0, T]$ and defines a function $u : [t_0, T] \to \mathbb{R}^n$. Moreover, we observe that for all $t \in [t_0, T]$ and $k \geq K$ we have

$$\|u(t) - u_k(t)\| = \lim_{l\to\infty} \|u_l(t) - u_k(t)\| \leq \frac{\varepsilon}{2} < \varepsilon,$$

which shows that the sequence $u_k : [t_0, T] \to \mathbb{R}^n, k \in \mathbb{N}_0$, converges uniformly.

Finally, from Analysis 1 we know that if $u_k : [t_0, T] \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to a function $u : [t_0, T] \to \mathbb{R}$ then u is continuous. This statement can be easily be generalized for vector-valued functions $[t_0, T] \to \mathbb{R}^n$ by replacing the absolute value by a norm in $\mathbb{R}^n$.

Hence, the limit u is continuous, i.e. $u \in C^0([t_0, T]; \mathbb{R}^n)$ which concludes the proof. □

Since the norms $\|\cdot\|_{max}$ and $\|\cdot\|_{exp}$ are equivalent, it follows that $C^0([t_0, T]; \mathbb{R}^n)$ with $\|\cdot\|_{exp}$ is also a Banach space.

We saw that the IVP (3.1) can be reformulated as the integral equation (3.2), and showing that a solution of (3.1) exists is equivalent to showing that the operator $\mathcal{T}$ in (3.4) has a fixed point. We will show the latter by applying Banach's Fixed Point Theorem for contraction mappings. It provides a sequence of successive approximations that converges to the unique fixed point.

Theorem 3.9 (Banach's Fixed Point Theorem)**.** *Let $A \subset V$ be a non-empty, closed subset of a Banach space $(V, \|\cdot\|)$. We assume that $\mathcal{T} : A \to V$ is a contraction mapping, i.e. there exists $\theta \in [0, 1)$ such that*

$$\|\mathcal{T}(u) - \mathcal{T}(v)\| \leq \theta\|u - v\| \qquad \forall u, v \in A, \tag{3.5}$$

and $\mathcal{T}(A) \subset A$. Then, there exists a unique fixed point $v^* \in A$ of $\mathcal{T}$, i.e. $\mathcal{T}(v^*) = v^*$.

Moreover, for every $v_0 \in A$ the sequence $(v_k)_{k\in\mathbb{N}_0}$ of successive approximations,

$$v_{k+1} = \mathcal{T}(v_k), \quad k \in \mathbb{N}_0,$$

converges to v^* as $k \to \infty$. In particular, the following estimate holds,

$$\|v_k - v^*\| \leq \frac{\theta^k}{1-\theta}\|v_0 - v_1\| \qquad \forall k \in \mathbb{N}. \tag{3.6}$$

Proof. By induction it follows that $v_k \in A$ for all $k \in \mathbb{N}_0$, as $\mathcal{T}(A) \subset A$ and $v_0 \in A$.

Next, we show by induction that

$$\|v_{k+1} - v_k\| \leq \theta^k\|v_1 - v_0\| \qquad \forall k \in \mathbb{N}_0. \tag{3.7}$$

For $k = 0$ it is certainly satisfied. We assume that the inequality has been shown for $k \in \mathbb{N}$. Then, using (3.5) for $k + 1$ we obtain

$$\|v_{k+2} - v_{k+1}\| = \|\mathcal{T}(v_{k+1}) - \mathcal{T}(v_k)\| \leq \theta\|v_{k+1} - v_k\| \leq \theta^{k+1}\|v_1 - v_0\|,$$

where we used (3.7) in the last step. Consequently, (3.7) holds for $k+1$ which concludes the proof of this estimate.

Let now $l > k$, then

$$\begin{aligned} \|v_l - v_k\| &= \Big\|\sum_{i=k}^{l-1}(v_{i+1} - v_i)\Big\| \leq \sum_{i=k}^{l-1}\|v_{i+1} - v_i\| \overset{(3.7)}{\leq} \|v_1 - v_0\|\sum_{i=k}^{l-1}\theta^i \\ &\leq \|v_1 - v_0\|\theta^k\sum_{i=0}^{\infty}\theta^i \leq \frac{\theta^k}{1-\theta}\|v_1 - v_0\|, \end{aligned} \tag{3.8}$$

where we used the geometric series $\sum_{i=0}^{\infty}\theta^i = \frac{1}{1-\theta}$. Consequently, $(v_k)_{k\in\mathbb{N}_0}$ is a Cauchy sequence and, as V is a Banach space, it converges to an element $v^* \in V$. Moreover, since $v_k \in A$ for all $k \in \mathbb{N}_0$ and A is closed, it follows that $v^* \in A$.

Furthermore, since $v_{k+1} = \mathcal{T}(v_k)$, taking the limit $k \to \infty$ we conclude that

$$v^* = \lim_{k\to\infty} v_{k+1} = \lim_{k\to\infty}\mathcal{T}(v_k) = \mathcal{T}(v^*),$$

i.e. v^* is a fixed point of $\mathcal{T}$. Here, we used that $\lim_{k\to\infty}\mathcal{T}(v_k) = \mathcal{T}(v^*)$, which follows from the estimate $\|\mathcal{T}(v^*) - \mathcal{T}(v_k)\| \le \theta\|v^* - v_k\|$ and the convergence of $(v_k)_{k\in\mathbb{N}_0}$ to v^*.

To show (3.6) we observe that for $l \ge 1$ we have

$$\|v_l - v^*\| = \|v_l - \lim_{k\to\infty} v_k\| = \lim_{k\to\infty}\|v_l - v_k\| \overset{(3.8)}{\le} \frac{\theta^l}{1-\theta}\|v_1 - v_0\|.$$

Finally, to show uniqueness let w^* be another fixed point. Then, it follows that

$$\|v^* - w^*\| = \|\mathcal{T}(v^*) - \mathcal{T}(w^*)\| \overset{(3.6)}{\le} \theta\|v^* - w^*\|,$$

which implies that $\|v^* - w^*\| = 0$, and consequently, $v^* = w^*$. □

3.3 Global existence

Here and in the sequel, $\|\cdot\|$ denotes an arbitrary norm in $\mathbb{R}^n$. To apply Banach's Fixed Point Theorem we need the following lemma.

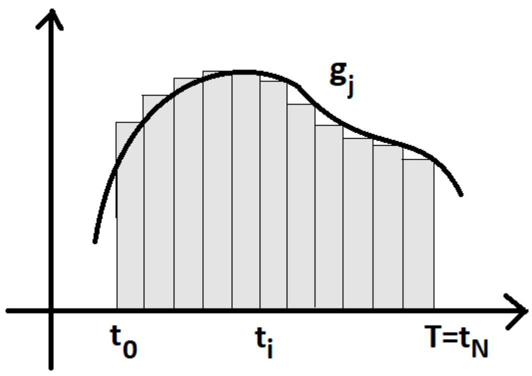
Lemma 3.10. *Let $t_0 \in \mathbb{R}, T > t_0$ and $g : [t_0, T] \to \mathbb{R}^n$ be a continuous function. Then,*

$$\left\| \int_{t_0}^{T} g(s)ds \right\| \le \int_{t_0}^{T} \|g(s)\|ds.$$

Proof. Let $\varepsilon > 0$. We consider an approximation of the integrals by Riemann sums,

$$\int_{t_0}^{T} g_j(t)dt = \sum_{i=1}^{N}(t_i - t_{i-1})g_j(t_i) + \delta_j, \qquad |\delta_j| < \varepsilon,\ j = 1,\dots,n, \tag{3.9}$$

where $t_i = t_0 + i\frac{T-t_0}{N}, i = 1,\dots,N$.



Similarly, as $\|g\| : [t_0, T] \to \mathbb{R}$ is continuous, we have

$$\int_{t_0}^{T} \|g(t)\|dt = \sum_{i=1}^{N}(t_i - t_{i-1})\|g(t_i)\| + \mu, \qquad |\mu| < \varepsilon. \tag{3.10}$$

Let $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$. Then, the triangle inequality and the approximations above imply that

$$\left\| \int_{t_0}^{T} g(t)dt \right\| \overset{(3.9)}{\leq} \|d\| + \sum_{i=1}^{N} (t_i - t_{i-1})\|g(t_i)\| \overset{(3.10)}{\leq} \int_{t_0}^{T} \|g(t)\|dt + \|\delta\| + |\mu|.$$

Finally, since $\|\delta\|$ and $|\mu|$ can be arbitrarily small, the inequality follows. □

We aim to apply Banach's Fixed Point Theorem to show that the operator $\mathcal{T}$ in (3.4) has a unique fixed point. The following Lipschitz condition for f will imply that $\mathcal{T}$ is a contraction.

Definition 3.11. Let $D \subset \mathbb{R}^{n+1}$ and $f : D \to \mathbb{R}^n$ be continuous. We say that f satisfies a **Lipschitz condition** in D if there exists $L > 0$ such that

$$\|f(t,u) - f(t,v)\| \leq L\|u - v\| \qquad \forall (t,u), (t,v) \in D. \tag{3.11}$$

The constant $L > 0$ is called the **Lipschitz constant** of f.

Theorem 3.12. *Let $t_0 \in \mathbb{R}, T > t_0$ and $u_0 \in \mathbb{R}^n$. We assume that the function $f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the Lipschitz condition* (3.11) *in $D = [t_0, T] \times \mathbb{R}^n$. Then, there exist a unique solution $u : [t_0, T] \to \mathbb{R}^n$ of the IVP* (3.1)*,*

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(t_0) &= u_0. \end{aligned}$$

Moreover, the sequence of successive approximations $(u_k)_{k \in \mathbb{N}_0}$ defined in (3.3)*,*

$$u_{k+1}(t) = u_0 + \int_{t_0}^{t} f(s, u_k(s))ds, \qquad t \in [t_0, T],\ k \in \mathbb{N}_0,$$

converges uniformly to u on $[t_0, T]$.

Proof. We consider the Banach space of continuous functions $C^0([t_0, T]; \mathbb{R}^n)$ with the exponentially weighted norm $\|u\|_{exp} = \max_{t \in [t_0, T]}\{\|u(t)\|e^{-2Lt}\}$. We could equally work with the maximum norm, but using the exponentially weighted norm turns out to be more convenient.

We aim to apply Banach's fixed point theorem (Theorem 3.9) to show that the mapping $\mathcal{T} : C^0([t_0, T]; \mathbb{R}^n) \to C^0([t_0, T]; \mathbb{R}^n)$ defined in (3.2),

$$\mathcal{T}(u)(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds, \qquad u \in C^0([t_0, T]; \mathbb{R}^n),$$

has a unique fixed point. To show that $\mathcal{T}$ is a contraction in $(C^0([t_0, T]; \mathbb{R}^n), \|\cdot\|_{exp})$ let $u, v \in C^0([t_0, T]; \mathbb{R}^n)$. Then, Lemma 3.10 and the Lipschitz condition imply that

$$\begin{aligned} \|\mathcal{T}(u)(t) - \mathcal{T}(v)(t)\| &= \left\| \int_{t_0}^{t} f(s, u(s)) - f(s, v(s))ds \right\| \\ &\leq \int_{t_0}^{t} \|f(s, u(s)) - f(s, v(s))\|ds \leq \int_{t_0}^{t} L\|u(s) - v(s)\|ds \\ &= L \int_{t_0}^{t} \left\|(u(s) - v(s))e^{-2Ls}\right\| e^{2Ls} ds, \end{aligned}$$

for all $t \in [t_0, T]$. Using the fact that $\|(u(s) - v(s))e^{-2Ls}\| \leq \|u - v\|_{exp}$ for all $s \in [t_0, T]$ it follows that

$$\|\mathcal{T}(u)(t) - \mathcal{T}(v)(t)\| \leq \|u - v\|_{exp} L \int_{t_0}^{t} e^{2Ls} ds \leq \frac{1}{2}\|u - v\|_{exp} e^{2Lt} \qquad \forall t \in [t_0, T].$$

We conclude that

$$\|\mathcal{T}(u)(t) - \mathcal{T}(v)(t)\| e^{-2Lt} \leq \frac{1}{2}\|u - v\|_{exp},$$

and taking the maximum over all $t \in [t_0, T]$ yields

$$\|\mathcal{T}(u) - \mathcal{T}(v)\|_{exp} \leq \frac{1}{2}\|u - v\|_{exp},$$

i.e. $\mathcal{T}$ is a contraction in the Banach space $(C^0([t_0, T]; \mathbb{R}^n), \| \cdot \|_{exp})$.

Hence, by Theorem 3.9 there exists a unique fixed point u^* of $\mathcal{T}$,

$$u^*(t) = u_0 + \int_{t_0}^{t} f(s, u^*(s)) ds,$$

and the sequence of successive approximations (3.3) converges uniformly to u^* on $[t_0, T]$. In fact, $(u_n)_{n \in \mathbb{N}_0}$ converges to u with respect to the exponentially weighted norm and this is equivalent to the convergence with respect to the maximum norm. Moreover, the convergence with respect to the maximum norm implies the uniform convergence (see the proof of Proposition 3.8).

Finally, by Lemma 3.1, the fixed point u^* is the unique solution of the IVP (3.1) on the interval $[t_0, T]$. □

The proof of Theorem 3.12 is *constructive*, i.e. we can explicitly compute the sequence of successive approximations and obtain the solution of the IVP by passing to the limit. However, only in simple cases this method allows us to find an explicit solution (see exercises). For numerical approximations, the Picard iteration is not very suitable and there are other approximation schemes that are easier to implement and that require less computer memory.

The global Lipschitz condition for f in the previous theorem is a restrictive assumption that excludes many nonlinear ODEs. In the next section we relax this hypothesis and prove a local existence and uniqueness result for more general functions f.

3.4 Local existence

We prove the general local existence and uniqueness result in two steps. First, we show local existence for functions f that are Lipschitz continuous in particular subsets.

Theorem 3.13. *Let $t_0 \in \mathbb{R}, \hat{t} > t_0$ and $u_0 \in \mathbb{R}^n$. Moreover, let $D = [t_0, \hat{t}] \times B$, where $B = \{u \in \mathbb{R}^n : \|u - u_0\| \leq r\}$ for some $r > 0$. We assume that $f : D \to \mathbb{R}^n$ is continuous and satisfies the Lipschitz condition* (3.11) *in D.*

Then, there exist a unique local solution u of the IVP

$$u'(t) = f(t, u(t)),$$
$$u(t_0) = u_0.$$

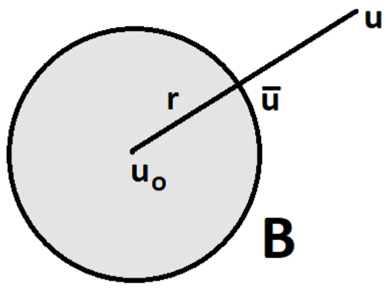
The solution exists on $[t_0, T]$, *where* $T = \hat{t}$ *or*

$$T = \frac{r}{\max_{(t,u)\in D} \|f(t,u)\|}.$$

Proof. First, we construct a function $\tilde{f} : [t_0, \hat{t}] \times \mathbb{R}^n \to \mathbb{R}^n$ that coincides with f on D and satisfies the global Lipschitz condition (3.11). To this end let

$$\tilde{f}(t,u) = \begin{cases} f(t,u) & (t,u) \in D, \\ f\left(t, u_0 + r\frac{u-u_0}{\|u-u_0\|}\right) & (t,u) \in [t_0, \hat{t}] \times (\mathbb{R}^n \setminus B). \end{cases}$$

Note that if $u \notin B$ then $\bar{u} = u_0 + r\frac{u-u_0}{\|u-u_0\|}$ satisfies $\|\bar{u} - u_0\| = r$.



If $u, v \in B$ the Lipschitz continuity of f in D immediately implies that

$$\|\tilde{f}(t,u) - \tilde{f}(t,v)\| \leq L\|u - v\|.$$

Next, we assume that $u, v \notin B$. Then, the Lipschitz continuity of f in D implies that

$$\begin{aligned} \|\tilde{f}(t,u) - \tilde{f}(t,v)\| &\leq L \left\| u_0 + r\frac{u-u_0}{\|u-u_0\|} - u_0 - r\frac{v-u_0}{\|v-u_0\|} \right\| \\ &\leq Lr \left\| \frac{u-u_0}{\|u-u_0\|} - \frac{v-u_0}{\|u-u_0\|} + \frac{v-u_0}{\|u-u_0\|} - \frac{v-u_0}{\|v-u_0\|} \right\| \\ &\leq Lr \left(\frac{\|u-v\|}{\|u-u_0\|} + \|v - v_0\| \frac{|\, \|v-u_0\| - \|u-u_0\| \,|}{\|v-u_0\| \, \|u-u_0\|} \right) \\ &\leq Lr\frac{2\|u-v\|}{\|u-u_0\|} \leq 2L\|u-v\|, \end{aligned}$$

where we used that $\|u - u_0\| > r$ and $|\, \|v - u_0\| - \|u - u_0\| \,| \leq \|u - v\|$.

Finally, we assume that $u \in B$ and $v \notin B$. Then, the Lipschitz continuity of f in D implies that

$$\begin{aligned} \|\tilde{f}(t,u) - \tilde{f}(t,v)\| &\leq L \left\| u - u_0 - r\frac{v-u_0}{\|v-u_0\|} \right\| = L \left\| u - v + v - u_0 - r\frac{v-u_0}{\|v-u_0\|} \right\| \\ &\leq L\left(\|u - v\| + \left\| v - u_0 - r\frac{v-u_0}{\|v-u_0\|} \right\| \right) \leq L\left(\|u-v\| + \frac{\|(v-u_0)(\|v-u_0\| - r)\|}{\|v-u_0\|} \right) \\ &= L\left(\|u-v\| + \|v - \bar{v}\|\right) \leq 2L\|u-v\|, \end{aligned}$$

where we used that $u \in B$ and $\operatorname{dist}(v, B) = \|v - \bar{v}\|$ in the last step.

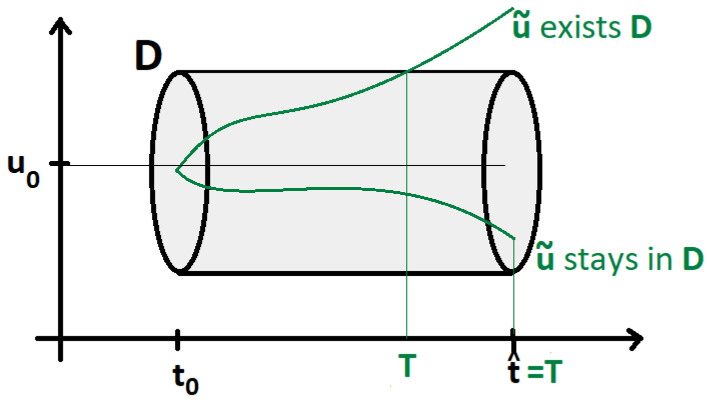
Therefore, combining all cases it follows that $\tilde{f}$ is satisfies the Lipschitz condition on $\widetilde{D} = [t_0, \hat{t}] \times \mathbb{R}^n$ and hence, by Theorem 3.12 there exists a unique solution of the IVP

$$\begin{aligned} \tilde{u}'(t) &= \tilde{f}(t, \tilde{u}(t)), \\ \tilde{u}(t_0) &= u_0, \end{aligned} \tag{3.12}$$

on $[t_0, \hat{t}]$. Moreover, $\tilde{u}$ satisfies the integral equation (3.2), and consequently,

$$\begin{aligned} \|\tilde{u}(t) - u_0\| &\overset{(3.10)}{\leq} \int_{t_0}^{t} \|\tilde{f}(s, \tilde{u}(s))\| ds \leq \sup_{(t,u) \in [t_0, \hat{t}] \times \mathbb{R}^n} \left\{ \|\tilde{f}(t, u)\| \right\} (t - t_0) \\ &\leq \max_{(t,u) \in D} \left\{ \|f(t, u)\| \right\} (t - t_0), \end{aligned} \tag{3.13}$$

for all $t \in [t_0, \hat{t}]$.



Let $T > t_0$ be the largest number such that $\tilde{u}(t) \in D$ for all $t \in [t_0, T]$. Then, the above estimate implies that

$$T = \hat{t} \qquad \text{or} \qquad T \geq t_0 + \frac{r}{\max_{(t,u) \in D} \{\|f(t, u)\|\}}.$$

Since $f = \tilde{f}$ on D this implies that $\tilde{u}$ is a solution of the original IVP on $[t_0, T]$. Conversely, every solution u of the original IVP satisfies the auxiliary problem (3.12) as long as $u(t) \in D$ and hence, it must coincide with $\tilde{u}$ on the interval $[t_0, T]$. □

We now use this theorem to prove the main result. In particular, we show that a local Lipschitz condition of f implies the *local existence* of a solution, i.e. the existence of a solution in a sufficiently small time interval around t_0.

Definition 3.14. Let $D \subset \mathbb{R}^{n+1}$ and the function $f : D \to \mathbb{R}^n$ be continuous. We say that f satisfies a **local Lipschitz condition** in D if for every $(\hat{t}, \hat{u}) \in D$ there exist a neighborhood $U = \left\{ (t, u) \in \mathbb{R}^{n+1} : |t - \hat{t}| \leq \rho,\ \|u - \hat{u}\| \leq \rho \right\}$, for some $\rho > 0$, such that

$$\|f(t, u) - f(t, v)\| \leq L\|u - v\| \qquad \forall (t, u), (t, v) \in U \cap D.$$

In this case, the Lipschitz constant L can depend on the neighborhood U.

The local Lipschitz condition is typically not easy to verify directly. However, there is a property that is easy to check and that implies the local Lipschitz condition.

Let $D \subset \mathbb{R}^m$ be open. We recall that a function $g : D \to \mathbb{R}^n$ is *continuously differentiable* if all partial derivatives $\partial_{y_j} g_i$, $j = 1, \ldots, m, i = 1, \ldots, n$, exist and are continuous on D. In this case, we denote by $g'(y), Dg(y)$ or $J_g(y)$ the Jacobian matrix of g in $y \in D$, i.e.

$$g'(y) = Dg(y) = J_g(y) = \begin{pmatrix} \partial_{y_1} g_1(y) & \cdots & \partial_{y_m} g_1(y) \\ \vdots & \ddots & \vdots \\ \partial_{y_1} g_n(y) & \cdots & \partial_{y_m} g_n(y) \end{pmatrix}.$$

Proposition 3.15. Let $D \subset \mathbb{R}^{n+1}$ be open and $f : D \to \mathbb{R}^n$, $(t, u) \mapsto f(t, u)$, be continuous and continuously differentiable with respect to u in D, i.e. the partial derivatives $\partial_{u_j} f_i$, $i, j = 1, \ldots, n$, exist and are continuous in D. Then, f satisfies a local Lipschitz condition in D.

Proof. Let $(t_0, u_0) \in D$. Since D is open, there exists $\rho > 0$ such that

$$D_0 = \{(t, u) \in \mathbb{R}^{n+1} : |t - t_0| \leq \rho, \ \|u - u_0\| \leq \rho\} \subset D.$$

Moreover, as f is continuously differentiable with respect to u we obtain

$$\begin{aligned} f(t, u) - f(t, v) &= \int_0^1 \frac{d}{d\theta} \left(f(t, v + \theta(u - v)) \right) d\theta \\ &= \int_0^1 D_u f(t, v + \theta(u - v))(u - v) d\theta, \end{aligned}$$

for all $(t, u), (t, v) \in D_0$. Here, $D_u f(y)$ denotes the Jacobian matrix of f with respect to u in y. Taking the maximum norm in $\mathbb{R}^n$ and using Lemma 3.10 it follows that

$$\|f(t, u) - f(t, v)\|_{max} \leq \int_0^1 \|D_u f(t, v + \theta(u - v))(u - v)\|_{max} d\theta.$$

Moreover, for all $i = 1, \ldots, n$ we have

$$\begin{aligned} \Big| \sum_{j=1}^n \partial_{u_j} f_i(t, v + \theta(u - v))(u_j - v_j) \Big| &\leq \max_{i,j=1,\ldots,n} \left\{ \max_{(t,w) \in D_0} |\partial_{u_j} f_i(t, w)| \right\} \sum_{j=1}^n |u_j - v_j| \\ &\leq nC\|u - v\|_{max}, \end{aligned}$$

where

$$C = \max_{i,j=1,\ldots,n} \left\{ \max_{(t,w) \in D_0} |\partial_{u_j} f_i(t, w)| \right\}.$$

Here, we used the continuity of the functions $\partial_{u_j} f_i$. Indeed, as $\partial_{u_j} f_i$ is continuous and the set D_0 is compact, $\partial_{u_j} f_i$ attains a maximum on D_0, for all $i, j = 1, \ldots, n$. Consequently, it follows that

$$\|f(t, u) - f(t, v)\|_{max} \leq \int_0^1 nC\|u - v\|_{max} d\theta = nC\|u - v\|_{max}.$$

This shows that f satisfies a local Lipschitz condition in D_0 with respect to the maximum norm. Since all norms in $\mathbb{R}^n$ are equivalent, it follows that f satisfies a local Lipschitz condition with respect to any norm in $\mathbb{R}^n$. □

Theorem 3.16 (Picard-Lindelöf)**.** *Let $D \subset \mathbb{R}^{n+1}$ be open and $f : D \to \mathbb{R}^n$ be continuous. If f satisfies a local Lipschitz condition in D, then for every $(t_0, u_0) \in D$ there exist a unique solution u of the IVP*

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(t_0) &= u_0, \end{aligned}$$

on an interval $[t_0 - \delta, t_0 + \delta]$, for some $\delta > 0$.

Proof. Let $(t_0, u_0) \in D$. Since D is open and f satisfies a local Lipschitz condition in D, there exist $\rho > 0$ such that $D_0 = \{(t, u) : |t - t_0| \le \rho, \|u - u_0\| \le \rho\} \subset D$ and f satisfies the Lipschitz condition in D_0. Consequently, Theorem 3.13 implies that there exists a unique solution u_+ of the IVP on an interval $[t_0, t_0 + \hat{\delta}]$, for some $\hat{\delta} > 0$.

To obtain a solution for $t < t_0$ we define the functions $\bar{u}(t) = u(2t_0 - t)$ and $\bar{f}(t, u) = -f(2t_0 - t, u)$. Then, $\bar{u}$ satisfies the IVP

$$\begin{aligned} \bar{u}'(t) &= u'(2t_0 - t)(-1) = -f(2t_0 - t, u(2t_0 - t)) = \bar{f}(t, \bar{u}(t)), \\ \bar{u}(t_0) &= u_0. \end{aligned} \tag{3.14}$$

Moreover, $\bar{f}$ satisfies the Lipschitz condition in D_0 and hence, by Theorem 3.13, there exists a unique solution $\bar{u}$ of the IVP (3.14) on an interval $[t_0, t_0 + \bar{\delta}]$, for some $\bar{\delta} > 0$. We conclude that $u_-(t) = \bar{u}(2t_0 - t)$ is the unique solution of the original IVP on the interval $[t_0 - \bar{\delta}, t_0]$.

Finally, combining the two pieces, i.e. setting $u = u_+$ on $[t_0, t_0 + \hat{\delta}]$ and $u = u_-$ on $[t_0 - \bar{\delta}, t_0]$, we obtain a solution u of the IVP on an interval $[t_0 - \delta, t_0 + \delta]$, where $\delta = \min\{\bar{\delta}, \hat{\delta}\}$. □

Corollary 3.17. *Let $D \subset \mathbb{R}^{n+1}$ be open. Moreover, let $f : D \to \mathbb{R}^n$ be continuous and continuously differentiable with respect to u in D. Then, for all $(t_0, u_0) \in D$ there exists a unique solution of the IVP*

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(t_0) &= u_0, \end{aligned}$$

on an interval $[t_0 - \delta, t_0 + \delta]$, for some $\delta > 0$.

Proof. This is an immediate consequence of Proposition 3.15 and Theorem 3.16. □

These results provide the *local* existence and uniqueness of solutions. The maximal interval of existence as well as the interval on which the solution is unique may depend on the initial time t_0 and the initial value u_0. The following two examples illustrate this.

Example 3.18. Consider the following IVP

$$u'(t) = t^2 u^2(t), \qquad u(t_0) = u_0,$$

where $t_0, u_0 \in \mathbb{R}$.

The function $f : \mathbb{R}^2 \to \mathbb{R}$, $(t, u) \mapsto t^2 u^2$, is continuous and continuously differentiable with respect to u in $\mathbb{R}^2$ and hence, by Proposition 3.15, it satisfies a local Lipschitz condition in $\mathbb{R}^2$. Consequently, by Corollary 3.17 there exists a unique solution of the IVP on an interval $[t_0 - \delta, t_0 + \delta]$, for some $\delta > 0$. However, we do not know how large the interval of existence is.

We remark that f *does not* satisfy a Lipschitz condition in any subset of the form $[t_0 - T, t_0 + T] \times \mathbb{R}$, $T > 0$. Indeed, we observe that

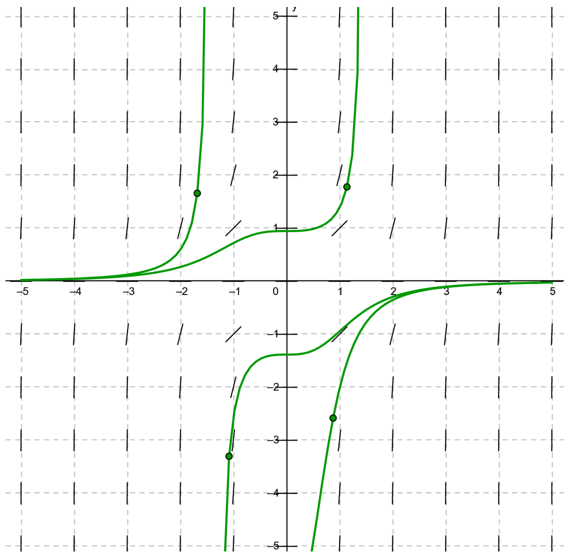
$$|f(t,u) - f(t,v)| = t^2|u^2 - v^2| = t^2|u+v||u-v| \qquad u, v \in \mathbb{R}, t \in [t_0 - T, t_0 + T].$$

Hence, the (global) Lischitz condition (3.11) cannot be satisfied and Theorem 3.12 is not applicable in this case.

We observe that $u^* = 0$ is a steady state of the ODE and this constant solution exists for all $t \in \mathbb{R}$. Moreover, if $tu(t) \neq 0$, then $u'(t) > 0$, i.e. u is strictly increasing. The ODE is separable and we can solve the IVP explicitly. In fact, we obtain

$$u(t) = \frac{3}{\alpha - t^3} \qquad \text{for} \begin{cases} t^3 < \alpha = t_0^3 + \frac{3}{u_0} & \text{if } u_0 > 0, \\ t^3 > \alpha & \text{if } u_0 > 0. \end{cases}$$

Since local uniqueness holds, these are the only solutions of the IVP. The following figure shows the slope field and the graphs of several solutions.



Example 3.19. Consider the following IVP

$$u'(t) = -t\sqrt{|u(t)|}, \qquad u(t_0) = u_0,$$

where $t_0, u_0 \in \mathbb{R}$.

The function $f : \mathbb{R}^2 \to \mathbb{R}, (t,u) \mapsto -t\sqrt{|u|}$ is continuous and continuously differentiable with respect to u in any point $(t_0, u_0) \in \mathbb{R}^2$ with $u_0 \neq 0$. Hence, if $u_0 \neq 0$, then by Corollary 3.17, there exists a unique solution of the IVP on an interval $[t_0 - \delta, t_0 + \delta]$, for some $\delta > 0$. However, if $u_0 = 0$, Corollary 3.17 cannot be applied to conclude the existence and uniqueness of solutions (and neither Theorem 1.6 for separable ODEs).

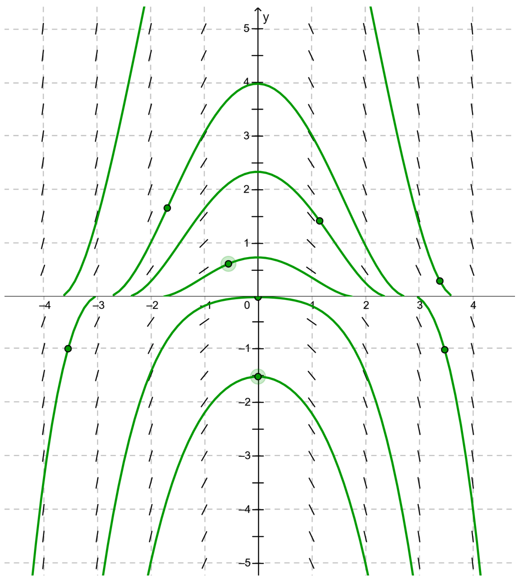
We observe that $u^* = 0$ is a steady state of the ODE and this constant solution exists for all $t \in \mathbb{R}$. Moreover, the ODE is separable and we can solve the IVP explicitly. In fact, considering positive and negative solutions separately, we obtain

$$u(t) = \frac{1}{16}(\alpha - t^2)^2 \qquad \text{for } t^2 \le \alpha = t_0^2 + 4\sqrt{|u_0|}, \qquad \text{if } u_0 > 0,$$
$$u(t) = -\frac{1}{16}(t^2 - \beta)^2 \qquad \text{for } t^2 > \alpha, \qquad \text{if } u_0 < 0.$$

We consider the case $u_0 > 0$. Then, there exists a unique solution u on the interval $[-\sqrt{\alpha}, \sqrt{\alpha}]$ and this solution can be extended for all $t \in \mathbb{R}$, but the extension is not unique. In fact, for arbitrary $\beta_1, \beta_2 > \alpha$, we can extend u by zero on the intervals $[-\sqrt{\beta_1}, -\sqrt{\alpha}]$ and $[\sqrt{\alpha}, \sqrt{\beta_2}]$ and by

$$u(t) = -\frac{1}{16}(t^2 - \beta_1)^2, \qquad t \le -\sqrt{\beta_1},$$
$$u(t) = -\frac{1}{16}(t^2 - \beta_2)^2, \qquad t \ge -\sqrt{\beta_2}.$$

The following figure shows the graphs of the solutions.



Proposition 3.20. Let $D \subset \mathbb{R}^{n+1}$ be open and $f : D \to \mathbb{R}^n$ be continuous and satisfy a local Lipschitz condition in D. Assume that $u_1 : I \to \mathbb{R}^n$ and $u_2 : J \to \mathbb{R}^n$, $I, J \subset \mathbb{R}$ intervals, are solutions of the ODE

$$u'(t) = f(t, u(t)).$$

If there exists $\hat{t} \in I \cap J$ such that $u_1(\hat{t}) = u_2(\hat{t})$, then $u_1 \equiv u_2$ on $I \cap J$.

Proof. Let $I_0 = I \cap J$, and $\widetilde{I_0} = \{t \in I_0 : u_1(t) = u_2(t)\}$. Then, $\widetilde{I_0} \neq \emptyset$ as $\hat{t} \in \widetilde{I_0}$. We need to show that $I_0 = \widetilde{I_0}$.

First, we observe that $\widetilde{I_0} \subset I_0$ is relatively closed. Indeed, let $(t_k)_{k \in \mathbb{N}_0}$ be a sequence in $\widetilde{I_0}$ such that $t_k \to \tilde{t}$ as $k \to \infty$. Then, as u_1 and u_2 are continuous, it follows that

$$u_1(\tilde{t}) = u_1(\lim_{k\to\infty} t_k) = \lim_{k\to\infty} u_1(t_k) = \lim_{k\to\infty} u_2(t_k) = u_2(\lim_{k\to\infty} t_k) = u_2(\tilde{t}),$$

which implies that $\tilde{t} \in \widetilde{I_0}$.

Moreover, $\widetilde{I_0} \subset I_0$ is relatively open. Indeed, let $t_0 \in \widetilde{I_0}$. Then, by Theorem 3.16 there exists a unique solution $u : [t_0 - \delta, t_0 + \delta] \to \mathbb{R}^n$, $\delta > 0$, of the IVP with $u(t_0) = u_0 = u_1(t_0) = u_2(t_0)$. By the uniqueness of solutions, it follows that $u_1|_{[t_0-\delta,t_0+\delta]\cap I_0} = u_2|_{[t_0-\delta,t_0+\delta]\cap I_0}$.

Consequently, $\widetilde{I_0} \subset I_0$ is relatively closed and relatively open and this implies that $\widetilde{I_0} = I_0$. □

Remark. The local Lipschitz condition in Theorem 3.16 is a sufficient condition for the local existence and uniqueness of solutions, but it is not necessary. There are examples of IVPs for which a unique local solution exists, but the function f does not satisfy a local Lipschitz condition.

Moreover, the local existence of solutions can already be shown if f is merely continuous on an open neighborhood around (t_0, u_0). This result is known as *Peano's Theorem* and its proof is more involved than the proof of Theorem 3.16. Peano's Theorem, however, does not guarantee the uniqueness of solutions, as shown, e.g. in Example 3.19.

3.5 Gronwall's Lemma and perturbation results

The following result is one of several different versions of Gronwall's Lemma which is important in many applications. It allows to derive bounds for functions that satisfy an integral inequality.

Lemma 3.21. *Let $t_0 \in \mathbb{R}, T > t_0$, the functions $g, a, b : [t_0, T] \to \mathbb{R}$ be continuous and $b \geq 0$ in $[t_0, T]$. If g satisfies the inequality*

$$g(t) \leq a(t) + \int_{t_0}^{t} b(s)g(s)ds \qquad \forall t \in [t_0, T], \tag{3.15}$$

then,

$$g(t) \leq a(t) + \int_{t_0}^{t} e^{\int_s^t b(r)dr} a(s)b(s)ds \qquad \forall t \in [t_0, T]. \tag{3.16}$$

If, in addition, a is continuously differentiable, then

$$g(t) \leq a(t_0)e^{\int_{t_0}^t b(s)ds} + \int_{t_0}^{t} e^{\int_s^t b(r)dr} a'(s)ds \qquad \forall t \in [t_0, T]. \tag{3.17}$$

Proof. We define $h(t) := \int_{t_0}^t g(s)b(s)ds$ and set $v(t) = g(t) - h(t)$. Then, (3.15) is equivalent to the inequality $v(t) \leq a(t)$ $\forall t \in [t_0, T]$. Moreover, it follows that

$$h'(t) = g(t)b(t) = v(t)b(t) + h(t)b(t) \qquad \forall t \in [t_0, T],$$

and $h(t_0) = 0$. Hence, h satisfies a linear scalar ODE and Theorem 1.9 implies that

$$h(t) = \int_{t_0}^{t} e^{\int_s^t b(r)dr} v(s)b(s)ds.$$

Finally, $v(t) \leq a(t)$ and $b(t) \geq 0$ imply that

$$g(t) = v(t) + h(t) = v(t) + \int_{t_0}^{t} e^{\int_s^t b(r)dr} v(s)b(s)ds \leq a(t) + \int_{t_0}^{t} e^{\int_s^t b(r)dr} a(s)b(s)ds,$$

for all $t \in [t_0, T]$, which proves (3.16).

To show (3.17) we integrate by parts. In fact, we observe that

$$\int_{t_0}^{t} e^{\int_s^t b(r)dr} a(s)b(s)ds = \int_{t_0}^{t} -\frac{d}{ds}\left(e^{\int_s^t b(r)dr}\right) a(s)ds$$
$$= -a(t) + a(t_0)e^{\int_{t_0}^t b(s)ds} + \int_{t_0}^{t} e^{\int_s^t b(r)dr} a'(s)ds,$$

which implies (3.17). □

We now apply Gronwall's Lemma to show a perturbation result. This is important in applications as we expect that if we perturb the function f or the initial value u_0 slightly, then the difference between the solutions of the original problem and the perturbed problem should be small.

For simplicity we formulate the results assuming that the function f satisfies the (global) Lipschitz condition (3.11). However, similar results hold if we assume a local Lipschitz condition for f.

Theorem 3.22. *Let $t_0 \in \mathbb{R}, T > t_0$ and $u_0, \tilde{u}_0 \in \mathbb{R}^n$. We assume that the functions $f, \tilde{f} : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and satisfy the Lipschitz condition* (3.11) *in $D = [t_0, T] \times \mathbb{R}^n$.*

Then, there exist unique solutions $u, \tilde{u} : [t_0, T] \to \mathbb{R}^n$ of the IVPs

$$u'(t) = f(t, u(t)), \qquad \tilde{u}'(t) = \tilde{f}(t, \tilde{u}(t)),$$
$$u(t_0) = u_0, \qquad \tilde{u}(t_0) = \tilde{u}_0,$$

and the following estimate holds,

$$\|u(t) - \tilde{u}(t)\| \leq e^{L(t-t_0)}\|u_0 - \tilde{u}_0\| + \int_{t_0}^{t} e^{L(t-s)}\|f(s, \tilde{u}(s)) - \tilde{f}(s, \tilde{u}(s))\|ds,$$

for all $t \in [t_0, T]$.

Proof. By Theorem 3.13, there exist unique solutions u and $\tilde{u}$ of the IVPs and the solutions are defined on $[t_0, T]$. By Lemma 3.1 the solutions satisfy the integral equation (3.2) and hence, we obtain for their difference

$$u(t) - \tilde{u}(t) = u_0 - \tilde{u}_0 + \int_{t_0}^{t} (f(s, u(s)) - \tilde{f}(s, \tilde{u}(s)))ds$$
$$= u_0 - \tilde{u}_0 + \int_{t_0}^{t} (f(s, u(s)) - f(s, \tilde{u}(s)))ds + \int_{t_0}^{t} (f(s, \tilde{u}(s)) - \tilde{f}(s, \tilde{u}(s)))ds$$

for all $t \in [t_0, T]$. Furthermore, the triangle inequality and Lemma 3.10 now imply that

$$\|u(t) - \tilde{u}(t)\|$$
$$\leq \|u_0 - \tilde{u}_0\| + \int_{t_0}^{t} \|f(s, u(s)) - f(s, \tilde{u}(s))\|ds + \int_{t_0}^{t} \|f(s, \tilde{u}(s)) - \tilde{f}(s, \tilde{u}(s))\|ds$$
$$\leq \|u_0 - \tilde{u}_0\| + L\int_{t_0}^{t} \|u(s) - \tilde{u}(s)\|ds + \int_{t_0}^{t} \|f(s, \tilde{u}(s)) - \tilde{f}(s, \tilde{u}(s))\|ds,$$

where we used the Lipschitz condition of f in the last step. Finally, Gronwall's Lemma (Lemma 3.21) applied to $g(t) = \|u(t) - \tilde{u}(t)\|$, $b \equiv L$ and

$$a(t) = \|u_0 - \tilde{u}_0\| + \int_{t_0}^{t} \|f(s, \tilde{u}(s)) - \tilde{f}(s, \tilde{u}(s))\| ds,$$

implies the estimate. □

An immediate consequence of the previous theorem is the continuity of solutions with respect to initial values.

Corollary 3.23. *Let $t_0 \in \mathbb{R}, T > t_0$ and $u_0, \tilde{u}_0 \in \mathbb{R}^n$. We assume that the function $f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies the Lipschitz condition* (3.11) *in $D = [t_0, T] \times \mathbb{R}^n$. Consider the ODE*

$$u'(t) = f(t, u(t))$$

and let $u : [t_0, T] \to \mathbb{R}^n$ denote the unique solution of the IVP with $u(t_0) = u_0$ and let $\tilde{u} : [t_0, T] \to \mathbb{R}^n$ denote the unique solution of the IVP with $\tilde{u}(t_0) = \tilde{u}_0$. Then,

$$\|u(t) - \tilde{u}(t)\| \leq e^{L(t-t_0)} \|u_0 - \tilde{u}_0\| \qquad \forall t \in [t_0, T].$$

Proof. This is an immediate consequence of Theorem 3.22 with $f = \tilde{f}$. □

The previous corollary implies the *well-posedness* of the IVP. J. Hadamard introduced this notion and called a problem well-posed if a solution exists, the solution is unique and solutions depend continuously on the initial conditions.

3.6 ODEs of higher order

So far, we only considered ODEs of first order. We now look at ODEs of higher order.

Let $D \subset \mathbb{R}^{n+1}$ and $f : D \to \mathbb{R}$ be a continuous function. An equation of the form

$$u^{(n)}(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t)) \tag{3.18}$$

is called explicit **ODE of n-th order**, $n \in \mathbb{N}$, where $u^{(n)}$ denotes the n-th derivative of u.

Definition 3.24. A **solution** of (3.18) is a function $u : I \to \mathbb{R}$, $I \subset \mathbb{R}$ an interval, such that

- u is n-times differentiable,
- $(t, u(t), u'(t), \dots, u^{(n-1)}(t)) \in D$ for all $t \in I$,
- $u^{(n)}(t) = f(t, u(t), u'(t), \dots, u^{(n-1)}(t))$ for all $t \in I$.

An ODE of n-th order can be reduced to a system of n first order ODEs. In fact, let $v_1 = u, v_2 = u', \dots, v_n = u^{(n-1)}$. Then, $v = \left(v_1, \dots, v_n\right)$ satisfied the first order system

$$\begin{aligned} v_1'(t) &= v_2(t), \\ v_2'(t) &= v_3(t), \\ &\vdots \\ v_n'(t) &= f(t, v_1(t), \dots, v_n(t)). \end{aligned} \tag{3.19}$$

Lemma 3.25. *If $u : I \to \mathbb{R}$ is a solution of* (3.18)*, then the function $v : I \to \mathbb{R}^n$ defined above is a solution of* (3.19)*.*

If $v : I \to \mathbb{R}^n$ is a solution of the system (3.19)*, then $u = v_1 : I \to \mathbb{R}$ is a solution of* (3.18)*.*

Proof. Let $u : I \to \mathbb{R}$ be a solution of (3.18) and the function $v : I \to \mathbb{R}^n$ be defined as above. If we define $F : D \to \mathbb{R}^n$ by

$$F(t, v) = \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ v_n \\ f(t, v_1, \ldots, v_n) \end{pmatrix},$$

we observe that v is a solution of (3.19) according to Definition 1.1.

Let now v be a solution of (3.19). Then, by the definition of v and the differentiability of v it follows that $v_2 = v_1' = u'$, $v_3 = v_2' = u''$, $\ldots$, $v_n = v_{n-1}' = u^{(n-1)}$. Therefore, since v_n is differentiable, it follows that u is n-times differentiable and by the last equation in (3.19), it follows that u satisfies (3.18). □

In the same way, we can reformulate n-th order systems of ODEs as systems of first order. Therefore, as every n-th order ODE can be reduced to a system of first order ODEs, it suffices to develop the theory for first order systems.

Example 3.26. Systems of second order ODEs are important in physics. For instance, let $x(t)$ denote the position of a point mass in $\mathbb{R}^3$ at time $t \geq 0$. Then, $x'(t)$ denotes the velocity of the point mass and $x''(t)$ the acceleration of the point mass at time t.

By Newton's law we have

$$mx''(t) = F(t, x(t), x'(t)),$$

where m is the mass and F the force depending on time, position and velocity.

This system of three second order ODEs is equivalent to the system of six first order ODEs

$$\begin{aligned} x'(t) &= v(t), \\ v'(t) &= \frac{1}{m} F(t, x(t), x'(t)), \end{aligned}$$

where $v(t) = x'(t)$.

Theorem 3.27. *Let $D \subset \mathbb{R}^{n+1}$ be open and let $f : D \to \mathbb{R}, (t, v) \mapsto f(t, v)$, be continuous and satisfy a local Lipschitz condition in D. Then, for every $(t_0, u_0, u_1, \ldots, u_{n-1}) \in D$ there exists a unique solution of the IVP*

$$\begin{aligned} u^{(n)}(t) &= f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)), \\ u(t_0) &= u_0,\ u'(t_0) = u_1,\ \ldots,\ u^{(n-1)}(t_0) = u_{n-1}, \end{aligned}$$

on an interval $[t_0 - \delta, t_0 + \delta]$, $\delta > 0$.

Proof. This is an immediate consequence of Lemma 3.25 and Theorem 3.16. □

3.7 Exercises

E3.1 **Banach spaces**

Let $t_0 \in \mathbb{R}, T > t_0$ and $C^1([t_0, T]; \mathbb{R})$ denote the space of all continuously differentiable functions $u : [t_0, T] \to \mathbb{R}$. We introduce the norm

$$\|u\|_{C^1([t_0,T];\mathbb{R})} := \|u\|_{max} + \|u'\|_{max},$$

where the maximum norm is defined as $\|u\|_{max} = \max_{t\in[t_0,T]}\{|u(t)|\}$.

Show that $\|\cdot\|_{C^1([t_0,T];\mathbb{R})}$ indeed defines a norm and that $C^1([t_0, T]; \mathbb{R})$ with the norm $\|\cdot\|_{C^1([t_0,T];\mathbb{R})}$ is a Banach space.

E3.2 **Banach's fixed point theorem**

Let $y_0 \in \mathbb{R}$. Use Banach's fixed point theorem to prove that the iterative sequence $(y_n)_{n\in\mathbb{N}_0}$ defined by

$$y_{n+1} = \frac{1}{2}y_n + 4, \qquad n \in \mathbb{N}_0,$$

converges and determine the limit of the sequence.

E3.3 **Picard iteration**

Use the method of successive approximations to determine the solution of the IVP

$$u'(t) = \frac{2u(t)}{t}, \qquad u(1) = 1.$$

To this end, find a formula for the approximations $u_n, n \in \mathbb{N}_0$, prove it by induction and then pass to the limit $n \to \infty$.

E3.4 **Lipschitz continuity**

Let $f : I \to \mathbb{R}$, $I \subset \mathbb{R}$ an interval, be continuous. The function f is (globally) *Lipschitz continuous* if there exists a constant $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y| \qquad \forall x, y \in I.$$

The function f is *locally Lipschitz continuous* if for every $x_0 \in I$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le L|x - y| \qquad \forall x, y \in [x_0 - \delta, x_0 + \delta] \cap I.$$

(a) Show that if I is open and f is continuously differentiable, then f is locally Lipschitz continuous.

(b) Give a simple counterexample that shows that Lipschitz continuity does not imply that a function is continuously differentiable.

(c) Give an example of a function that is locally Lipschitz continuous but not globally Lipschitz continuous.

E3.5 **Symmetry**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy a local Lipschitz condition. Assume that

$$f(-t,u) = -f(t,u) \qquad \forall (t,u) \in \mathbb{R}^2.$$

Let $r > 0$. Use the theorem by Picard-Lindelöf to show that if $u : [-r,r] \to \mathbb{R}$ is a solution of the ODE

$$u'(t) = f(t,u(t)),$$

then, for the reflection of u over the u-axis, i.e. $\bar{u}(t) = u(-t)$, it follows that $\bar{u} = u$ on $[-r,r]$.

E3.6 **Lipschitz condition**

(a) Investigate whether the following function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies a local Lipschitz condition in $\mathbb{R}^2$,

$$f(t,u) = \sin(t)\sqrt[3]{|u|}, \qquad (t,u) \in \mathbb{R}^2.$$

(b) Given an example (that is different from examples in the lecture notes) for an IVP that possesses more than one solution.

E3.7 **Existence and uniqueness**

Without solving them, show that there exists a unique local solution of the following IVPs.

(a) The *Lotka–Volterra model* describes the dynamics of two interacting biological species, where one is the predator and the other one the prey. The system of ODEs is of the form

$$\begin{aligned} u'(t) &= \alpha u(t) - \beta u(t)v(t), \\ v'(t) &= \delta v(t)u(t) - \gamma v(t), \\ u(0) &= u_0, \qquad v(0) = v_0, \end{aligned}$$

where α, β, γ and δ are positive constants and $u_0, v_0 \geq 0$. Moreover, u denotes the density of the prey population and v the density of the predator population. We will discuss and analyze this model later in the course.

(b) The following ODE describes a *simple harmonic oscillator*, i.e. an oscillator that is neither driven through an external force nor damped,

$$\begin{aligned} u''(t) &= -\omega^2 u(t) \\ u(0) &= u_0, \qquad u'(0) = u_1, \end{aligned}$$

where $\omega, u_0, u_1 \in \mathbb{R}$. Moreover, u denotes the displacement from its equilibrium position. *Hint: First, rewrite the IVP as an IVP for a first order system of ODEs. To this end define $v := u'$, and consider the ODE system for $\begin{pmatrix} u \\ v \end{pmatrix}$.*

E3.8 **Maximal interval of existence**

Consider the IVP

$$\begin{aligned} u'(t) &= \frac{u^2}{1+t^2} \\ u(t_0) &= u_0, \qquad (t_0, u_0) \in \mathbb{R}^2. \end{aligned}$$

(a) What can you say about the existence and uniqueness of solutions of the IVP?

(b) Find explicit solutions and sketch the graph of the solutions depending on the initial value. On which intervals exist the solutions?

(c) Determine the behavior of the solutions as t tends to the endpoints of the maximal intervals of existence.

E3.9 **ODEs of higher order**

(a) Reformulate the following ODE as a system of first order ODEs,

$$u'''(t) - 2u'(t) + 3u''(t) - u(t) = 5.$$

(b) Let $(u_0, u_1) \in \mathbb{R}^2$. Show that there exists a unique solution u of the IVP

$$\begin{aligned} u''(t) &= t^3 \sin(u^2(t)), \\ u(0) &= u_0, \quad u'(0) = u_1, \end{aligned}$$

that exists on an interval $[-\delta, \delta]$, for some $\delta > 0$.

E3.10 **Free fall with a parachute**

Assuming that the friction force is proportional to the velocity, the motion of a free falling body of mass 1 under the influence of a constant external force g is described by the IVP

$$\begin{aligned} x''(t) + \gamma x'(t) &= g, \\ x(0) &= x_0, \\ x'(0) &= x_1, \end{aligned}$$

where $x_0, x_1 \in \mathbb{R}$. Solve the IVP with $x_0 = x_1 = 0$ and interpret the result.

Chapter 4

First-order linear systems

4.1 Linear systems

Definition 4.1. A system of first order ODEs of the from

$$u'(t) = A(t)u(t) + f(t), \tag{4.1}$$

where $A : J \to \mathbb{R}^{n\times n}$ and $f : J \to \mathbb{R}^n$, $J \subset \mathbb{R}$ an interval, are continuous functions, is called **first order linear system**.

The system (4.1) is a **system with constant coefficients** if $A(t) = A \in \mathbb{R}^{n\times n}$ for all $t \in J$. Moreover, the system is called **homogeneous** if $f \equiv 0$, and **inhomogeneous** otherwise.

Here, we call $A = (a_{ij})_{1\le i,j\le n} : J \to \mathbb{R}^{n\times n}$ continuous if all coefficients $a_{ij} : J \to \mathbb{R}$, $i, j = 1, \ldots, n$, are continuous. As for vector-valued functions, integrals and derivatives of matrix-valued functions are defined component-wise.

In case of linear systems, the right hand side in (4.1) is a linear function of the unknown, vector-valued function u. Written component-wise, the system takes the form

$$\begin{aligned} u_1'(t) &= a_{11}(t)u_1(t) + \cdots + a_{1n}(t)u_n(t) + f_1(t), \\ &\vdots \\ u_n'(t) &= a_{n1}(t)u_1(t) + \cdots + a_{nn}(t)u_n(t) + f_n(t). \end{aligned}$$

The linear structure immediately implies the following *superposition principle* for homogeneous systems.

Proposition 4.2. Let $u : I \to \mathbb{R}^n$ and $v : I \to \mathbb{R}^n$ be two solutions of the linear homogeneous system (4.1), i.e. $f \equiv 0$. Then, $w = au + bv$, for arbitrary $a, b \in \mathbb{R}$, is also a solution.

Proof. Using that u and v satisfy the homogeneous ODE we conclude that

$$w'(t) = au'(t) + bv'(t) = aA(t)u(t) + bA(t)v(t) = A(t)(au(t) + bv(t)) = A(t)w(t),$$

for all $t \in I$. □

Let $||\cdot||$ denote an arbitrary norm in $\mathbb{R}^n$. The *induced matrix norm* in $\mathbb{R}^{n\times n}$ is defined by

$$||A|| = \max_{x\in\mathbb{R}^n,\ x\neq 0} \frac{||Ax||}{||x||}, \qquad A \in \mathbb{R}^{n\times n}.$$

The induced matrix norm $||A||$ is the smallest constant $c \geq 0$ such that

$$||Ax|| \leq c||x|| \qquad \forall x \in \mathbb{R}^n.$$

Moreover, it has the following properties,

$$\begin{aligned} ||A + B|| &\leq ||A|| + ||B||, \\ ||AB|| &\leq ||A||\,||B||, \end{aligned}$$

for all $A, B \in \mathbb{R}^{n\times n}$.

Theorem 4.3. *Let $t_0 \in \mathbb{R}, T > 0$ and $J = [t_0 - T, t_0 + T]$. Moreover, we assume that $A : J \to \mathbb{R}^{n\times n}$ and $f : J \to \mathbb{R}^n$ are continuous and $u_0 \in \mathbb{R}^n$. Then, there exists a unique solution u of the IVP*

$$\begin{aligned} u'(t) &= A(t)u(t) + f(t), \\ u(t_0) &= u_0, \end{aligned}$$

and the solution exists on the interval J.

Proof. We observe that the function $\tilde{f} : J \times \mathbb{R}^n \to \mathbb{R}^n$, $\tilde{f}(t, u) = A(t)u + f(t)$ is continuous and satisfies

$$\begin{aligned} ||\tilde{f}(t, u) - \tilde{f}(t, v)|| &= ||A(t)(u - v)|| \leq ||A(t)||\,||u - v|| \\ &\leq \max_{t\in J}\{||A(t)||\}\,||u - v||, \end{aligned}$$

for all $(t, u), (t, v) \in J \times \mathbb{R}^n$. Since A is continuous and the interval J is compact, the maximum is attained. Therefore, $\tilde{f}$ satisfied the Lipschitz condition in $D = J \times \mathbb{R}^n$.

Theorem 3.12 now implies that there exists a unique solution of the IVP on the interval $[t_0, t_0 + T]$. The unique solution on the interval $[t_0 - T, t_0]$ can be constructed as in the proof of Theorem 3.16. □

4.2 Systems with constant coefficients

In this section, we analyze systems with constant coefficients. We first consider homogeneous IVPs, i.e.

$$\begin{aligned} u'(t) &= Au(t), \\ u(t_0) &= u_0, \end{aligned} \tag{4.2}$$

where $t_0 \in \mathbb{R}, u_0 \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a matrix with constant coefficients $a_{ij} \in \mathbb{R}$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

In the case $n = 1$, we obtain an IVP for a linear homogeneous scalar ODE

$$u'(t) = au(t),$$
$$u(t_0) = u_0,$$

where $t_0 \in \mathbb{R}, u_0 \in \mathbb{R}$ and $a \in \mathbb{R}$. The solution is given by $u(t) = u_0 e^{a(t-t_0)}$ and exists for all $t \in \mathbb{R}$.

Remark 4.4. We recall that the exponential function can be written as a power series,

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}, \qquad t \in \mathbb{R},$$

that is absolutely convergent and the convergence radius of this series is infinity. Moreover, power series $\sum_{k=0}^{\infty} c_k t^k$, $c_k \in \mathbb{R}, t \in \mathbb{R}$, are differentiable (within their radius of convergence) and we have

$$\left(\sum_{k=0}^{\infty} c_k t^k\right)' = \sum_{k=0}^{\infty} \left(c_k t^k\right)' = \sum_{k=1}^{\infty} c_k k t^{k-1}.$$

Using the power series representation for the exponential function we now solve the homogeneous ODE system (4.2). To this end we introduce *matrix exponentials*. Here and in the sequel, we denote the identity matrix and the zero matrix by

$$\mathrm{Id} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \qquad 0 = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}.$$

Definition 4.5. For $A \in \mathbb{R}^{n\times n}$ we define

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k,$$

where $A^0 = e^0 = Id$.

Lemma 4.6. *For all $A \in \mathbb{R}^{n\times n}$ the series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ converges in $\mathbb{R}^{n\times n}$, i.e. the matrix* $\exp(A) = e^A$ *is well-defined.*

Proof. Let $m := \max\{|a_{ij}| : i, j = 1, \dots, n\}$. Then, we can estimate the entries of the matrices in the series by

$$\left|\left(\frac{1}{k!} A^k\right)_{ij}\right| \le \left(\frac{1}{k!} \begin{pmatrix} m & \cdots & m \\ \vdots & \ddots & \vdots \\ m & \cdots & m \end{pmatrix}^k\right)_{ij} = \frac{n^{k-1} m^k}{k!}.$$

Since the series

$$\sum_{k=0}^{\infty} \frac{n^{k-1} m^k}{k!} = \frac{1}{n} e^{nm}$$

converges and is an upper bound for $\left|\left(\frac{1}{k!} A^k\right)_{ij}\right|$, for all $i, j = 1, \dots, n$, it follows the convergence of the series $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ in $\mathbb{R}^{n\times n}$. □

Theorem 4.7. *Let $A \in \mathbb{R}^{n\times n}, t_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}^n$. Then, there exists a unique solution of the homogeneous IVP* (4.2) *namely,*

$$u(t) = e^{(t-t_0)A}u_0 \qquad \forall t \in \mathbb{R}.$$

In particular, we have

$$\frac{d}{dt}e^{tA} = Ae^{tA} \qquad \forall t \in \mathbb{R}.$$

Proof. Without loss of generality we can assume that $t_0 = 0$. By Lemma 4.8, each component $\left(e^{tA}\right)_{ij}$ is a power series that converges for all $t \in \mathbb{R}$. Moreover, since power series are differentiable, we obtain

$$\begin{aligned} u'(t) = \left(e^{tA}u_0\right)' &= \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{1}{k!}(tA)^k u_0\right) = \sum_{k=0}^{\infty}\frac{d}{dt}\left(\frac{1}{k!}(tA)^k u_0\right) \\ &= \sum_{k=1}^{\infty}\left(\frac{1}{(k-1)!}t^{k-1}A^k u_0\right) = A\sum_{k=0}^{\infty}\left(\frac{1}{k!}(tA)^k u_0\right) = Ae^{tA}u_0 = Au(t), \end{aligned}$$

and $e^{0A}u_0 = \mathrm{Id}u_0 = u_0$. Finally, the solution exists for all $t \in \mathbb{R}$ and the uniqueness of solutions follows from Theorem 4.3. □

Next, we derive some properties of the function e^{tA} that we will need to solve the inhomogeneous problem.

Lemma 4.8. *Let $A, B \in \mathbb{R}^{n\times n}$. Then, the following properties hold:*

(i) If A and B commute, i.e. $AB = BA$, then

$$e^{t(A+B)} = e^{tA}e^{tB}.$$

(ii) $(e^{tA})^{-1} = e^{-tA}$ *for all $t \in \mathbb{R}$.*

Proof. By induction, it follows that $A^kB = BA^k$ for all $k \in \mathbb{N}_0$, and consequently, we have

$$\sum_{k=0}^{m}\frac{t^k}{k!}A^kB = \sum_{k=0}^{m}\frac{t^k}{k!}BA^k = B\sum_{k=0}^{m}\frac{t^k}{k!}A^k.$$

Taking the limit $m \to \infty$ is follows that

$$e^{tA}B = \lim_{m\to\infty}\sum_{k=0}^{m}\frac{t^k}{k!}A^kB = B\lim_{m\to\infty}\sum_{k=0}^{m}\frac{t^k}{k!}A^k = Be^{tA}.$$

Moreover, we observe that for $u_0 \in \mathbb{R}^n$ the function $u(t) = e^{tA}e^{tB}u_0$ satisfies

$$\begin{aligned} u'(t) &= \left(\frac{d}{dt}e^{tA}\right)e^{tB}u_0 + e^{tA}\left(\frac{d}{dt}e^{tB}\right)u_0 \\ &= Ae^{tA}e^{tB}u_0 + e^{tA}Be^{tB}u_0 = (A+B)e^{tA}e^{tB}u_0 = (A+B)u(t), \end{aligned}$$

where we used the property above and Theorem 4.7. Consequently, u is the unique solution of the IVP $u'(t) = (A+B)u(t)$, $u(0) = u_0$. By Theorem 4.7, it follows that $u(t) = e^{t(A+B)}u_0$, which implies the statement.

The property (ii) now follows from (i) by taking $B = -A$. Indeed, we obtain

$$e^{tA}e^{-tA} - \mathrm{Id} = 0,$$

which implies (ii). □

Note that the statement (i) in Lemma 4.8 implies that

$$e^{tA}e^{sA} = e^{(t+s)A} \qquad \text{for all } t, s \in \mathbb{R}.$$

We now consider inhomogeneous IVPs,

$$\begin{aligned} u'(t) &= Au(t) + f(t), \\ u(t_0) &= u_0, \end{aligned} \tag{4.3}$$

where $A \in \mathbb{R}^{n\times n}$ and $u_0 \in \mathbb{R}^n$. Moreover, $f : J \to \mathbb{R}^n$ is continuous, $J \subset \mathbb{R}$ is an interval and $t_0 \in J$. The inhomogeneous IVP can be solved very similarly to the scalar case using a variation of constants formula.

Theorem 4.9. *Let $A \in \mathbb{R}^{n\times n}$, $J \subset \mathbb{R}$ be an interval, $f : J \to \mathbb{R}^n$ be continuous, $t_0 \in J$ and $u_0 \in \mathbb{R}^n$. Then, there exists a unique solution $u : J \to \mathbb{R}^n$ of the IVP* (4.3)*, and it is given by*

$$u(t) = e^{(t-t_0)A}u_0 + \int_{t_0}^{t} e^{(t-s)A} f(s)ds, \qquad t \in J. \tag{4.4}$$

Proof. We observe that $u(t_0) = e^{0A}u_0 = u_0$. Moreover, using Lemma 4.8 we can rewrite the integral equation as

$$\begin{aligned} u(t) &= e^{tA}e^{-t_0A}u_0 + \int_{t_0}^{t} e^{tA}e^{-sA} f(s)ds \\ &= e^{tA}\left(e^{-t_0A}u_0 + \int_{t_0}^{t} e^{-sA} f(s)ds\right). \end{aligned}$$

Differentiating the equation we obtain

$$\begin{aligned} u'(t) &= Ae^{tA}\left(e^{-t_0A}u_0 + \int_{t_0}^{t} e^{-sA} f(s)ds\right) + e^{tA}\left(e^{-tA}f(t)\right) \\ &= A\left(e^{(t-t_0)A}u_0 + \int_{t_0}^{t} e^{(t-s)A} f(s)ds\right) + f(t) = Au(t) + f(t), \end{aligned}$$

where we used Theorem 4.7 and Lemma 4.8.

Finally, the solution exists for all $t \in J$ and the uniqueness of solutions follows from Theorem 4.3. □

4.3 Computing matrix exponentials

We found a solution formula for linear systems with constant coefficients. The representation formula involves matrix exponentials. We now discuss methods that allow us to compute them.

The results in the previous sections of this chapter remain true if we consider complex matrices and complex-valued solutions. Hence, we formulate the following properties for the more general case of complex matrices.

Lemma 4.10. *Let $A, B \in \mathbb{C}^{n\times n}$ and $t \in \mathbb{R}$.*

(i) If there exists $S \in \mathbb{C}^{n\times n}$ invertible such that $A = SBS^{-1}$, then

$$e^{tA} = Se^{tB}S^{-1}.$$

(ii) If A is a block diagonal matrix of the form $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with $A_1 \in \mathbb{C}^{k\times k}$ and $A_2 \in \mathbb{C}^{(n-k)\times(n-k)}$, then

$$e^{tA} = \begin{pmatrix} e^{tA_1} & 0 \\ 0 & e^{tA_2} \end{pmatrix}.$$

(iii) For diagonal matrices we have

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} \implies e^{tA} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{C}$.

(iv) If A is a Jordan block, i.e.

$$A = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix},$$

for some $\lambda \in \mathbb{C}$, then we have

$$e^{tA} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t} & \cdots & \frac{t^{n-1}}{(n-1)!}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{t^2}{2!}e^{\lambda t} \\ \vdots & & \ddots & e^{\lambda t} & te^{\lambda t} \\ 0 & \cdots & \cdots & 0 & e^{\lambda t} \end{pmatrix}.$$

Proof. (i) We observe that $(SBS^{-1})^k = SB^kS^{-1}$ for all $k \in \mathbb{N}_0$ (proof by induction). Hence, the statement immediately follows from Definition 4.5 by observing that

$$e^{tA} = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{t^k}{k!}(SBS^{-1})^k = S \lim_{m\to\infty} \left(\sum_{k=0}^{m} \frac{t^k}{k!} B^k \right) S^{-1} = S e^{tA} S^{-1}.$$

(ii) We observe that

$$A^k = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}^k = \begin{pmatrix} B^k & 0 \\ 0 & C^k \end{pmatrix} \qquad \forall k \in \mathbb{N}_0,$$

and hence, the statement follows from Definition 4.5 similarly as in (ii).

(iii) This property follows by iteration from (ii).

(iv) We write

$$E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

and observe that $A = \lambda \mathrm{Id} + E$. By Lemma 4.8(i) and (iii), it follows that

$$e^{tA} = e^{t\lambda \mathrm{Id} + tE} = e^{t\lambda \mathrm{Id}} e^{tE} = e^{t\lambda} e^{tE}.$$

Moreover, we note that $E^k, k \le n-1$, has entries equal to 1 on the k-th upper diagonal and zeros otherwise, and $E^k = 0$ for all $k \ge n$ (see tutorials). Hence, we conclude that

$$e^{tE} = \sum_{k=0}^{\infty} \frac{1}{k!}(tE)^k = \mathrm{Id} + tE + \frac{t^2}{2}E^2 + \cdots + \frac{t^{n-1}}{(n-1)!}E^{n-1}$$

$$= \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{t^2}{2!} \\ \vdots & & \ddots & 1 & t \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

and the statement follows.

□

By Lemma 4.10, we can now compute the matrix exponentials $e^{tA}, A \in \mathbb{C}^{n\times n}$, as follows:

- If $A \in \mathbb{C}^{n\times n}$ possesses n linearly independent eigenvectors, then it is *diagonizable*, i.e. there exists an invertible matrix $S \in \mathbb{C}^{n\times n}$ and a diagonal matrix

$$B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} \in \mathbb{C}^{n\times n},$$

such that $A = SBS^{-1}$. Here, λ_j, $j = 1, \dots, n$, are the eigenvalues of A, and the j-th column of S is the corresponding eigenvector. By Lemma 4.10, the matrix exponential is then given by

$$e^{tA} = S \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{pmatrix} S^{-1}.$$

- Not all matrices are diagonizable, but from Linear Algebra we know that for every matrix $A \in \mathbb{C}^{n\times n}$ there exists a *Jordan normal form* J, i.e. it can be written as $A = SJS^{-1}$, where $S \in \mathbb{C}^{n\times n}$ is invertible and $J \in \mathbb{C}^{n\times n}$ is of the form

  $$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_m \end{pmatrix}, \qquad J_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ 0 & \lambda_k & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_k \end{pmatrix} \in \mathbb{C}^{n_k\times n_k},$$

  with $n_1 + \cdots + n_m = n$ and $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A. The same eigenvalue λ_k can appear in several Jordan blocks J_k. If all Jordan blocks have dimension one, the matrix is diagonizable. However, if there exists a block with dimension $n_k \geq 2$, then A is not diagonizable and called *defective*.

  By Lemma 4.10, the matrix exponential is given by

  $$e^{tA} = Se^{tJ}S^{-1}, \qquad e^{tJ} = \begin{pmatrix} e^{tJ_1} & 0 & \cdots & 0 \\ 0 & e^{tJ_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{tJ_m} \end{pmatrix},$$

  where

  $$e^{tJ_k} = \begin{pmatrix} e^{\lambda_k t} & te^{\lambda_k t} & \frac{t^2}{2!}e^{\lambda_k t} & \cdots & \frac{t^{n-1}}{(n-1)!}e^{\lambda_k t} \\ 0 & e^{\lambda_k t} & te^{\lambda t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{t^2}{2!}e^{\lambda_k t} \\ \vdots & & \ddots & e^{\lambda_k t} & te^{\lambda_k t} \\ 0 & \cdots & \cdots & 0 & e^{\lambda_k t} \end{pmatrix}, \qquad k = 1, \dots, m.$$

Remark. We mention a few facts about the Jordan decomposition, for further details we refer to standard textbooks on Linear Algebra.

We recall that the *algebraic multiplicity* $m(\lambda)$ of an eigenvalue is the multiplicity of λ as zero of the characteristic polynomial $\det(A - \lambda\mathrm{Id})$. It equals the number of times the eigenvalue λ occurs in the diagonal of the Jordan normal form J of a matrix.

The *geometric multiplicity* $m_g(\lambda)$ of an eigenvalue is the dimension of the corresponding eigenspace. It equals the number of linearly independent eigenvectors corresponding to the eigenvalue λ. It holds that $m_g(\lambda) \leq m(\lambda) \leq n$. If $m(\lambda) = m_g(\lambda)$, the eigenvalue is called *half-simple*. In this case, the corresponding Jordan block is a diagonal matrix.

Once we found the diagonalization or Jordan normal form of a matrix, we can compute the matrix exponentials and use it in the solution formula for linear systems with constant coefficients.

Example 4.11. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}.$$

The characteristic polynomial is $\det(A - \lambda \mathrm{Id}) = (\lambda - 3)(\lambda + 2)$ and hence, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$. Moreover, the corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix},$$

and they are linearly independent. Consequently, we have

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix},$$

and we obtain

$$e^{tA} = S e^{tD} S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} \frac{4}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{3t} + \frac{1}{5}e^{-2t} & \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t} \\ \frac{4}{5}e^{3t} - \frac{4}{5}e^{-2t} & \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t} \end{pmatrix}.$$

Example 4.12. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then, the IVP

$$u'(t) = Au(t), \qquad u(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

has a unique solution, namely,

$$u(t) = e^{tA} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^t \cos t + 2e^t \sin t \\ -e^t \sin t + 2e^t \cos t \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Indeed, the characteristic polynomial is $\det(A - \lambda \mathrm{Id}) = \lambda^2 - 2\lambda + 2 = 0$, and hence, the eigenvalues are $\lambda_1 = 1 - i$ and $\lambda_2 = 1 + i$. The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix},$$

and we obtain

$$S = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \qquad D = \begin{pmatrix} 1 - i & 0 \\ 0 & 1 + i \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}i \\ \frac{1}{2} & -\frac{1}{2}i \end{pmatrix}.$$

Therefore,

$$e^{tA} = S e^{tD} S^{-1} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{(1-i)t} & 0 \\ 0 & e^{(1+i)t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2}i \\ \frac{1}{2} & -\frac{1}{2}i \end{pmatrix} = \begin{pmatrix} e^t \cos t & e^t \sin t \\ -e^t \sin t & e^t \cos t \end{pmatrix},$$

where we used that $e^{it} = \cos t + i \sin t$. Theorem 4.7 now implies the statement.

Example 4.13. We consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}.$$

The characteristic polynomial is $\det(A - \lambda \mathrm{Id}) = (\lambda + 1)^2$ and hence, there exists one eigenvalue $\lambda = -1$ with multiplicity two. Moreover, the corresponding eigenvector is $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We find the Jordan normal form

$$A = SJS^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix},$$

and hence, the matrix exponential is

$$e^{tA} = S e^{tJ} S^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} e^{-t} + 2te^{-t} & -te^{-t} \\ 4te^{-t} & e^{-t} - 2te^{-t} \end{pmatrix}.$$

Remark 4.14. Let $A \in \mathbb{C}^{n \times n}$ and $A = SJS^{-1}$ be its Jordan decomposition.

- The matrix exponential is then given by $e^{tA} = S e^{tJ} S^{-1}$ and taking the induced matrix norm we conclude that $\|e^{tA}\| \le \|S\| \, \|e^{tJ}\| \, \|S^{-1}\|$. Consequently, we have

$$\frac{1}{c} \|e^{tJ}\| \le \|e^{tA}\| \le c \|e^{tJ}\|, \tag{4.5}$$

  where $c = \|S\| \, \|S^{-1}\|$.

- If J is a Jordan matrix we can compute the induced matrix norm corresponding to the maximum norm as follows,

$$\|e^{tJ}\|_{max} = \max_{1 \le k \le m} \|e^{tJ_k}\|_{max} = \max_{1 \le k \le m} |e^{t\lambda_k}| \sum_{j=0}^{n_k - 1} \frac{|t|^j}{j!}, \tag{4.6}$$

  where $J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_m \end{pmatrix}$.

4.4 General theory

We now consider IVPs for linear systems with variable coefficients such as introduced in Section 4.1,

$$\begin{aligned} u'(t) &= A(t)u(t) + f(t), \\ u(t_0) &= u_0, \end{aligned} \tag{4.7}$$

where $u_0 \in \mathbb{R}^n$, $A : J \to \mathbb{R}^{n \times n}$ and $f : J \to \mathbb{R}^n$ are continuous functions and $J \subset \mathbb{R}$ is an interval such that $t_0 \in J$. As for systems with constant coefficients, we first solve the homogeneous problem

$$u'(t) = A(t)u(t), \tag{4.8}$$

and afterwards the inhomogeneous problem by deriving a suitable variation of constants formula.

By Theorem 4.3, for every $u_0 \in \mathbb{R}^n$ there exists a unique solution $u : J \to \mathbb{R}^n$ of the IVP (4.7). Moreover, for homogeneous problems the superposition principle holds (Proposition 4.2).

Theorem 4.15. *Let $A : J \to \mathbb{R}^{n\times n}$ be continuous. Then, the solutions $u : J \to \mathbb{R}^n$ of the homogeneous problem* (4.8) *form an n-dimensional real vector space $\mathcal{L}$. In particular, the mapping*

$$u_0 \mapsto u(t; t_0, u_0)$$

where $u(\,\cdot\,; t_0, u_0)$ denotes the unique solution of the IVP (4.8) *with $u(t_0) = u_0$, defines a linear isomorphism between $\mathbb{R}^n$ and the vector space of solutions $\mathcal{L}$.*

Proof. Consider the set of solutions $\mathcal{L} = \{u \in C^1(J; \mathbb{R}^n) : u \text{ solution of (4.8)}\}$ of the homogeneous system. By the superposition principle (Proposition 4.2), they form a linear subspace of the space of continuously differentiable functions $C^1(J; \mathbb{R}^n)$.

We define the mapping $\Phi_{t_0} : \mathcal{L} \to \mathbb{R}^n$ by $\Phi_{t_0}(u) := u(t_0)$, which is certainly linear. Next, we show that it is bijective. By Theorem 4.3, for every $u_0 \in \mathbb{R}^n$ there exists a unique solution of the IVP

$$u'(t) = A(t)u(t), \qquad u(t_0) = u_0.$$

Since $\Phi_{t_0}(u) = u(t_0) = u_0$, it follows that Φ_{t_0} is surjective.

Furthermore, let $u \in \mathcal{L}$ be such that $\Phi_{t_0} = u(t_0) = 0$ and consider $v \equiv 0$. Then, u and v are both solutions of the IVP

$$u'(t) = A(t)u(t), \qquad u(t_0) = 0,$$

and by the uniqueness of solutions, it follows that $u \equiv v \equiv 0$ on J. This shows that the kernel of Φ_{t_0} is zero and consequently, Φ_{t_0} is injective.

This proves that Φ_{t_0} is an isomorphism. In particular, the dimension of the subspace $\mathcal{L}$ equals the dimension of $\mathbb{R}^n$. □

An immediate consequence is the following property.

Corollary 4.16. *Let the assumptions of Theorem 4.15 be satisfied. If u is a solution of* (4.8) *with $u(\hat{t}) = 0$ for some $\hat{t} \in J$, then $u \equiv 0$ in J.*

Proof. We observe that $v \equiv 0$ is a solution of the ODE. Therefore, $u : J \to \mathbb{R}^n$ and $v : J \to \mathbb{R}^n$ are both solutions of the ODE that satisfy $u(\hat{t}) = v(\hat{t}) = 0$. By Proposition 3.20, they must coincide on J, i.e. $u \equiv v \equiv 0$. □

Our aim is to find a basis of the solution space $\mathcal{L} = \{u \in C^1(J; \mathbb{R}^n) : u \text{ solution of (4.8)}\}$. We call k solutions $u_1, \dots, u_k$ of the ODE (4.8) **linearly dependent**, if there exists $c_1, \dots, c_k \in \mathbb{R}$ with $|c_1| + \cdots + |c_k| > 0$ such that

$$c_1 u_1 + \cdots + c_k u_k = 0. \tag{4.9}$$

Otherwise, the solutions $u_1, \dots, u_k$ are called **linearly independent**.

Note that by Corollary 4.16, the equation (4.9) holds in one point $\hat{t} \in J$ if and only if it holds in the entire interval J.

Definition 4.17. The functions $u_1, \dots, u_n \in \mathcal{L}$ form a **fundamental system** for the homogeneous system (4.8) if $\{u_1, \dots, u_n\}$ is a basis of $\mathcal{L}$.

A matrix function $U : J \to \mathbb{R}^{n\times n}$ is called **fundamental matrix** for the homogeneous system (4.8) if the columns of U form a fundamental system for (4.8).

Proposition 4.18. (i) Let U be a fundamental matrix for the homogeneous system (4.8). Then, every solution of the homogeneous system is of the form $u(t) = U(t)c$, for some $c \in \mathbb{R}^n$.

(ii) The matrix $U : J \to \mathbb{R}^{n\times n}$ is a fundamental matrix for (4.8) if and only if U satisfies the matrix differential equation

$$U'(t) = A(t)U(t) \qquad \forall t \in J,$$

and there exists $\hat{t} \in J$ such that the matrix $U(\hat{t})$ is regular (i.e. invertible).

Proof. (i) If $U = (u_1 \cdots u_n)$ is a fundamental matrix, then $\{u_1, \dots, u_n\}$ is a basis of $\mathcal{L}$. Consequently, if u is a solution of (4.8), then there exist $c_1, \dots, c_n \in \mathbb{R}$ such that

$$u(t) = c_1 u_1(t) + \cdots + c_n u_n(t) = (u_1(t) \cdots u_n(t)) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = U(t)c,$$

for some $c \in \mathbb{R}^n$.

(ii) $\Rightarrow$: Let $U(t) = (u_1(t) \cdots u_n(t))$ be a fundamental matrix. Then, $u_1, \dots, u_n$ is a fundamental system for the homogeneous system (4.8). Consequently,

$$\begin{aligned} U'(t) &= (u_1'(t) \cdots u_n'(t)) = (A(t)u_1(t) \cdots A(t)u_n(t)) \\ &= A(t)(u_1(t) \cdots u_n(t)) = A(t)U(t). \end{aligned}$$

Moreover, $u_1, \dots, u_n$ are linearly independent in $\mathcal{L}$, which implies that $u_1(\hat{t}), \dots, u_n(\hat{t})$ are linearly independent in $\mathbb{R}^n$, for all $\hat{t} \in J$. This implies that the matrix $U(\hat{t}) = (u_1(\hat{t}) \cdots u_n(\hat{t}))$ is regular.

$\Leftarrow$: If $U = (u_1 \cdots u_n)$ satisfies the matrix differential equation, then the i-th column u_i satisfies the ODE $u_i'(t) = A(t)u_i(t)$, $i = 1, \dots, n$. Moreover, $u_1(\hat{t}), \dots, u_n(\hat{t})$ are linearly independent in $\mathbb{R}^n$ if and only if $u_1, \dots, u_n$ are linearly independent in $\mathcal{L}$ by Corollary 4.16. Consequently, $u_1, \dots, u_n$ form a fundamental system for (4.8), which implies that $U = (u_1 \cdots u_n)$ is a fundamental matrix. □

Example 4.19. Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable, $\lambda_1, \dots \lambda_n \in \mathbb{R}$ be its eigenvalues and $v_1, \dots, v_n$ be the corresponding eigenvectors, i.e.

$$A = SDS^{-1} = S \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}, \qquad S = (v_1 \cdots v_n).$$

Then, $U(t) = Se^{tD} = S \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$ is a fundamental matrix for the homogeneous system with constant coefficients $u'(t) = Au(t)$. In particular, every solution is of the form

$$u(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n, \qquad c_1, \dots, c_n \in \mathbb{R}.$$

Indeed, we observe that

$$U'(t) = SDe^{tD} = SDS^{-1}S\,e^{tD} = AS\,e^{tD} = AU(t),$$

and $\det(U(t)) = \det(S)\det(e^{tD}) \neq 0$ for all $t \in \mathbb{R}$. Consequently, U is a fundamental matrix by Proposition 4.18.

We remark that another fundamental matrix is $\widetilde{U}(t) = S\,e^{tD}S^{-1.}$

Definition 4.20. Let $A : J \to \mathbb{R}^{n\times n}$ be continuous. If $U = (u_1 \cdots u_n) : I \to \mathbb{R}^{n\times n}$ is a solution of the matrix differential equation

$$U'(t) = A(t)U(t),$$

its determinant $w(t) = \det U(t)$ is called the **Wronskian** of the solution system $u_1, \ldots, u_n$.

We remark that in the previous definition we do not require that U is a fundamental matrix.

Theorem 4.21. *Let $A : J \to \mathbb{R}^{n\times n}$ be continuous and $U : J \to \mathbb{R}^{n\times n}$ be a solution of the matrix differential equation*

$$U'(t) = A(t)U(t).$$

Then, the Wronskian $w(t) = \det U(t)$ satisfies the scalar ODE

$$w'(t) = tr(A(t))w(t), \qquad t \in J,$$

where $tr(A(t)) = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)$ is the trace of A. Consequently, we have

$$w(t) = w(t_0)e^{\int_{t_0}^{t} tr(A(s))ds}, \qquad \textit{for some } t \in J.$$

Proof. A proof can be found, e.g. in [6]. □

Theorem 4.21 allows to compute the Wronskian without knowing the fundamental matrix U. In particular, if we choose the fundamental matrix $\widetilde{U}(t)$ with initial value $\widetilde{U}(t_0) = \mathrm{Id}$, then $w(t) = \det \widetilde{U}(t)$ is given by

$$w(t) = e^{\int_{t_0}^{t} \mathrm{tr}(A(s))ds}.$$

Corollary 4.22. *Let the hypothesis of Theorem 4.21 be satisfied. Then, the Wronskian w is either $w \equiv 0$ in J or $w(t) \neq 0$ for all $t \in J$.*

In particular, U is a fundamental matrix for the homogeneous system (4.8) *if and only if $w(t) = \det U(t) \neq 0$ for all $t \in J$ and U satisfies $U'(t) = A(t)U(t)$.*

Proof. This is an immediate consequence of Theorem 4.21 and Proposition 4.18. □

If U satisfies the matrix differential equation $U'(t) = A(t)U(t)$, then the non-vanishing of the Wronskian is a necessary and sufficient condition for U being a fundamental matrix.

In general, it is not easy to find a fundmental matrix. In certain cases, Theorem 4.21 allows to determine the fundamental matrix, if one particular solution of the ODE system in known. This is illustrated in the following example.

Example 4.23. Consider the ODE

$$x''(t) = p(t)x'(t) + q(t)x(t), \tag{4.10}$$

where $p, q : \mathbb{R} \to \mathbb{R}$ are continuous. It can be rewritten as the first order system

$$u'(t) = A(t)u(t), \qquad A(t) = \begin{pmatrix} 0 & 1 \\ q(t) & p(t) \end{pmatrix}, \quad u(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$$

We assume that one solution x is known and aim to find a fundamental matrix for the system,

$$U(t) = (x(t)y(t)) = \begin{pmatrix} x(t) & y(t) \\ x'(t) & y'(t) \end{pmatrix},$$

i.e. we need to find a second linearly independent solution y.

By Theorem 4.21, the Wronskian $w(t) = \det U(t) = x(t)y'(t) - x'(t)y(t)$ satisfies

$$w(t) = w(t_0)e^{\int_{t_0}^t p(s)ds}.$$

If $x \neq 0$ (at least on some interval), then we can find y by solving the ODE

$$y'(t) = \frac{x'(t)}{x(t)}y(t) + \frac{1}{x(t)}w(t_0)e^{\int_{t_0}^t p(s)ds},$$

which is a scalar linear ODE for y that we can solve explicitly.

We conclude that all solutions of the ODE (4.10) are of the form

$$z(t) = c_1x(t) + c_2y(t), \qquad c_1, c_2 \in \mathbb{R}.$$

Finally, we consider IVPs for linear inhomogeneous systems (4.7), i.e.

$$\begin{aligned} u'(t) &= A(t)u(t) + f(t), \\ u(t_0) &= u_0, \end{aligned}$$

where $u_0 \in \mathbb{R}^n$ and $A : J \to \mathbb{R}^{n\times n}$ and $f : J \to \mathbb{R}^n$ are continuous.

As for scalar equations and systems with constant coefficients we will obtain the solution of the inhomogeneous problem by a suitable variation of constants formula.

Theorem 4.24. *Every solution of the inhomogeneous problem* (4.7) *can be written in the form*

$$u(t) = u_p(t) + u_h(t),$$

where u_p is one particular solution of the inhomogeneous system and u_h a solution of the homogeneous system (4.8).

Proof. Let u_p be one particular solution of the inhomogeneous system and u be an arbitrary solution of the inhomogeneous system. Then, their difference satisfies the homogeneous system

$$u'(t) - u_p'(t) = A(t)(u(t) - u_p(t)),$$

i.e. $u - u_p \in \mathcal{L}$. □

The problem therefore reduces to the problem of finding one particular solution of the inhomogeneous problem. To this end we use the *variation of constants method*. We recall that if U is a fundamental matrix of the homogeneous problem, then all solutions of the homogeneous system are of the form $u(t) = U(t)c$, for some $c \in \mathbb{R}^n$. We now vary the constant and make the ansatz

$$u(t) = U(t)c(t). \tag{4.11}$$

In particular, we aim to find a function c such that u solves the inhomogeneous system. We obtain

$$u'(t) = U'(t)c(t) + U(t)c'(t) = A(t)U(t)c(t) + U(t)c'(t) = A(t)u(t) + U(t)c'(t),$$

and hence, u is a solution of the inhomogeneous system if c satisfies

$$U(t)c'(t) = f(t).$$

Since U is a fundamental matrix, the Wronskian $w(t) = \det U(t) \neq 0, t \in J$, and hence, the inverse matrix $(U(t))^{-1}$ exists and is continuous in J. Multiplying the above equation from the left by $(U(t))^{-1}$ we obtain

$$c'(t) = (U(t))^{-1} f(t)$$

and integrating from t_0 to t is follows that

$$c(t) = c(t_0) + \int_{t_0}^{t} (U(s))^{-1} f(s) ds.$$

Finally, by (4.11) the solution u is given by

$$u(t) = U(t)c(t_0) + U(t) \int_{t_0}^{t} (U(s))^{-1} f(s) ds,$$

where we need to choose $c(t_0)$ such that u satisfies the initial value.

Theorem 4.25. *Let $A : J \to \mathbb{R}^{n\times n}$ and $f : J \to \mathbb{R}^n$ be continuous and $u_0 \in \mathbb{R}^n$. Then, there exists a unique solution of the IVP* (4.7)*,*

$$\begin{aligned} u'(t) &= A(t)u(t) + f(t), \\ u(t_0) &= u_0, \end{aligned}$$

which is given by

$$u(t) = U(t)(U(t_0))^{-1} u_0 + U(t) \int_{t_0}^{t} (U(s))^{-1} f(s) ds,$$

where U is a fundamental matrix for the homogeneous system.

Proof. By Corollary 4.22, U satisfies $U'(t) = A(t)U(t), t \in J$, and $U(t)$ is regular for all $t \in J$. Consequently, the inverse matrix $(U(t))^{-1}$ exists and the solution formula is well-defined. We obtain

$$\begin{aligned} u'(t) &= U'(t)(U(t_0))^{-1} u_0 + U'(t) \int_{t_0}^{t} (U(s))^{-1} f(s) ds + U(t)(U(t))^{-1} f(t) \\ &= U'(t)\left((U(t_0))^{-1} u_0 + \int_{t_0}^{t} (U(s))^{-1} f(s) ds\right) + f(t) \\ &= A(t)u(t) + f(t), \end{aligned}$$

for all $t \in J$, i.e. u satisfies the ODE. Moreover, $u(t_0) = u_0$, i.e. u is a solution of the IVP. The uniqueness of solutions follows from Theorem 4.3. □

Example 4.26. Consider the linear inhomogeneous system

$$u'(t) = \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ t \end{pmatrix}, \qquad t > 0,$$

with initial value $u(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

We first observe that $U(t) = \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix}$ is a fundamental matrix for the homogeneous system. Indeed, $\det U(t) = -t^2 \neq 0$ for all $t > 0$ and

$$A(t)U(t) = \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix} \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} = \begin{pmatrix} 2t & 1 \\ 2 & 0 \end{pmatrix} = U'(t),$$

and hence, Corollary 4.22 implies that U is a fundamental matrix.

We can now compute the solution of the inhomogeneous IVP using the variation of constants formula in Theorem 4.25. We observe that

$$(U(t))^{-1} = -\frac{1}{t^2} \begin{pmatrix} 1 & -t \\ -2t & t^2 \end{pmatrix}$$

and consequently,

$$\begin{aligned} u(t) &= U(t)(U(t_0))^{-1}u_0 + U(t) \int_{t_0}^{t} (U(s))^{-1} f(s) ds \\ &= \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} \int_1^t \frac{-1}{s^2} \begin{pmatrix} 1 & -s \\ -2s & s^2 \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} ds \\ &= \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} t^2 & t \\ 2t & 1 \end{pmatrix} \int_1^t \begin{pmatrix} 1 \\ -s \end{pmatrix} ds = \begin{pmatrix} \frac{1}{2}t^3 - 2t^2 + \frac{5}{2}t \\ \frac{3}{2}t^2 - 4t + \frac{5}{2} \end{pmatrix}. \end{aligned}$$

4.5 Exercises

E4.1 **Matrix exponentials**

(a) Compute e^{tA} if $A \in \mathbb{R}^{n \times n}$ is of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}.$$

(b) Find an example of two matrices $A, B \in \mathbb{R}^{2 \times 2}$ such that

$$e^{A+B} \neq e^A e^B.$$

E4.2 **Induced matrix norm**

Consider $\mathbb{R}^n$ with the maximum norm $\|\cdot\|_{max}$. Show that the induced matrix norm is given by

$$\|A\| = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|, \qquad A \in \mathbb{R}^{n\times n}.$$

E4.3 **Matrix exponentials**

Compute e^{tA} and e^{tB} for the following matrices:

$$A = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

E4.4 **Initial value problems**

Consider the matrices A and B in Problem 1.

(a) Compute the solution of the IVP

$$u'(t) = Au(t), \qquad u(0) = \begin{pmatrix} -3 \\ 3 \end{pmatrix}.$$

(b) Compute the solution of the IVP

$$u'(t) = Bu(t) + \begin{pmatrix} t \\ 0 \end{pmatrix}, \qquad u(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

E4.5 **Cauchy-Euler equation**

Let $a, b \in \mathbb{R}$. The Cauchy-Euler equation is the ODE

$$x''(t) + \frac{b}{t}x'(t) + \frac{a}{t^2}x(t) = 0, \qquad t > 0.$$

We assume that $a < \frac{1}{4}(1-b)^2$. Find the solution of the ODE and determine the behavior of the solution as t tends to infinity.

To this end, first transform the equation into a linear, second order ODE by introducing the change of variables $s = \ln t$, i.e. $t = e^s$.

E4.6 **Fundamental matrix**

Consider the ODE

$$x''(t) + \frac{2}{t}x'(t) + x(t) = 0, \qquad t > 0. \tag{4.12}$$

One solution of the ODE is given by the function $x(t) = \frac{\sin(t)}{t}$. Rewrite the second order ODE as a system of first order ODEs and find the fundamental matrix of this system. Finally, determine the solution of the original ODE (4.12) corresponding to the initial values $x(\pi) = 1$ and $x'(\pi) = 0$.

Hint: See Example 4.22.

E4.7 **Inhomogeneous IVP**

Consider the ODE system

$$u'(t) = -\frac{1}{2t}u(t) + \frac{1}{2t^2}v(t) + t,$$
$$v'(t) = \frac{1}{2}u(t) + \frac{1}{2t}v(t) + t^2,$$

for $t > 0$.

(a) Show that $U(t) = \begin{pmatrix} 1 & \frac{1}{t} \\ t & -1 \end{pmatrix}$ is a fundamental matrix for the homogeneous system.

(b) Compute the solution of the inhomogeneous IVP with $u(1) = 1, v(1) = 2$.

E4.8 **Fundamental matrix**

Find a fundamental matrix for the ODE system

$$u'(t) = \begin{pmatrix} -1 & 2e^{3t} \\ 0 & -2 \end{pmatrix} u(t).$$

<u>*Hint:*</u> *Note that the solutions can be computed explicitly.*

Chapter 5

Stability and linearization

In this chapter we analyze the qualitative behavior of *autonomous* ODE systems,

$$u'(t) = f(u(t)), \tag{5.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. In general, we cannot find explicit solutions, but we can often derive qualitative properties of solutions without explicitly solving the ODE. In particular, we aim to study equilibria and their stability properties. Before we consider general nonlinear autonomous systems, we analyze linear systems.

Phase portraits

A solution $u(t) = (u_1(t), \dots, u_n(t)), t \in I$, $I \subset \mathbb{R}$ an interval, of (5.1) can be viewed as a parametetrization of a curve in $\mathbb{R}^n$. This curve is called **trajectory** or **orbit** of the solution, and $\mathbb{R}^n$ is called the **phase space** (or **phase plane** if $n = 2$). Drawing several trajectories corresponding to different initial conditions and indicating the orientation of the curve by small arrows, we obtain the **phase portrait** for the ODE. The arrows indicate in which direction the solutions transverse the curve as time increases.

Phase portraits illustrate the qualitative behavior of solutions such as equilibria and their stability or periodicity of solutions and their longtime behavior.

5.1 Stability of equilibria

Here and in the sequel, we assume that a unique solution of the initial value problem

$$\begin{aligned} u'(t) &= f(u(t)), \\ u(t_0) &= u_0, \end{aligned} \tag{5.2}$$

exists for every $t_0 \in \mathbb{R}, u_0 \in \mathbb{R}^n$, and that the solution $u = u(\cdot; t_0, u_0) : [t_0, \infty) \to \mathbb{R}^n$ exists for all $t \geq t_0$.

Recall that every zero $u^* \in \mathbb{R}^n$ of the function f corresponds to an equilibrium of the ODE, i.e. to a constant solution $u \equiv u^*$ of the ODE $u'(t) = f(u(t))$.

Definition 5.1. Let $u^* \in \mathbb{R}^n$ be an equilibrium of the ODE (5.1) and let $\|\cdot\|$ denote an arbitrary norm in $\mathbb{R}^n$.

- We call the equilibrium u^* **stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|u_0 - u^*\| < \delta \quad \text{implies that} \quad \|u(t) - u^*\| < \varepsilon \quad \forall t \geq t_0.$$

- We call the equilibrium u^* **unstable** if u^* is not stable.
- We call the equilibrium u^* **asymptotically stable** if it is stable and if there exists $\eta > 0$ such that

$$\|u_0 - u^*\| < \eta \quad \text{implies that} \quad \lim_{t\to\infty} u(t) = u^*.$$

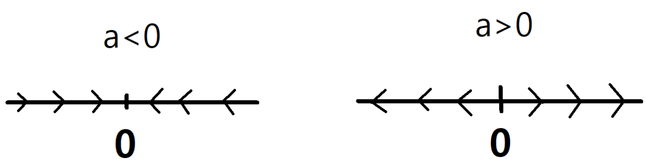
Since all norms in $\mathbb{R}^n$ are equivalent, the concept of stability is independent of the norm we choose. Furthermore, we remark that stability as defined above is a *local property*. A steady state is stable if solutions starting close to u^* remain close u^* for all times, and solutions even converge to u^* if the equilibrium is asymptotically stable. The behavior of solutions that do not start close to the equilibrium u^* is not addressed in these definitions.

Example 5.2. Consider the scalar ODE

$$u'(t) = au(t),$$

where $a \in \mathbb{R}$. Then, all solutions are of the form $u(t) = ce^{at}$, for some $c \in \mathbb{R}$. If $a \neq 0$ then $u^* = 0$ is the only equilibrium. If $a = 0$, then every solution is an equilibrium. Moreover, the steady state $u^* = 0$ is stable if $a \leq 0$, unstable if $a > 0$ and asymptotically stable if $a < 0$.

The phase portraits for this ODE with $a \neq 0$ are sketched below.



5.2 Linear systems

For autonomous linear homogeneous systems, i.e. systems of the form

$$u'(t) = Au(t),$$

where $A \in \mathbb{R}^{n\times n}$, the zero solution $u^* = 0$ is an equilibrium. Moreover, it is the only equilibrium if the matrix A is non-singular. In this case, we can easily characterize the stability properties of the equilibrium $u^* = 0$.

Classification for two-dimensional systems

We first discuss two-dimensional systems whose phase portraits can be fully classified. We consider

$$u'(t) = Au(t), \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.3}$$

where $a, b, c, d \in \mathbb{R}$. Then, $u^* = 0$ is an equilibrium, and all solutions of the ODE are of the form

$$u(t) = e^{At}\hat{c}, \qquad t \in \mathbb{R},$$

for some constant $\hat{c} \in \mathbb{R}^2$.

Normal forms

We aim to characterize the phase portraits for the linear ODE (5.3). We recall that two matrices $A, B \in \mathbb{R}^{2\times 2}$ are **equivalent** (or **similar**) if there exists an invertible matrix $S \in \mathbb{R}^{2\times 2}$ such that $A = SBS^{-1}$. Moreover, similar matrices have the same eigenvalues, determinant and trace.

Every non-singular matrix $A \in \mathbb{R}^{2\times 2}$ is equivalent to one of the following *normal forms*

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad A_3 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where λ, μ, α and β are real numbers such that $\mu \neq 0$, $\lambda \neq 0$ and $\beta \neq 0$. Note that zero is not an eigenvalue as A is not singular.

If A and B are similar matrices and u is a solution of the ODE

$$u'(t) = Au(t),$$

then, $v = S^{-1}u$ satisfies the ODE

$$v'(t) = Bv(t), \qquad B = S^{-1}AS.$$

Under such transformations the phase portraits are subject to an affine transformation (e.g. circles transform into ellipses), but the qualitative behavior of solutions (e.g. the stability of equilibria) of both systems is the same. Therefore, to analyze the qualitative behavior of the ODE (5.3) with a non-singular matrix A, it suffices to study the behavior of the system for the normal forms A_1, A_2 and A_3.

Consider the matrix A in (5.3) and assume it is non-singular, i.e. $\det(A) \neq 0$. Its characteristic polynomial is

$$\det(A - \lambda \mathrm{Id}) = \lambda^2 - \lambda \mathrm{tr}(A) + \det(A),$$

where $\mathrm{tr}(A) = a + d$ is the trace of A and $\det(A) = ad - bc$. Hence, we obtain the eigenvalues

$$\lambda_1 = \frac{1}{2}\Big(\mathrm{tr}(A) + \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}\Big), \qquad \lambda_2 = \frac{1}{2}\Big(\mathrm{tr}(A) - \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}\Big),$$

and can distinguish the following cases:

- If $\mathrm{tr}(A)^2 > 4\det(A)$ we have two real eigenvalues λ_1 and λ_2, the matrix is diagonizable and similar to A_1. In fact, if $S = (v_1 v_2)$ is the matrix whose columns are the eigenvectors v_1 and v_2 corresponding to λ_1 and λ_2, then $A = SA_1S^{-1}$.
- If $\mathrm{tr}(A)^2 < 4\det(A)$ we have two conjugate complex eigenvalues $\lambda_1 = \lambda$ and $\lambda_2 = \bar{\lambda}$ and the matrix is similar to A_3. In fact, if v and $\bar{v}$ are the corresponding (complex) eigenvectors, then with $S = (v\bar{v})$ being the matrix with columns v and $\bar{v}$ we have $A = S\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}S^{-1}$. However,

we aim to find a real normal form. To this end we write $v = v_1 + iv_2$ and $\lambda = \alpha + i\beta$ with $\beta > 0$. Splitting the equation $Av = \lambda v$ into the real and imaginary parts we obtain

$$\begin{cases} Av_1 = \alpha v_1 - \beta v_2 \\ Av_2 = \alpha v_2 + \beta v_1 \end{cases} \iff A(v_1 v_2) = (Av_1 Av_2) = (v_1 v_2)\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Since $v, \bar{v}$ are linearly independent and can be represented as a linear combination of v_1 and v_2 ($v_1 = \frac{1}{2}(v + \bar{v})$, $v_2 = -\frac{i}{2}(v - \bar{v})$), the vectors v_1 and v_2 are linearly independent as well. Hence, the matrix $\widetilde{S} = (v_1 v_2)$ is regular and we have $A = \widetilde{S} A_3 \widetilde{S}^{-1}$.

- Finally, if $\mathrm{tr}(A)^2 = 4\det(A)$, the matrix A has only one eigenvalue $\lambda \in \mathbb{R}$. If there exists two linearly independent eigenvectors, then A is similar to A_1 with $\lambda = \mu$. Otherwise, A is defective and similar to A_2. In fact, in this case there exists one eigenvector v_1 corresponding to the eigenvector λ and $v_2 \in \mathbb{R}^2$ such that $(A - \lambda \mathrm{Id})v_2 = v_1$. Then, the matrix $S = (v_1 v_2)$ satisfies $A = SA_2S^{-1}$.

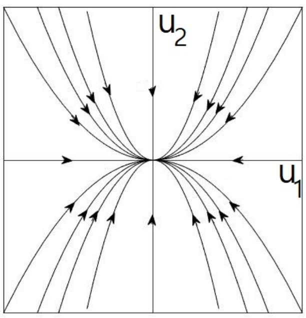
Phase portraits

Up to rotations and scaling, the phase portrait of the ODE (5.3) with a non-singular matrix A is determined by the phase portraits corresponding to A_1, A_2 or A_3. Hence, it suffices to determine the phase portraits for these normal forms. We plot the phase portraits for the matrices A_1, A_2 and A_3, the case that A is singular is left as an exercise (see case (v)).

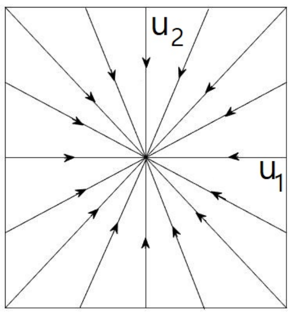
(i) For the matrix A_1, if $\mu \leq \lambda < 0$ or $\mu \geq \lambda > 0$, all solutions are of the form $u_1(t) = c_1 e^{\lambda t}$, $u_2(t) = c_2 e^{\mu t}$, $t \in \mathbb{R}$, for some constants $c_1, c_2 \in \mathbb{R}$. The solution curves are determined by

$$\left(\frac{u_1}{c_1}\right)^{\mu} = \left(\frac{u_2}{c_2}\right)^{\lambda}, \qquad c_1, c_2 \neq 0, \ \frac{u_1}{c_1}, \frac{u_2}{c_2} > 0.$$

If $\lambda = \mu$ the curves are straight lines, and potential curves otherwise.



$\mu < \lambda < 0$

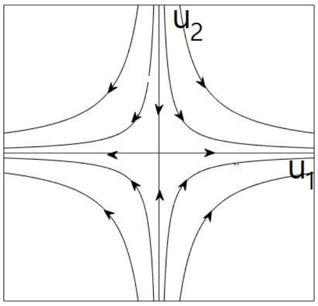


$\mu = \lambda < 0$

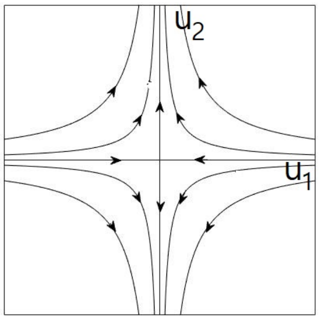
If $\mu \leq \lambda < 0$, then $\|u(t)\| = 0$ as $t \to \infty$, i.e. the solutions tend to $u^* = 0$ as t tends to infinity. In this case, the equilibrium $u^* = 0$ is asymptotically stable and called a **stable node**.

If the signs of the eigenvalues are reversed, i.e. $\mu \geq \lambda > 0$, the arrows point in the opposite direction and $\|u(t)\| \to \infty$ as $t \to \infty$. Then, $u^* = 0$ is unstable and called an **unstable node**.

(ii) For the matrix A_1, if λ and μ have opposite signs, the solutions are again given by $u_1(t) = c_1 e^{\lambda t}$, $u_2(t) = c_2 e^{\mu t}, t \in \mathbb{R}$, for some constants $c_1, c_2 \in \mathbb{R}$, and the solution curves are determined by the same relation as in (i). However, $\frac{\mu}{\lambda} < 0$ and the phase portrait looks essentially different. We have two trajectories pointing to the origin, all other solutions satisfy $\|u(t)\| \to \infty$ as $t \to \infty$. In this case, the origin is an unstable equilibrium and called a **saddle**.



$\lambda > 0 > \mu$

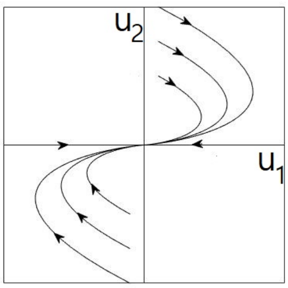


$\lambda < 0 < \mu$

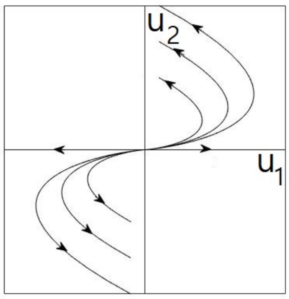
(iii) For the matrix A_2, all solutions are of the form $u_1(t) = (c_1 + tc_2)e^{\lambda t}$, $u_2(t) = c_2 e^{\lambda t}, t \in \mathbb{R}$, for some constants $c_1, c_2 \in \mathbb{R}$. If $c_1 = 0$, i.e. we start with an initial value $u_1(0) = 0, u_2(0) = c_2$, then we have $u_1(t) = tu_2(t)$. The trajectories are then determined by

$$\lambda u_1 = u_2 \ln\left(\frac{u_2}{c_2}\right), \qquad \text{if } c_2 \neq 0.$$

If $\lambda < 0$, then $\|u(t)\| \to 0$ as $t \to \infty$, i.e. the solutions tend to $u^* = 0$ as t tends to infinity. In this case, the equilibrium $u^* = 0$ is asymptotically stable and called a **stable node**. If the sign of the eigenvalue λ is reversed, the arrows point in the opposite direction and $\|u(t)\| \to \infty$ as $t \to \infty$. Then, the equilibrium $u^* = 0$ is unstable and called an **unstable node**.



$\lambda < 0$



$\lambda > 0$

(iv) For the matrix A_3, the solutions are of the form

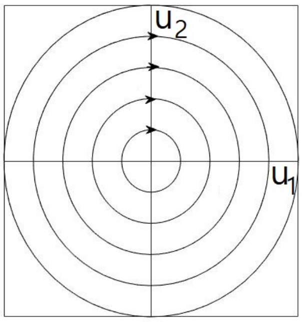
$$u_1(t) = (c_1 \cos(\beta t) - c_2 \sin(\beta t))e^{\alpha t},$$
$$u_2(t) = (c_1 \sin(\beta t) + c_2 \cos(\beta t))e^{\alpha t},$$

$t \in \mathbb{R}$, for some constants $c_1, c_2 \in \mathbb{R}$. We could equally write the solutions in complex notation

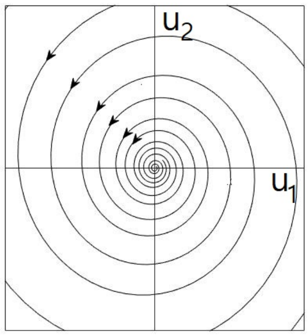
$$u_1(t) = ce^{\alpha t - i\beta t}, \qquad u_2(t) = ce^{\alpha t + i\beta t}, \qquad c \in \mathbb{C}.$$

We observe that $\|u(t)\| = \|c\| e^{\alpha t}$, $t \in \mathbb{R}$. If $\alpha = 0$, the solution curves are circles around the origin and the sign of β determines in which direction the curves are traversed. In this case, $u^* = 0$ is stable, but not asymptotically stable, and called a **center**.

If $\alpha < 0$ the solution curves spiral around the origin and converge to $u^* = 0$ as time tends to infinity. In this case, the origin is asymptotically stable and called a **stable spiral**. Finally, if $\alpha > 0$, the solution curves spiral around the origin, but $\|u(t)\| \to \infty$ as $t \to \infty$. In this case, $u^* = 0$ is unstable and called an **unstable spiral**.



$\alpha = 0,\ \beta > 0$



$\alpha < 0,\ \beta < 0$

(v) Finally, if $A \neq 0$ is not invertible, then A is similar to

$$A_4 = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad A_5 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with $\lambda \neq 0$. For the phase portraits, we refer to the exercises.

In the case of matrix A_4, the steady state $u^* = 0$ is stable if $\lambda < 0$, but not asymptotically stable, and unstable if $\lambda > 0$. For the matrix A_5, the origin $u^* = 0$ is unstable.

Summarizing, we observe that if all eigenvalues of the matrix A in (5.3) satisfy $\mathrm{Re}(\lambda) < 0$, the origin $u^* = 0$ is asymptotically stable, while the origin is unstable if there exists an eigenvalue with $\mathrm{Re}(\lambda) > 0$.

Example 5.3. The following scalar second order ODE

$$x''(t) + 2\alpha x'(t) + \beta x(t) = 0$$

describes a *damped oscillation*. We assume that the initial values are $x(0) = x_0$, $x'(0) = x_1$ with $x_0, x_1 \in \mathbb{R}$. Here, x describes the displacement from equilibrium, $2\alpha > 0$ is the damping coefficient and $\beta > 0$ the spring constant.

Setting $y = x'$ we can rewrite the ODE as first order system,

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta & -2\alpha \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The eigenvalues and eigenvectors of the matrix A are

$$\lambda_{1/2} = -\alpha \pm \sqrt{\alpha^2 - \beta}, \qquad v_i = \begin{pmatrix} 1 \\ \lambda_i \end{pmatrix}, \qquad i = 1, 2.$$

Since α and β are positive, we conclude that $\mathrm{Re}(\lambda_i) < 0, i = 1, 2$, and hence, the zero steady state is asymptotically stable. We can distinguish three cases.

- *Overdamping*: If $\alpha^2 > \beta$ both eigenvalues are negative and the solutions are of the form

  $$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \qquad t \in \mathbb{R},$$

  where the constants $c_1, c_2 \in \mathbb{R}$ are determined by the initial values. In this case, the origin is a stable node and the phase portrait is of the type (i).

- *Critical damping*: If $\alpha^2 = \beta$, the matrix A is defective and we have only one eigenvalue. The solutions are of the form

  $$x(t) = (c_1 + c_2 t)e^{-\alpha t}, \qquad t \in \mathbb{R},$$

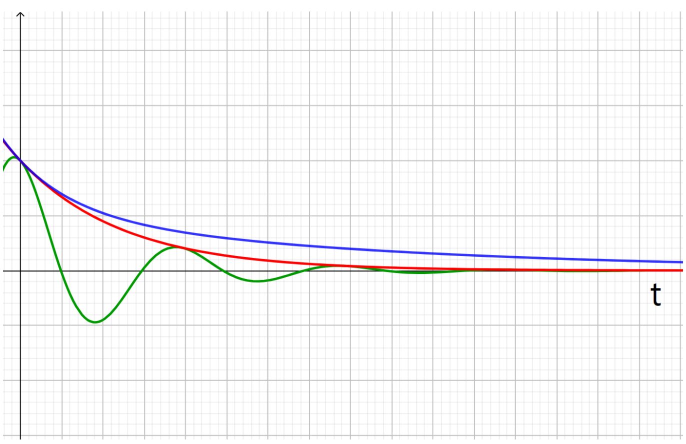
  and hence, the origin is a stable node and the phase portrait is of the type (iii).

- *Damped oscillation*: If $\alpha^2 < \beta$, we have two complex conjugate eigenvalues and the solutions are of the form

  $$x(t) = c_1 e^{-\alpha t}\cos(\omega t) + c_2 e^{-\alpha t}\sin(\omega t), \qquad t \in \mathbb{R},$$

  where $\omega = \sqrt{\beta - \alpha^2}$. In this case, the phase portrait is of the type (iv) and the origin is a stable spiral.

From the phase portraits we cannot determine how fast the solutions converge to the steady state. To compare the three cases we plot the graphs of the solutions versus time with the same values for x_0, x_1 and the parameter α, but we vary β. In the figure below the blue line corresponds to the case $\alpha^2 > \beta$ (overdamping), the red line to the case $\alpha^2 = \beta$ (critical damping) and the green line to the case $\alpha^2 < \beta$ (damped oscillations). We observe that the fastest decay occurs with critical damping.



Stability for n-dimensional systems

For two dimensional linear systems (5.3) we observed that the origin $u^* = 0$ is asymptotically stable if and only if $\text{Re}(\lambda) < 0$ for all eigenvalues λ of A and it is unstable if there exists an eigenvalue λ of A such that $\text{Re}(\lambda) > 0$. Moreover, the origin is stable if $\text{Re}(\lambda) \leq 0$ for all eigenvalues of A and the eigenvalues with $\text{Re}(\lambda) = 0$ are not defective. We aim to generalize these results now for n-dimensional linear systems.

We consider linear systems of the form,

$$u'(t) = Au(t), \tag{5.4}$$

where $A \in \mathbb{R}^{n\times n}$. Then, $u^* = 0$ is an equilibrium of the ODE and all solutions are of the form $u(t) = e^{tA}c$, for some $c \in \mathbb{R}^n$.

Remark 5.4. We observe that the origin is stable if and only if $\|e^{tA}\| \leq M$ for all $t \geq 0$, for some constant $M \geq 1$, where $\|\cdot\|$ denotes the induced matrix norm. Moreover, it is asymptotically stable, if and only if $\|e^{At}\| \to 0$ as $t \to \infty$ (see exercises).

We use these estimates to prove the following theorem which generalizes the properties we derived in the previous subsection for higher dimensional systems.

Theorem 5.5. *Consider the ODE* (5.4).

(i) The stationary solution $u^* = 0$ is stable if and only if every eigenvalue λ of A satisfies $Re(\lambda) \leq 0$ and eigenvalues with $Re(\lambda) = 0$ are not defective.

(ii) The stationary solution $u^* = 0$ is asymptotically stable if and only if every eigenvalue λ of A satisfies $Re(\lambda) < 0$.

(iii) The stationary solution $u^* = 0$ is unstable if and only if there exists an eigenvalue λ of A such that $Re(\lambda) > 0$, or if there exists an eigenvalues with $Re(\lambda) = 0$ that is defective.

Proof. There exists an invertible matrix $S \in \mathbb{C}^{n\times n}$ such that $A = SJS^{-1}$, where

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_m \end{pmatrix}, \qquad J_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ 0 & \lambda_k & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_k \end{pmatrix} \in \mathbb{C}^{n_k\times n_k},$$

where $n_1 + \cdots + n_m = n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ are the eigenvalues of A. By (4.5) in Remark 4.14 we have

$$\frac{1}{c}\|e^{tJ}\| \leq \|e^{tA}\| \leq c\|e^{tJ}\|,$$

for some constant $c > 0$. Moreover, using the maximum norm we have

$$\|e^{tJ}\|_{max} = \max_{1\leq k\leq m} \|e^{tJ_k}\|_{max} = \max_{1\leq k\leq m} |e^{t\lambda_k}| \sum_{j=0}^{n_k-1} \frac{|t|^j}{j!},$$

see (4.6) in Remark 4.14. Let $\alpha = \max\{\mathrm{Re}(\lambda_k) : k = 1, \dots, m\}$, where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of A with the corresponding dimensions $n_1, \dots, n_m$ of the Jordan blocks.

To show (i) we observe that the estimates (4.5) and (4.6) above imply that $\|e^{tA}\| \le M$ for all $t \ge 0$ if and only if $\|e^{tJ}\| \le \widetilde{M}$ for all $t \ge 0$. This is equivalent to the conditions $\alpha \le 0$ and $n_k = 1$ if there exists an eigenvalue λ_k with $\mathrm{Re}(\lambda_k) = 0$.

Moreover, to prove (ii) we note that $\|e^{tA}\| \to 0$ as $t \to \infty$ if and only if $\|e^{tJ}\| \to 0$ as $t \to \infty$. This is equivalent to $\alpha < 0$, which shows the second statement about asymptotic stability.

Finally, statement (iii) follows from (i) since $u^* = 0$ is unstable, if it is not stable. □

Proposition 5.6. If $u^* = 0$ is an asymptotically stable equilibrium of (5.4) (i.e. $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of A), then

$$\|e^{tA}\| \le Me^{-at} \qquad \text{for all } t \ge 0,$$

for some constants $M \ge 1$ and $a > 0$ such that $\max\{\mathrm{Re}(\lambda) : \lambda \text{ eigenvalue of } A\} < -a < 0$.

Proof. Let $\widetilde{A} = a\mathrm{Id} + A$, then $\|e^{tA}\| = e^{-ta}\|e^{t\widetilde{A}}\|$. Moreover, $\widetilde{A}$ has the eigenvalues $\tilde{\lambda} = a + \lambda$, where λ is an eigenvalue of A with $\mathrm{Re}(\tilde{\lambda}) = a + \mathrm{Re}(\lambda) < 0$. Consequently, by Remark 5.4 we conclude that $\|e^{t\widetilde{A}}\| \le M$ for all $t \ge 0$, for some constant $M \ge 1$. □

Finally, we remark that stability for linear systems is not a local but a *global property*. In fact, if u is a solution, then au is also a solution for all $a \in \mathbb{R}$. This is different for nonlinear systems.

5.3 Nonlinear systems

Before we analyze the stability for equilibria of general nonlinear systems we consider systems with a *principal linear part*,

$$u'(t) = Au(t) + g(u(t)), \tag{5.5}$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix with constant coefficients and $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and satisfies

$$\lim_{u\to 0} \frac{\|g(u)\|}{\|u\|} = 0, \tag{5.6}$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$. The latter condition implies that $g(0) = 0$, and hence, $u^* = 0$ is still an equilibrium of system (5.5).

The following theorem states that stability properties of the steady state $u^* = 0$ of the linear system $u' = Au$ are preserved for the perturbed system (5.5). Here, it is essential that the perturbation $g(u)$ is small compared to u as u tends to zero.

Theorem 5.7. *Consider the ODE* (5.5)*. Let* $g : \mathbb{R}^n \to \mathbb{R}^n$ *be continuously differentiable and satisfy* (5.6)*. Then, the following statements hold:*

(i) If $\mathrm{Re}(\lambda) < 0$ *for all eigenvalues* λ *of* A*, then the equilibrium* $u^* = 0$ *is asymptotically stable.*

(ii) If there exists an eigenvalue λ *of* A *with* $\mathrm{Re}(\lambda) > 0$*, then the equilibrium* $u^* = 0$ *is unstable.*

Proof. We will only prove the first statement (i). The proof of the second statement can be found in [6]. It is rather technical and will be omitted.

Assume that $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of A. Then, by Proposition 5.6, we have

$$\|e^{tA}\| \leq Me^{-at} \qquad \forall t \geq 0,$$

for some constants $M \geq 1$ and $a > 0$. Condition (5.6) implies that there exists $\delta > 0$ such that

$$\|g(v)\| \leq \frac{a}{2M}\|v\| \qquad \text{for all } v \in \mathbb{R}^n : \|v\| \leq \delta. \tag{5.7}$$

To show the asymptotic stability of $u^* = 0$ we will prove that for all $\varepsilon \leq \delta$ and $u_0 \in \mathbb{R}^n$ such that $\|u_0\| < \frac{\varepsilon}{M}$, there exists a unique solution u of the IVP (5.5) with $u(0) = u_0$, the solution exists for all $t \geq 0$ and satisfies

$$\|u(t)\| < e^{-\frac{1}{2}at}\varepsilon \qquad \forall t \geq 0. \tag{5.8}$$

Let now $T > 0$ be arbitrary. Since g is continuously differentiable and does not depend on time t, g satisfies a Lipschitz condition in $D = [0, T] \times B$, where $B = \{v \in \mathbb{R}^n : \|v\| \leq \varepsilon\}$. This implies that the function $u \mapsto Au + g(u)$ satisfies a Lipschitz condition in D as well. Consequently, Theorem 3.13 implies that there exists a unique solution u of the IVP, the solution exists on an interval $[0, \widetilde{T}]$ and $\|u(t)\| \leq \varepsilon$ for all $t \in [0, \widetilde{T}]$, where $\widetilde{T} = T$, or $0 < \widetilde{T} < T$ and $\|u(\widetilde{T})\| = \varepsilon$.

The variation of constants formula for linear inhomogeneous systems (Theorem 4.9) now implies that

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}g(u(s))ds.$$

Taking the norm and using Lemma 3.10 we obtain

$$\begin{aligned}
\|u(t)\| &\leq \|e^{tA}\|\,\|u_0\| + \int_0^t \|e^{(t-s)A}\|\,\|g(u(s))\|ds \\
&\leq e^{-at}M\|u_0\| + \int_0^t Me^{-a(t-s)}\frac{a}{2M}\|u(s)\|ds \\
&\leq e^{-at}M\|u_0\| + \int_0^t \frac{a}{2}e^{-a(t-s)}\|u(s)\|ds,
\end{aligned}$$

where we used Proposition 5.6 and the estimate (5.7). We can rewrite this estimate as

$$e^{at}\|u(t)\| \leq M\|u_0\| + \int_0^t \frac{a}{2}e^{as}\|u(s)\|ds.$$

Hence, Gronwall's Lemma (Lemma 3.21) applied to $G(t) = e^{at}\|u(t)\|$ implies that

$$G(t) \leq M\|u_0\| + \frac{a}{2}\int_0^t e^{\frac{a}{2}(t-s)}M\|u_0\|ds = M\|u_0\|e^{\frac{at}{2}}.$$

Consequently, we conclude that

$$\|u(t)\| \leq e^{-\frac{a}{2}t}M\|u_0\| < e^{-\frac{a}{2}t}\varepsilon \qquad \forall t \in [0, \widetilde{T}].$$

This shows that $\|u(\widetilde{T})\| < \varepsilon$, and consequently, $\widetilde{T} = T$. Finally, since $T > 0$ was arbitrary, it follows that the solution exists for all $t \geq 0$ and u satisfies the estimate (5.8) for all $t \geq 0$. □

Remark 5.8. We remark that no conclusion can be made if the eigenvalues λ of the matrix A satisfy $\max\{\mathrm{Re}(\lambda) : \lambda \text{ eigenvalue of } A\} = 0$. In this case, the stability of $u^* = 0$ strongly depends on the specific form of the nonlinearity g.

Theorem 5.7 is limited to a very particular class of nonlinear ODEs. We now extend the result for general autonomous systems. We consider the ODE (5.1),

$$u'(t) = f(u(t)),$$

and assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f(u^*) = 0$ for some $u^* \in \mathbb{R}$. This implies that u^* is an equilibrium of the ODE.

We *linearize* f *around* u^*. To this end let $w \in \mathbb{R}^n$ with $\|w\|$ small and $A = f'(u^*)$ be the Jacobian matrix of f in u^*, i.e. $A = (a_{ij})_{1\le i,j\le n}$, where $a_{ij} = \frac{\partial f_i}{\partial u_j}(u^*)$. Then, Taylor's Theorem implies that

$$f(u^* + w) = f(u^*) + Aw + o(\|w\|) = Aw + o(\|w\|) \qquad \text{as } w \to 0,$$

where we used that $f(u^*) = 0$. We recall that a function $g \in o(h)$ as $w \to 0$ if $\lim_{w\to 0} \left\|\frac{g(w)}{h(w)}\right\| = 0$. Consequently, we have

$$\lim_{w\to 0} \frac{\|f(u^* + w) - Aw\|}{\|w\|} = 0.$$

Defining $g(w) := f(u^* + w) - Aw$, $w \in \mathbb{R}^n$, we observe that g is continuously differentiable and satisfies (5.6).

Let now u be a solution of the ODE system (5.1), $u'(t) = f(u(t))$, and let u^* be an equilibrium. We define

$$w(t) = u(t) - u^*,$$

and apply the approximation above to obtain

$$w'(t) = u'(t) = f(u^* + w(t)) = Aw(t) + g(w(t)).$$

Since $f(u^*) = 0$ it follows that

$$w'(t) = Aw(t) + g(w(t)).$$

The equation $w'(t) = Aw(t)$ with $A = f'(u^*)$ is called the *linearized equation* of the ODE (5.1) in u^*. The linearized equation often allows us to determine the stability of the steady state u^*. In fact, from Theorem 5.7 and the approximation above we derive the following result.

Theorem 5.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, u^* be an equilibrium of the ODE $u'(t) = f(u(t))$ and $A = f'(u^*)$. Then, the following holds:*

(i) If $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of A, then u^* is asymptotically stable.

(ii) If there exists an eigenvalue λ of A with $\mathrm{Re}(\lambda) > 0$, then u^* is unstable.

Proof. This is an immediate consequence of Theorem 5.7 and the derivation of the linearization around u^* above. □

Note that the *stability* of the linearized equation is not enough to draw a conclusion about the stability of the equilibrium of the nonlinear equation. For an example we refer to the exercises. However, if all eigenvalues of the matrix $A = f'(u^*)$ satisfy $\mathrm{Re}(\lambda) < 0$, then the equilibrium u^* is *asymptotically stable*, and the asymptotic stability transfers to the equilibrium of the nonlinear equation.

Example 5.10. Consider the ODE

$$u'(t) = f(u(t)) = \begin{pmatrix} -u_1 - u_2 + u_2^2 \\ u_1(1 + u_2^2) \end{pmatrix}, \qquad u = (u_1, u_2).$$

Then, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable. The steady states are $u^* = (0, 0)$ and $v^* = (0, 1)$, and the Jacobian matrix of f in (u_1, u_2) is

$$f'(u_1, u_2) = \begin{pmatrix} -1 & -1 + 2u_2 \\ 1 + u_2^2 & 2u_1u_2 \end{pmatrix}.$$

To determine the stability of the equilibria we compute the linearizations in u^* and v^*. The linearization of the ODE in u^* is

$$w'(t) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} w(t) = A_1 w(t).$$

The matrix A_1 has the eigenvalues $\lambda_{1/2} = \frac{-1 \pm i\sqrt{3}}{2}$. Hence, $\mathrm{Re}(\lambda_{1/2}) < 0$, which implies that u^* is asymptotically stable by Theorem 5.9.

The linearization of the ODE in v^* is

$$w'(t) = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} w(t) = A_2 w(t).$$

The matrix A_2 has the eigenvalues $\lambda_1 = 1, \lambda_2 = -2$. Hence, $\mathrm{Re}(\lambda_1) > 0$, which implies that v^* is unstable by Theorem 5.9.

Remark 5.11. If the Jabocian matrix $A = f'(u^*)$ in an equilibrium u^* has no eigenvalues whose real part is zero, not only the stability of the zero solution of the linearized equation $w'(t) = Aw(t)$ determines the stability of the steady state u^* of the original ODE, but also the topological structure of the phase portrait will be locally the same, including the direction of the trajectories (i.e. spirals, nodes and saddle points in the two-dimensional case). This remarkable property is known as the *Hartman-Grobman Theorem*:

Let $f : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ open, be continuously differentiable and $f(u^*) = 0$ for some $u^* \in D$. If $f'(u^*)$ has no eigenvalues λ with $\mathrm{Re}(\lambda) = 0$, then there exist neighborhoods U of u^* and V of 0 and a continuous bijection $\varphi : V \to U$ that maps the trajectories of the linearized equation $w'(t) = f'(u^*)w(t)$ onto the trajectories of the nonlinear equation $u'(t) = f(u(t))$.

Example 5.12. *Mathematical pendulum.*

We assume a point mass m is attached to a massless, non-stretchable cord suspended from a pivot point and swings freely without friction and energy loss under the influence of gravity. The movement of the pendulum is then described by the ODE

$$\varphi''(t) + c\sin(\varphi(t)) = 0, \tag{5.9}$$

where $c = \frac{g}{l}$, g is the gravitational constant, l the length of the cord and φ the angle between the vertical and the cord.

Setting $\psi = \varphi'$ we can rewrite the ODE (5.9) as first order system

$$\begin{pmatrix} \varphi'(t) \\ \psi'(t) \end{pmatrix} = \begin{pmatrix} \psi(t) \\ -\sin(\varphi(t)) \end{pmatrix} = f(\varphi(t), \psi(t)).$$

The steady states are $(0,0)$ and $(\pi,0)$, and the matrices in the corresponding linearizations are

$$A_1 = f'(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_2 = f'(\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The linearized equation $w' = A_1 w$ is the harmonic oscillator $w'' + w = 0$, see Example 5.3 with $\alpha = 0$ and $\beta = 1$. The eigenvalues of A_1 are $\lambda_{1,2} = \pm i$ and hence, the steady state $(0,0)$ is a center of the linearized equation and the trajectories are circles around the origin. However, Theorem 5.9 cannot be applied to draw a conclusion about the stability of the nonlinear system.

On the other hand, the matrix A_2 has the eigenvalues $\lambda_{1,2} = \pm 1$ and hence, the steady state $(\pi,0)$ is a saddle of the linearized system $w' = A_2 w$. By Theorem 5.9 and the Hartmann-Grobman theorem, the equilibrium $(\pi,0)$ is also a saddle for the nonlinear system.

5.4 Exercises

E5.1 **Singular matrix**

Consider the linear two-dimensional ODE

$$u'(t) = Au(t),$$

with a matrix $A \in \mathbb{R}^{2\times 2}$. If A is not invertible (and not the zero matrix), then A is similar to one of the following two matrices:

$$A_4 = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad A_5 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where $\lambda \neq 0$. Determine all equilibria of the ODEs

$$u'(t) = A_4 u(t) \qquad \text{and} \qquad u'(t) = A_5 u(t)$$

and plot the phase portraits for both systems. Are the equilibria stable, asymptotically stable or unstable?

E5.2 **Classification in 2D**

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

be invertible and $D = \det(A) = ad - bc$ and $T = \operatorname{tr}(A) = a + d$ be its trace and determinant.

Determine in the $T - D$-plane the regions in which the different phase portraits (i)-(iv) plotted in the lecture notes occur and distinguish the stable and unstable cases.

Hint: *Plot the parabola* $T^2 = 4D$.

E5.3 **Classification of equilibria**

Consider the following matrices

$$B_1 = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}.$$

(a) Determine whether the equilibrium $u^* = 0$ of the ODE

$$u'(t) = B_i u(t), \qquad i = 1, 2,$$

is a stable/unstable node, a stable/unstable spiral, a saddle or a center.

(b) Sketch the phase portrait for the ODEs (up to rotation and scaling).

E5.4 **Linear system with constant coefficients**

Consider the ODE

$$u''(t) - 4u'(t) + 4u(t) = 0.$$

Reformulate the ODE as first order system. Find a fundamental matrix for this system and verify that it indeed is one.

Is the equilibrium $u^* = 0$ stable, asymptotically stable or unstable (for the original ODE)? Sketch the phase portrait for the resulting system (up to scalings and rotations).

Hint: *See Example 4.19.*

E5.5 **Scalar ODE**

Consider the scalar ODE

$$u'(t) = 5u(t)(u(t) - 1)(u(t) - 2).$$

(a) Determine the equilibria and sketch the phase portrait of the ODE.

(b) Sketch the graph of the solution corresponding to the following initial data

$$u(0) = u_0 \in \left\{-1, \frac{1}{4}, \frac{3}{2}, \frac{5}{2}\right\}.$$

(c) Determine whether the equilibria are stable, asymptotically stable or unstable.

E5.6 **Stability for linear systems [2 points]**

Consider the linear system

$$u'(t) = Au(t), \qquad A \in \mathbb{R}^{n\times n},$$

with equilibrium $u^* = 0$. Show that $u^* = 0$ is stable according to Definition 5.1 if and only if $\|e^{tA}\| \leq C$, for some constant $C \geq 1$.

Hint: *You can use (without proof) that*

$$\|e^{tA}\| = \max\left\{\|e^{tA}v\| : v \in \mathbb{R}^n, \|v\| \leq 1\right\}.$$

E5.7 **Nonlinear system**

Determine the equilibria of the system

$$x'(t) = (3 - y(t))x(t),$$
$$y'(t) = (1 + x(t) - y(t))y(t),$$

and determine whether they are stable, unstable or asymptotically stable.

E5.8 **Stability of nonlinear systems**

Show that the stability of the linearized equation is not enough to make a conclusion about the stability of an equilibrium of a nonlinear equation. To this end, consider the following ODEs

$$u'(t) = u^3(t), \qquad v'(t) = -v^3(t).$$

<u>Hint:</u> Recall Theorem 5.9. Determine the equilibria of the ODEs and linearize the equations in these steady states. Then, solve the ODEs explicitly.

Chapter 6

Qualitative theory of ODE systems

In this chapter we continue studying qualitative properties of solutions of ODE systems. In particular, we analyze whether solutions exist globally and study properties of orbits of solutions.

6.1 Global versus finite time existence

Consider the IVP

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(t_0) &= u_0, \end{aligned} \tag{6.1}$$

where $(t_0, u_0) \in D$, $D \subset \mathbb{R}^{n+1}$ is open and $f : D \to \mathbb{R}^n$ is continuous and satisfies a local Lipschitz condition in D. By Theorem 3.16, there exists a unique solution $u : I_0 \to \mathbb{R}^n$ of the IVP (6.1), where $I_0 = [t_0 - \delta_0, t_0 + \delta_0]$, for some $\delta_0 > 0$.

Let now $t_1 := t_0 + \delta_0$ and $u_1 := u(t_1)$, then $(t_1, u_1) \in D$. Therefore, Theorem 3.16 implies that there exists a unique solution $v : I_1 \to \mathbb{R}^n$, where $I_1 = [t_1 - \delta_1, t_1 + \delta_1]$, for some $\delta_1 > 0$, of the IVP

$$\begin{aligned} v'(t) &= f(t, v(t)), \\ v(t_1) &= u_1. \end{aligned}$$

By Proposition 3.20, we have $u \equiv v$ on $I_0 \cap I_1$. Hence, we can define

$$\bar{u}(t) = \begin{cases} u(t), & t \in I_0, \\ v(t), & t \in [t_1, t_1 + \delta_1]. \end{cases}$$

The function $\bar{u}$ is a solution of the original IVP (6.1) and called a **continuation of the solution** u. In the same way, we can continue solutions to the left, i.e. for $t \le t_0$.

Theorem 6.1. *Consider the IVP* (6.1). *Then, there exists an open interval $I_{max} \subset \mathbb{R}$ and a solution $u_{max} : I_{max} \to \mathbb{R}^n$ of* (6.1) *such that for all solutions $u : I \to \mathbb{R}^n$, $I \subset \mathbb{R}$ an interval, of* (6.1) *we have $I \subset I_{max}$ and $u_{max}\big|_I = u$.*

Proof. Let

$$\mathcal{J} := \{I \subset \mathbb{R} \text{ interval} : t_0 \in I, \text{there exists a solution } u_I : I \to \mathbb{R}^n \text{ of (6.1)}\}$$

and set $I_{max} := \bigcup_{I\in\mathcal{J}} I$. Then, $I_{max} \subset \mathbb{R}$ is an interval and $t_0 \in \mathbb{R}$. Moreover, I_{max} is open, since otherwise, if an endpoint of the interval was contained in I_{max} we could extend the solution as above. This would be a contradiction to the definition of I_{max}.

Finally, we define $u_{max}(t) := u_I(t), t \in I, I \in \mathcal{J}$. Due to the uniqueness of solutions, u_{max} is well-defined. Moreover, u_{max} is a solution of the IVP (6.1) and I_{max} is maximal. □

Recall that I_{max} is called the *maximal interval of existence*. By Theorem 6.1, the interval I_{max} is of the form $(-\infty, b), (a, b), (a, \infty)$ or $(-\infty, \infty)$, for some $-\infty < a < t_0 < b < \infty$. If $I_{max} = (-\infty, \infty)$, then u_{max} is called a **global solution** of the IVP.

For the solution u of the IVP (6.1) with maximal interval of existence I_{max} we have the following possibilities:

- The solution exists for all $t \in [t_0, \infty)$.
- There exists $t_0 < b < \infty$ such that $\|u(t)\| \to \infty$ as $t \to b$, i.e. the solution blows up in finite time.
- There exists $t_0 < b < \infty$ such that

  $$\operatorname{dist}((t, u(t)), \partial D) \to 0 \qquad \text{as } t \to b,$$

  where ∂D denotes the boundary of D and

  $$\operatorname{dist}(v, \partial D) = \inf\{\|v - w\| : w \in \partial D\},$$

  for $v \in D$, i.e. the solution approaches the boundary of D.

That these are the only possibilities follows from the following theorem. Analogous statements hold for the behavior of the solution for $t \leq t_0$.

Theorem 6.2. *Let u be the solution of the IVP* (6.1) *with maximal interval of existence I_{max}. Then, there exists no compact subset $K \subset D$ such that the positive graph of the solution $\Gamma^+ = \{(t, u(t)) : t \in I_{max}, t \geq t_0\}$ satisfies $\Gamma^+ \subset K$.*

Proof. Assume that $K \subset D$ is compact and $\Gamma^+ \subset K$. Since K is bounded and I_{max} is open, we conclude that $I_{max} = (a, b)$ or $I_{max} = (-\infty, b)$, for some $-\infty < a < t_0 < b < \infty$. Let $t, \tilde{t} \in [t_0, b)$ and $M := \max\{\|f(s, v)\| : (s, v) \in K\}$. The maximum is attained, since K is compact and f is continuous. We observe that

$$\|u(t) - u(\tilde{t})\| = \left\| \int_{\tilde{t}}^{t} f(s, u(s))ds \right\| \leq \int_{\tilde{t}}^{t} \|f(s, u(s))\|ds \leq M|t - \tilde{t}|,$$

which implies that the limit $\lim_{t\to b} u(t) =: u_b < \infty$ exists. Indeed, if $(t_m)_{m\in\mathbb{N}_0}$ is a sequence such that $t_m \to b$ as $m \to \infty$, then $(t_m)_{m\in\mathbb{N}_0}$ is a Cauchy sequence and the estimate above implies that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|u(t_k) - u(t_m)\| \leq M\|t_k - t_m\| < \varepsilon, \qquad \forall k, m \geq N.$$

This implies that $(u(t_m))_{m\in\mathbb{N}_0}$ is a Cauchy sequence in $\mathbb{R}^n$ and hence, it converges.

We can now continue u by $\tilde{u}(t) = u(t)$, $t \in [t_0, b)$ and $\tilde{u}(b) = u_b$. Since K is closed, we conclude that $(b, u_b) \in K$. Moreover, $\tilde{u}$ is a solution of the IVP as it satisfies

$$\tilde{u}(t) = u_b + \int_b^t f(s, u(s))ds \qquad \forall t \in [t_0, b].$$

We can now further continue the solution. In fact, by Theorem 3.16, there exists a unique solution $\bar{u} : [b - \delta, b + \delta] \to \mathbb{R}$, for some $\delta > 0$, of the IVP

$$u'(t) = f(u(t)), \qquad u(b) = u_b.$$

This contradicts the fact that $I_{max} = (a, b)$ or $I_{max} = (-\infty, b)$ and I_{max} is maximal. □

We remark that in the same way one can show that an analogous result holds for the negative graph $\Gamma^- = \{(t, u(t)) : t \in I_{max}, t \leq t_0\}$.

Corollary 6.3. *Consider the IVP* (6.1) *and assume that* $D = \mathbb{R}^{n+1}$. *If the solution satisfies*

$$\|u(t)\| \leq c \qquad \forall t \in I_{max} \cap [t_0, \infty),$$

for some $c \geq 0$*, then the solution exists for all* $t \in [t_0, \infty)$.

Proof. This follows immediately from Theorem 6.2. □

6.2 Qualitative properties of orbits

From now on, we consider autonomous ODEs, where the function f is defined for all $u \in \mathbb{R}^n$,

$$u'(t) = f(u(t)), \tag{6.2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies a local Lipschitz condition in $\mathbb{R}^n$.

By Theorem 6.2 and Theorem 3.16, for every $t_0 \in \mathbb{R}, u_0 \in \mathbb{R}^n$ there exists a unique solution of (6.2) with $u(t_0) = u_0$. Moreover, the solution either exists for all $t \geq t_0$ or there exists $t_0 < b < \infty$ such that the solution exists for all $[t_0, b)$ and $\|u(t)\| \to \infty$ as $t \to \infty$. In the same way, we can follow solutions backwards in time $t \leq t_0$, and either the solution exists for all $t \leq t_0$, or it exists on a finite interval $(a, t_0]$ and $\lim_{t \searrow a} \|u(t)\| = \infty$. Let I_{max} be the maximal interval of existence of a solution u. We consider the **trajectory** (or **orbit**) $\Gamma = \{u(t) : t \in I_{max}\}$ of the solution u.

Theorem 6.4. *Consider the ODE* (6.2)*. Through every* $u_0 \in \mathbb{R}^n$ *there passes exactly one trajectory. In particular, if two orbits* Γ *and* $\widetilde{\Gamma}$ *have one point in common, they must coincide, i.e.* $\Gamma \equiv \widetilde{\Gamma}$.

Proof. Let $t_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}^n$. Then, there exists a unique solution $u : I_{max} \to \mathbb{R}^n$ of (6.2) with $u(t_0) = u_0$, where I_{max} denotes the maximal interval of existence. Hence, there exists at least one orbit Γ passing through u_0.

Assume that $\tilde{u} : \widetilde{I}_{max} \to \mathbb{R}^n$ is another solution of (6.2) such that its orbit $\widetilde{\Gamma}$ passes through u_0 at time $\tilde{t} \in \widetilde{I}_{max}$, i.e.

$$\tilde{u}'(t) = f(\tilde{u}(t)), \qquad \tilde{u}(\tilde{t}) = u_0.$$

Let $v(t) := \tilde{u}(t - t_0 + \tilde{t})$, then

$$v'(t) = f(v(t)), \qquad v(t_0) = \tilde{u}(\tilde{t}) = u_0,$$

i.e. u and v are both solutions of the IVP (6.2) with $u(t_0) = v(t_0) = u_0$. The uniqueness of solutions implies that $u \equiv v$ on I_{max}. Consequently, $u(t) = \tilde{u}(t - t_0 + \tilde{t})$ for all $t \in I_{max}$, which implies that $\Gamma \subset \widetilde{\Gamma}$.

In the same way, we can show that $\widetilde{\Gamma} \subset \Gamma$, i.e. $\Gamma = \widetilde{\Gamma}$, which concludes the proof. □

Two different trajectories of solutions do not intersect, however, it can happen that trajectories are closed curves in the phase space. This situation corresponds to **periodic solutions**, i.e. there exists $T > 0$ such that the solution satisfies

$$u(t + T) = u(t) \qquad \forall t \in \mathbb{R}.$$

Unless the solution is constant (i.e. the condition holds for all $T > 0$ and the trajectory is a single point), the smallest $T > 0$ for which this property holds is well-defined and called the **period** of the solution.

Proposition 6.5. Let u be a solution of (6.2) such that $u(t_0) = u(t_0 + T)$ for some $t_0 \in \mathbb{R}$ and $T > 0$. Then, $u(t) = u(t + T)$ for all $t \in \mathbb{R}$, i.e. u must be periodic.

Proof. Assume that u is a solution of (6.2) such that $u(t_0) = u(t_0 + T)$. Then, the function $v(t) := u(t + T)$ is also a solution of (6.2) with $v(t_0) = u(t_0)$ and the uniqueness of solutions implies that $u \equiv v$. □

On the other hand, assume that $\widetilde{\Gamma}$ is a closed curve in $\mathbb{R}^n$ that does not contain any steady states and $u : \mathbb{R} \to \mathbb{R}^n$ is a solution of (6.2) with $u(t) \in \widetilde{\Gamma}$ for all $t \in \mathbb{R}$. Then, $\|u'(t)\| \geq c$ for all $t \in \mathbb{R}$ and some constant $c > 0$, and hence, the speed at which the solution moves along the curve Γ is strictly positive. It follows that u is periodic and the orbit of u is $\widetilde{\Gamma}$.

Trajectories of periodic solutions can be surrounded by trajectories of other periodic solutions, e.g. this occurs in the case of two-dimensional linear systems if the origin is a center, see Section 5.2. In case of nonlinear ODEs, it can also happen that periodic solutions attract or repel other nearby solutions. Then, the periodic solution is called a **limit cycle**.

We discuss one example of a nonlinear ODE in $\mathbb{R}^2$ that possesses periodic solutions, additional examples are discussed in the tutorials.

Example 6.6. The *Duffing equation* (without forcing) is a second order ODE of the form

$$x''(t) = x(t) - x^3(t) - \alpha x'(t).$$

It describes the motion of a damped oscillator, where $\alpha \geq 0$ is the damping parameter. It is used to model, e.g. an elastic pendulum whose spring's stiffness is described by the nonlinear force $x - x^3$.

We rewrite the ODE as first order system, $u_1 = x, u_2 = x'$,

$$u'(t) = \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} u_2(t) \\ u_1(t) - u_1^3(t) - \alpha u_2(t) \end{pmatrix} = f(u_1(t), u_2(t)).$$

It is an autonomous ODE system and the function f is defined on $\mathbb{R}^n$ and continuously differentiable (which implies that it satisfies a local Lipschitz condition in $\mathbb{R}^n$). There exist three equilibria, $u_1^* = (0, 0)$, $u_2^* = (1, 0)$, $u_3^* = (-1, 0)$. To determine the stability of the equilibria we compute the Jacobian matrix of f,

$$f'(u) = \begin{pmatrix} 0 & 1 \\ 1 - 3u_1^2 & -\alpha \end{pmatrix},$$

and determine the linearization in each equilibrium.

In $u_1^* = (0,0)$, we have

$$f'(u_1^*) = \begin{pmatrix} 0 & 1 \\ 1 & -\alpha \end{pmatrix},$$

and find the eigenvalues $-\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2+4}$. Since there exists an eigenvalue with positive real part, the origin is unstable. In $u_2^* = (1,0)$ and $u_3^* = (-1,0)$ we have

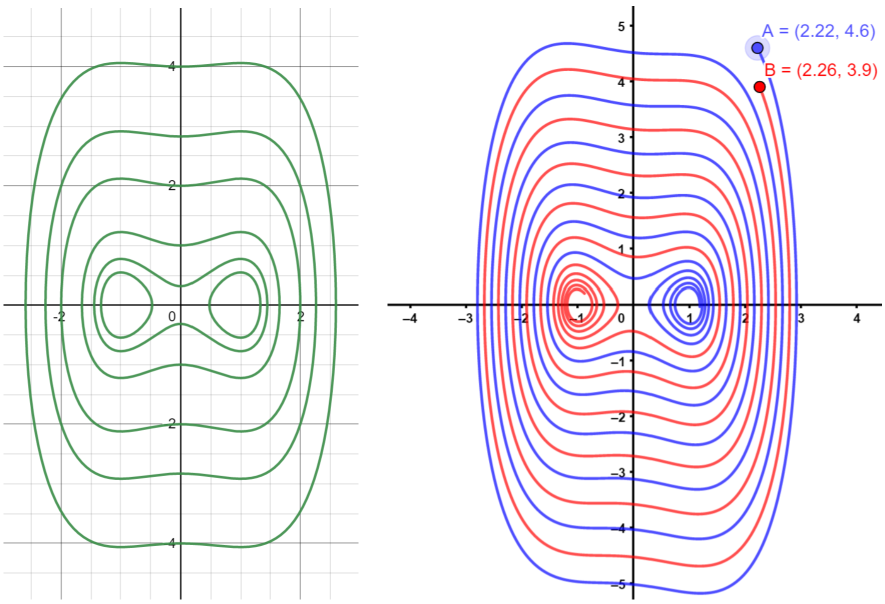
$$f'(u_2^*) = f'(u_3^*) = \begin{pmatrix} 0 & 1 \\ -2 & -\alpha \end{pmatrix},$$

and find the eigenvalues $-\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2-8}$. If $\alpha > 0$, both eigenvalues have a negative real part and hence, the steady states u_2^*, u_3^* are asymptotically stable. If $\alpha = 0$ the real parts of the eigenvalues are zero and we cannot apply Theorem 5.9.

We now analyze the case $\alpha = 0$ and will see that there exist periodic solutions. Even though we cannot explicitly compute the solutions, we can derive explicit expressions for the orbits. In fact, we observe that if $\alpha = 0$, the ODE implies that $(u_1 - u_1^3)u_1' = u_2 u_2'$. Integrating this equation we conclude that

$$E(u_1, u_2) = u_2^2 - u_1^2 + \frac{1}{2}u_1^4$$

satisfies $\frac{d}{dt}E(u_1(t), u_2(t)) = 0$. Consequently, the trajectories of the ODE correspond to the level sets of E, $E(u_1, u_2) = c$, for some $c \in \mathbb{R}$. These are closed curves, and therefore, if they do not contain equilibria (i.e. for $c \neq 0$ and $c \neq \frac{1}{2}$), the corresponding solutions are periodic. Several orbits are shown in the figure below.



In the right panel, trajectories of solutions for $\alpha > 0$ are plotted. We observe that as $t \to \infty$ these solutions converge to one of the asymptotically stable steady states u_1^* or u_2^*, and there do not exist periodic solutions. This can be shown as follows. Using the function E we observe that

$$\frac{d}{dt}E(u_1(t), u_2(t)) = (-2u_1(t) + 2u_1^3(t))u_1'(t) + 2u_2(t)u_2'(t) = -2\alpha u_2^2(t),$$

and hence, E is monotonically decreasing along solutions. On the other hand, if $u = (u_1, u_2)$ is a periodic solution with period $T > 0$, then we would have

$$0 = E(u_1(T), u_2(T)) - E(u_1(0), u_2(0)) = -2\alpha \int_0^T u_2^2(t)dt.$$

This implies that $u_2(t) = 0$ for all $t \in [0, T]$, which is a contradiction.

The function E allowed us to study the behavior of solutions along trajectories without knowing the solutions themselves. Such a function E is called *Lyapunov functions*. We introduce this concept in the next section and use it to formulate stability and instability results.

6.3 Lyapunov functions

We consider the autonomous ODE

$$u'(t) = f(u(t)), \tag{6.3}$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable in an open set $D \subset \mathbb{R}^n$, $0 \in D$ and $f(0) = 0$. Hence, the origin is an equilibrium of the ODE. Theorem 3.16 implies that for every $t_0 \in \mathbb{R}$ and $u_0 \in D$ there exists a unique solution of the ODE with $u(t_0) = u_0$.

We remark that the assumption $f(0) = 0$ is not restrictive and that the following results can easily be generalized for equilibria $u^* \neq 0$. Indeed, in this case we consider $\tilde{u} = u - u^*$, then $\tilde{u}' = u'$ and

$$\tilde{u}'(t) = \tilde{f}(\tilde{u}(t)), \qquad \tilde{f}(\tilde{u}) = f(\tilde{u} + u^*).$$

Definition 6.7. The zero steady state is called **exponentially stable**, if there exist positive constants $\delta, \gamma, c > 0$ such that for every solution u of (6.3) with $u(t_0) = u_0$ we have

$$\|u_0\| < \delta \qquad \text{implies that} \qquad \|u(t)\| \leq ce^{-\gamma(t-t_0)} \quad \text{for } t > t_0.$$

Here, we assume that solutions exists for all $t \geq t_0$.

Remark 6.8. If f is locally Lipschitz, the exponential stability implies the asymptotic stability of an equilibrium.

Indeed, for every $\varepsilon > 0$ there exists $a \geq 0$ such that $ce^{a\gamma} < \varepsilon$. Hence, if $\|u_0\| < \delta$ then the corresponding solution satisfies $\|u(t)\| < \varepsilon e^{-\gamma(t-t_0-a)}$ for all $t \in [t_0 + a, \infty)$. Moreover, Corollary 3.23 can be extended to functions $f : D \to \mathbb{R}^n$ that are continuously differentiable in an open set $D \subset \mathbb{R}^n$. This implies that there exists $\beta < \delta$ such that $\|u_0\| < \beta$ implies that $\|u(t)\| < \varepsilon$ for all $t \in [t_0, t_0 + a]$. Consequently, the inequality holds for all $t \geq t_0$, i.e. the equilibrium is stable and therefore also asymptotically stable.

Motivated by Example 6.6 we now introduce Lyapunov functions.

Definition 6.9. Consider the ODE (6.3). For a continuously differentiable function $V : D \to \mathbb{R}$ we define

$$\dot{V}(u) = \nabla V(u) \cdot f(u) = \sum_{i=1}^{n} V_{u_i}(u) f_i(u), \qquad u \in D.$$

Then, V is a **Lyapunov function** for the ODE (6.3) if

$$V(0) = 0, \qquad V(u) > 0 \quad \text{if} \quad u \neq 0, \qquad \dot{V}(u) \leq 0, \qquad u \in D.$$

We observe that if u is a solution of the ODE (6.3), then the chain rule implies that

$$\frac{d}{dt}V(u(t)) = \nabla V(u(t)) \cdot u'(t) = \nabla V(u(t)) \cdot f(u(t)) = \dot{V}(u(t)), \qquad t \geq t_0. \tag{6.4}$$

Therefore, $\dot{V}$ is typically called the *derivative of V along trajectories*. We can use Lyapunov functions to derive qualitative properties of trajectories without knowing the solutions explicitly. In particular, we have the following stability theorem.

Theorem 6.10. *Let $f : D \to \mathbb{R}^n$ be continuously differentiable, $f(0) = 0, 0 \in D$ and assume that there exists a Lyapunov function V for the ODE* (6.3)*. Then, the following statements hold:*

(i) If $\dot{V}(u) \leq 0$ for all $u \in D$ then the zero steady state is stable.

(ii) If $\dot{V}(u) < 0$ for all $u \in D \setminus \{0\}$ then the zero steady state is asymptotically stable.

(iii) If $\dot{V}(u) \leq -\alpha V(u)$ and $V(u) \geq c\|u\|^\beta$ for all $u \in D$, for some positive constants $\alpha, \beta, c > 0$, then the zero steady state is exponentially stable.

Proof. (i): Let $\varepsilon > 0$ be such that $\overline{B_\varepsilon(0)} \subset D$, where $\overline{B_\varepsilon(0)} = \{v \in \mathbb{R}^n : \|v\| \leq \varepsilon\}$. Moreover, let $\gamma > 0$ be such that $V(u) \geq \gamma$ for $\|u\| = \varepsilon$ and $0 < \delta < \varepsilon$ be such that $V(u) < \gamma$ for all $\|u\| < \delta$. Then, for solutions u corresponding to initial data $\|u_0\| < \delta$ by (6.4) it follows that $\varphi(t) = V(u(t))$ satisfies $\varphi'(t) \leq 0$. We conclude that $\varphi(t) \leq \varphi(t_0) < \gamma$. Since $V(u) \geq \gamma$ if $\|u\| = \varepsilon$, it follows that $\|u(t)\| < \varepsilon$ as long as the solution exists. By Corollary 6.3 we conclude that the solution exists for all $t \geq t_0$ and $\|u(t)\| < \varepsilon$ for all $t \geq t_0$ and we conclude that $u^* = 0$ is stable.

(ii): Let u be a solution of (6.3) and $\varphi(t) = V(u(t)$ as in (i). Then, the limit $\lim_{t\to\infty} \varphi(t) = \beta < \gamma$ exists (as φ is monotone and bounded), and $0 \leq \beta \leq \varphi(t) \leq \gamma$ for $t > t_0$. We aim to show that $\beta = 0$.

Assume that $\beta \neq 0$, then the set $A = \left\{u \in \overline{B_\varepsilon(0)} : \beta \leq V(u) \leq \gamma\right\}$ is a compact subset of $\overline{B_\varepsilon(0)} \setminus \{0\}$ and since $\dot{V}$ is continuous it attains a maximum on A. We conclude that $\max\{\dot{V}(u) : u \in A\} = -\alpha < 0$, for some $\alpha > 0$. Since the positive orbit of the solution lies in A, it follows that $\varphi'(t) \leq -\alpha$ which is a contradiction.

This shows that $\varphi(t) \to 0$ as $t \to \infty$, which implies that $u(t) \to 0$. Indeed, let $0 < \tilde{\varepsilon} < \varepsilon$ then the function V attains its minimum δ on the compact set $\{u \in \mathbb{R}^n : \tilde{\varepsilon} \leq \|u\| \leq \varepsilon\}$. Moreover, there exists $T > t_0$ such that $\varphi(t) \leq \delta$ for all $t \geq T$ and therefore, $\|u(t)\| \leq \tilde{\varepsilon}$ for all $t \geq T$.

(iii): As before, let u be a solution of (6.3) and $\varphi(t) = V(u(t)$. We observe that the hypotheses imply that $b\|u(t)\|^\beta \leq V(u(t)) = \varphi(t)$ and $\varphi'(t) \leq -\alpha\varphi(t)$. Gronwall's lemma (Lemma 3.21) implies that $\varphi(t) \leq \varphi(0)e^{-\alpha t}$, and we conclude that $\|u(t)\| \leq ce^{-\gamma t}$, where $\gamma = \frac{\alpha}{\beta} > 0$ and $c = \left(\frac{\varphi(0)}{b}\right)^{\frac{1}{\beta}}$. □

Using Lyapunov functions we can also derive an instability theorem.

Theorem 6.11. *Let $V : D \to \mathbb{R}$ be continuously differentiable, $V(0) = 0$ and $V(u_k) > 0$ for a sequence $(u_k)_{k\in\mathbb{N}}$ in D such that $u_k \to 0$ as $k \to \infty$.*

If $\dot{V}(u) > 0$ for $u \in D \setminus \{0\}$ or if $\dot{V}(u) \geq \lambda V(u)$ for all $u \in D$, for some $\lambda > 0$, then the zero steady state of (6.3) *is unstable. In particular, this holds if $V(u) > 0$ and $\dot{V}(u) > 0$ for $u \in D$.*

Proof. Let u be a solution of (6.3) with $u(t_0) = u_k$ and $\varphi(t) = V(u(t))$. Then, $\varphi(t_0) = \alpha > 0$. We consider the first case and choose $\varepsilon > 0$ such that $V(u) < \alpha$ for all $u \in \overline{B_\varepsilon(0)}$. Since $\varphi'(t) \geq 0$ we conclude that $\alpha = \varphi(t_0) \leq \varphi(t)$. This implies that $\|u(t)\| > \varepsilon$. Let now $\overline{B_r(0)}, r > \varepsilon$, be a closed ball contained in D. For $\varepsilon \leq \|u\| \leq r$ we have $\dot{V}(u) \geq \beta > 0$, which implies that $\varphi' \geq \beta$ as long as $u(t) \in B_r(0)$. Hence, we conclude that $\varphi(t) \geq \alpha + \beta t$ as long as $u(t) \in B_r(0)$. Since V attains a maximum in $\overline{B_r(0)}$, it is bounded in $\overline{B_r(0)}$. Therefore, the solution u has to exit the ball $\overline{B_r(0)}$ in finite time.

In the second case we have $\varphi'(t) \geq \lambda\varphi(t)$. Gronwall's lemma (Lemma 3.21) implies that $\varphi(t) \geq \alpha e^{\lambda t}$. Hence, as in the first case it follows that $\|u(t)\| > r$ for large t. Since the sequence $(u_k)_{k\in\mathbb{N}}$ converges to 0 as $k \to \infty$, there exist solutions with arbitrarily small initial values that exist the ball $\overline{B_r(0)}$ in finite time. □

There is no general recipe to construct Lyapunov functions. In a concrete application, one can only rely on known examples and intuition.

Example 6.12. • *Nonlinear oscillations without friction*

We consider the ODE

$$x''(t) + h(x(t)) = 0,$$

where $h : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $h(0) = 0$, and $xh(x) > 0$ for $x \neq 0$.. The solution x describes the movement of a point mass with mass 1, where $x = 0$ is the equilibrium configuration and $-h(x)$ is the size of the restoring force. Setting $u_1 = x$ and $u_2 = x'$ we rewrite the ODE as first order system

$$u_1'(t) = u_2(t),$$
$$u_2'(t) = -h(u_1(t)).$$

We consider the energy function

$$E(u) = E(u_1, u_2) = \frac{1}{2}u_2^2 + H(u_1), \qquad H(u_1) = \int_0^{u_1} h(s)ds,$$

which is the sum of the kinetic energy, $\frac{1}{2}u_2^2$, and potential energy, $H(u_1)$. We observe that

$$E(u) > 0 \qquad u \neq 0,$$
$$\dot{E}(u) = u_2 h(u_1) + (-h(u_1))u_2 = 0,$$

where the latter identity implies that the energy E is conserved. We note that E is a Lyapunov function and that Theorem 6.10 implies that $(u_1, u_2) = (0, 0)$ is stable.

• *Nonlinear oscillations with friction*

Including friction and considering a linear friction term $\mu x'$, for some $\mu > 0$, we obtain

$$x''(t) + \mu x'(t) + h(x(t)) = 0.$$

The ODE is equivalent to the first order system

$$u_1'(t) = u_2(t),$$
$$u_2'(t) = -\mu u_2(t) - h(u_1(t)).$$

Hence, the energy function now satisfies

$$\dot{E}(u) = -\mu u_2^2.$$

This shows that the energy is decreasing, as expected, and Theorem 6.10 implies that the zero steady state $(u_1, u_2) = (0, 0)$ is stable. We would even expect that it is asymptotically stable, but cannot make this conclusion based on Theorem 6.10. To shows this requires refined techniques.

6.4 Exercises

E6.1 **Stability via a different approach**

Consider the ODE system

$$\begin{aligned} u'(t) &= v(t) - \mu u(t)(u^2(t) + v^2(t)), \\ v'(t) &= -u(t) - \mu v(t)(u^2(t) + v^2(t)), \end{aligned}$$

where $\mu = \pm 1$.

(a) Determine the equilibrium of the ODE and explain why Theorem 5.9 cannot be applied to study the stability.

(b) To investigate whether the equilibrium is stable or unstable consider the function

$$\varphi(t) = E(u(t), v(t)) = u^2(t) + v^2(t).$$

Distinguish the cases $\mu = 1$ and $\mu = -1$.

E6.2 **Equilibria and limit cycles**

We aim to investigate the qualitative behavior of the following ODE system

$$\begin{aligned} u'(t) &= u(t) - v(t) - u(t)\sqrt{u^2(t) + v^2(t)}, \\ v'(t) &= u(t) + v(t) - v(t)\sqrt{u^2(t) + v^2(t)}. \end{aligned} \tag{6.5}$$

(a) Determine the steady states of system (6.5) and investigate whether they are stable, asymptotically stable or unstable.

(b) As in Problem 2 consider the function

$$\varphi(t) = E(u(t), v(t)) = u^2(t) + v^2(t)$$

and derive an ODE for φ. Determine the equilibria of this ODE, draw the phase portrait and investigate the stability of the equilibria.

(c) Use (b) to sketch the phase portrait for system (6.5).

(d) If $t_0 = 0$ and $u(0) = u_0, v(0) = v_0$ are given, does the corresponding solution exist for all $t \geq 0$? If $(u_0, v_0) \neq (0, 0)$ what is the asymptotic behavior of the solution as $t \to \infty$?

Chapter 7

Applications

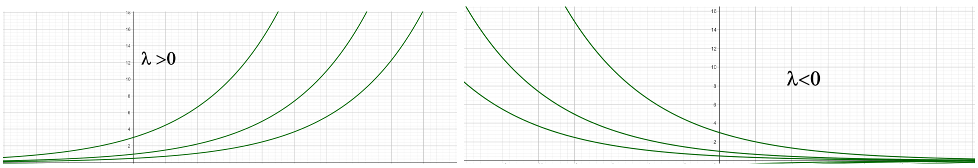
In this section we discuss and analyze two-dimensional ODE models applying the techniques and theory developed in the previous chapters.

7.1 Predator-prey models

We first consider models that describe the dynamics of two interacting populations. In Chapter 1, we already mentioned two simple models that are commonly used to describe the growth of single species populations. Let $u(t)$ denote the density of a population at time $t \geq 0$. Then, a simple model for its growth is the ODE

$$u'(t) = \alpha u(t),$$

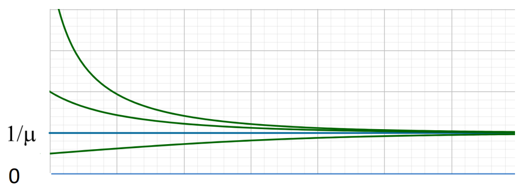
where $\alpha \in \mathbb{R}$ is the growth rate of the population, which is assumed to be constant. The model predicts exponential growth if $\alpha > 0$ and exponential decay of the population if $\alpha < 0$.



Exponential growth is observed in populations if resources are abundantly available, however, this is often not the case. The growth rate of a population typically decreases as the population size increases, since resources such as food and the available space become limited. This can be taken into account by considering growth rates that depend on the population size. We recall that a growth rate that decreases linearly in u leads to the ODE

$$u'(t) = \alpha u(t) - \beta u^2(t)$$

where we assume that $\alpha, \beta > 0$. This ODE is called the *logistic equation*. In population dynamics, it is also known as the **Verhulst model**, see Example 1.3.



We now aim to describe the dynamics of two interacting species: a prey population with density $u(t)$ and a predator population with density $v(t)$. We assume that for the prey population food is abundantly available and preys only die when killed by a predator. If the number of preys consumed by predators per unit time is proportional to uv, we obtain the ODE

$$u'(t) = \alpha_1 u(t) - \gamma_1 u(t)v(t),$$

where $\alpha_1 > 0$ is the growth rate of the prey population and $\gamma_1 > 0$ the attack rate. Furthermore, we assume that the predator population decreases at a constant rate $\alpha_2 > 0$ if there are no preys and it increases at a rate proportional to the number of consumed preys, i.e. to uv. These assumptions lead to the ODE

$$v'(t) = -\alpha_2 v(t) + \gamma_2 u(t)v(t),$$

where $\gamma_2 > 0$. Hence, the resulting ODE system is

$$\begin{aligned} u'(t) &= \alpha_1 u(t) - \gamma_1 u(t)v(t), \\ v'(t) &= -\alpha_2 v(t) + \gamma_2 u(t)v(t), \end{aligned} \tag{7.1}$$

and we assume that the initial values satisfy

$$u(0) = u_0 \geq 0, \qquad v(0) = v_0 \geq 0.$$

The model is known as the **predator-prey model** or the **Lotka-Volterra model**.

The Lotka-Volterra model is a nonlinear system of ODEs that cannot be solved explicitly. However, we can study the qualitative behavior of solutions. Since the solutions describe population densities, a first important property that the model should possess is that solutions remain non-negative. To shorten notations we define $f : \mathbb{R}^2 \to \mathbb{R}^2$ as the function on the right hand side of (7.1),

$$f(u, v) = \begin{pmatrix} \alpha_1 u - \gamma_1 uv \\ -\alpha_2 y + \gamma_2 uv \end{pmatrix}.$$

Proposition 7.1. For every $u_0 \geq 0$ and $v_0 \geq 0$ there exists a unique solution (u, v) of the Lotka-Volterra model (7.1), and the solutions satisfy $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in I_{max}$, where I_{max} is the maximal interval of existence. In particular, if the initial data are strictly positive, the solutions remain strictly positive for all $t \in I_{max}$.

Proof. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in $\mathbb{R}^2$, and therefore, by Corollary 3.17, there exists a unique local solution of the IVP.

If $u_0 = 0$ and $v_0 > 0$, then we obtain the solution $u(t) = 0, v(t) = e^{-\alpha_2 t} v_0 > 0$, for all $t \geq 0$. On the other hand, if $v_0 = 0$ and $u_0 > 0$, we observe that the corresponding solution is $v(t) = 0, u(t) = u_0 e^{\alpha_1 t} > 0$, for all $t \geq 0$. Consequently, the positive u-axis and the positive v-axis are trajectories of the ODE. The origin $(0, 0)$ is a steady state and hence, a single point trajectory. Since trajectories cannot intersect by Theorem 6.4, we conclude that every solution $(u, v) : I_{max} \to \mathbb{R}^2$ emanating from a positive initial value $u_0 > 0, v_0 > 0$ satisfies $u(t) > 0, v(t) > 0$ for all $t \in I_{max}$. □

Next, we analyze equilibria and their stability. We observe that the origin $(0, 0)$ and $(u^*, v^*) = (\frac{\alpha_2}{\gamma_2}, \frac{\alpha_1}{\gamma_1})$ are the steady states of the ODE. Furthermore, the Jacobian matrix is

$$f'(u, v) = \begin{pmatrix} \alpha_1 - \gamma_1 v & -\gamma_1 u \\ \gamma_2 v & -\alpha_2 + \gamma_2 u \end{pmatrix}.$$

Evaluating the matrix in the origin $(0,0)$ we obtain

$$f'(0,0) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix},$$

and hence, there exists an eigenvalue with positive real part. We conclude that $(0,0)$ is an unstable equilibrium by Theorem 5.9. Furthermore, in the positive steady state $(u^*, v^*) = (\frac{\alpha_2}{\gamma_2}, \frac{\alpha_1}{\gamma_1})$ we have

$$f'(u^*, v^*) = \begin{pmatrix} 0 & -\gamma_1 \frac{\alpha_2}{\gamma_2} \\ \gamma_2 \frac{\alpha_1}{\gamma_1} & 0 \end{pmatrix},$$

and hence, the eigenvalues of the matrix are purely imaginary. Therefore, Theorem 5.9 does not allow us to draw a conclusion about the stability of the steady state (u^*, v^*).

We will show that all solutions in the positive quadrant are periodic.

Proposition 7.2. All trajectories of the ODE (7.1) in the positive quadrant, except for the steady state of (u^*, v^*), are closed curves and hence, correspond to periodic solutions.

Proof. Assume that $u_0 > 0, v_0 > 0$, then the corresponding solutions are strictly positive. We observe that

$$\frac{u'(t)}{v'(t)} = \frac{\frac{\alpha_1}{v(t)} - \gamma_1}{-\frac{\alpha_2}{u(t)} + \gamma_2},$$

and consequently,

$$\left(\frac{\alpha_1}{v(t)} - \gamma_1\right) v'(t) = \left(-\frac{\alpha_2}{u(t)} + \gamma_2\right) u'(t).$$

Integrating the equation we observe that the trajectories are given by the curves $E(u, v) = c$, for some $c \in \mathbb{R}$, where

$$E(u, v) = \alpha_1 \ln(v) - \gamma_1 v + \alpha_2 \ln(u) - \gamma_2 u, \qquad u > 0, v > 0.$$

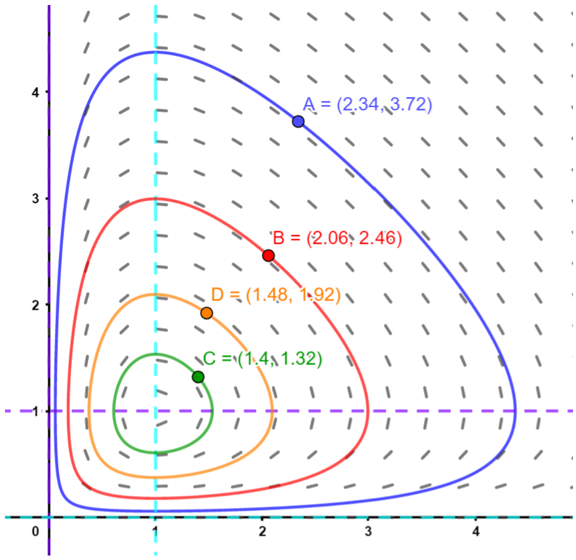
The equation $E(u, v) = c$ is equivalent to

$$\psi_1(u)\psi_2(v) = e^c, \qquad \psi_1(u) = \frac{u^{\alpha_2}}{e^{\gamma_2 u}}, \quad \psi_2(v) = \frac{v^{\alpha_1}}{e^{\gamma_1 v}},$$

where $u > 0, v > 0$. The function ψ_1 has one maximum in $\bar{u} = \frac{\alpha_2}{\gamma_2}$, ψ_2 has one maximum in $\bar{v} = \frac{\alpha_1}{\gamma_1}$ and $\psi_i(0) = \lim_{z\to\infty} \psi_i(z) = 0$, $i = 1, 2$. Hence, the product function $\psi_1\psi_2$ has a single maximum which is attained in the steady state $(u^*, v^*) = \left(\frac{\alpha_2}{\gamma_2}, \frac{\alpha_1}{\gamma_1}\right)$.

All other contour lines $\psi_1(u)\psi_2(v) = e^c, c \in \mathbb{R}$, are closed curves in the first quadrant around the steady state (u^*, v^*). Since these curves do not contain equilibria, they correspond to trajectories of periodic solutions, by Proposition 6.5. □

Remark. By Corollary 6.3 and Proposition 7.2 we conclude that all solutions emanating from non-negative initial data exist for all times $t \geq 0$.



A number of these trajectories are plotted in the figure above. Each trajectory corresponds to a periodic solution with some period $T > 0$. The dashed purple lines are the *null-isoclines* of u (i.e. the curves along which u' is zero) and the dashed turquoise lines the null-isoclines of v (i.e. the curves along which v' is zero). The equilibria correspond to intersections of dashed purple and dashed turquoise lines.

Even though the solutions and their periods are unknown, we can compute the time averages of the solutions over one period.

Proposition 7.3. Let (u, v) be a periodic solution of (7.1) of period $T > 0$. Then, the time averages over one period are given by

$$\bar{u} = \frac{1}{T}\int_0^T u(t)\,dt = \frac{\alpha_2}{\gamma_2}, \qquad \bar{v} = \frac{1}{T}\int_0^T v(t)dt = \frac{\alpha_1}{\gamma_1}.$$

Proof. From the first ODE in (7.1) we conclude that $\frac{u'(t)}{u(t)} = \alpha_1 - \gamma_1 v(t)$ and consequently,

$$\frac{1}{T}\int_0^T \frac{u'(t)}{u(t)}dt = \frac{1}{T}\int_0^T \alpha_1 - \gamma_1 v(t)dt = \alpha_1 - \gamma_1 \bar{v}.$$

On the other hand,

$$\int_0^T \frac{u'(t)}{u(t)}dt = \ln(u(T)) - \ln(u(0)) = 0,$$

since u is periodic, i.e. $u(0) = u(T)$. This implies that $\bar{v} = \frac{\alpha_1}{\gamma_1}$. In the same way, one can find the value for $\bar{u}$. □

All solutions in the positive quadrant are periodic and circle around the positive equilibrium (u^*, v^*). Moreover, the time averages $\bar{u}, \bar{v}$ of the solutions over one period are independent of the initial values and coincide with the coordinates of the steady state, $\bar{u} = u^*, \bar{v} = v^*$. The values of the initial data determine the amplitudes of the solutions.

Finally, we investigate the *effect of harvesting* on the predator-prey system assuming constant death rates $\beta_1, \beta_2 > 0$ for both populations, e.g. through fishing, hunting, pollution or pesticides. This leads to the modified model

$$\begin{aligned} u'(t) &= (\alpha_1 - \beta_1)u(t) - \gamma_1 u(t)v(t), \\ v'(t) &= -(\alpha_2 + \beta_2)v(t) + \gamma_2 u(t)v(t). \end{aligned} \tag{7.2}$$

If $\alpha_1 > \beta_1$, these additional terms do not modify the structure of the system, and only cause a change of the parameter values. In fact, the positive equilibrium of system (7.2) is shifted to $(\tilde{u}^*, \tilde{v}^*)$, where

$$\tilde{u}^* = \frac{\alpha_2 + \beta_2}{\gamma_2}, \qquad \tilde{v}^* = \frac{\alpha_1 - \beta_1}{\gamma_1}.$$

Hence, harvesting leads to an increase of the prey population and a decrease of the predator population.

Volterra's principle

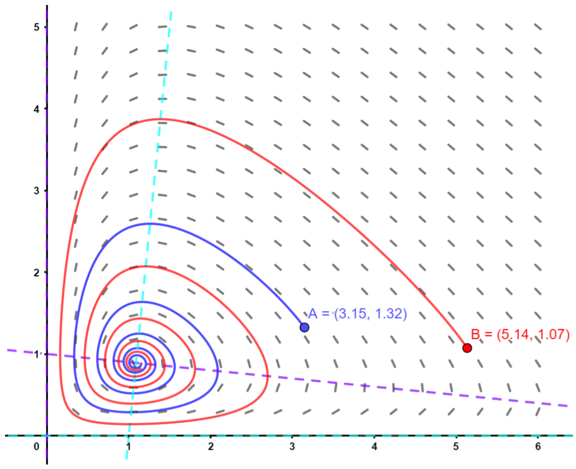
The observation concerning harvesting has important consequences for applications, e.g. for insecticide treatment. Suppose that in a greenhouse there is a prey insect population u (e.g. aphids) and a predator insect population v (e.g. ladybird beetles) that can be modeled by system (7.1). If insecticide is sprayed causing a constant decay of both insect populations, the time averages of the predator population decrease while the averages of the prey population increase, which is contrary to the intention.

This remarkable effect is known as **Volterra's principle**. Originally, Volterra proposed and analyzed the model (7.1) to explain data about fish populations in the Mediterranean Sea that were collected by the biologist D'Ancona during the First World War. In fact, it had been observed that the percentage of predatory fishes showed a large increase compared to the population of prey fishes. It seemed obvious that the reduced level of fishing during the war should be responsible. However, it was unclear why this would affect the predators and preys in a different way. Volterra's explanation was that the decrease of fisheries favored the predator population, shifting the equilibrium $(\bar{u}^*, \bar{v}^*)$ of the modified model (7.2) back to the equilibrium (u^*, v^*) of the original model (7.1).

The Lotka-Volterra model has been criticized for too simplistic modeling assumptions and has been further developed and improved in various directions. One way to improve the model is to take internal competition into account and to model the growth of the populations by the logistic equation. This leads to the modified system

$$\begin{aligned} u'(t) &= \alpha_1 u(t) - \kappa_1 u^2(t) - \gamma_1 u(t)v(t), \\ v'(t) &= -\alpha_2 v(t) - \kappa_2 v^2(t) + \gamma_2 u(t)v(t), \end{aligned} \tag{7.3}$$

where $\kappa_1, \kappa_2 > 0$. These parameters cause some damping and change the qualitative behavior of the model. In particular, if κ_1 and κ_2 are small compared to the other parameters, the solutions will spiral around the positive equilibrium (u^*, v^*), given by $u^* = \eta(\alpha_1\beta_2 + \alpha_2\gamma_1)$, $v^* = \eta(\alpha_1\gamma_2 - \alpha_2\beta_1)$, where $\eta = \frac{1}{\beta_1\beta_2 + \gamma_1\gamma_2}$, and converge to it as t tends to infinity. Two trajectories of the ODE (7.3) and the isoclines are shown in the figure below.



7.2 Competition models

Next, we consider models for two interacting populations where the species compete for a shared resource, e.g. a common nutrient. Let $u(t)$ and $v(t)$ denote the densities of the two populations at time $t \geq 0$. We assume that the growth of both species is described according to the Verhulst model and the species compete with each other. For both species we assume that the growth rate decreases as the population of the other species increases. This leads to the system

$$\begin{aligned} u'(t) &= \alpha_1 u(t) - \beta_1 u^2(t) - \gamma_1 u(t)v(t), \\ v'(t) &= \alpha_2 v(t) - \beta_2 v^2(t) - \gamma_2 u(t)v(t), \end{aligned} \tag{7.4}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ are the parameters modeling the logistic growth of u and v and $\gamma_1, \gamma_2 > 0$ describe the competition among the species. We assume that the initial values satisfy

$$u(0) = u_0 \geq 0, \qquad v(0) = v_0 \geq 0.$$

Depending on the parameter values, the model (7.4) can predict *coexistence* of both species, or *survival* of one species and *extinction* of the other one.

Proposition 7.4. For every $u_0 \geq 0$ and $v_0 \geq 0$ there exists a unique solution (u, v) of (7.4) with $u(0) = u_0, v(0) = v_0$, and the solutions satisfy $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in I_{max}$, where I_{max} is the maximal interval of existence. Moreover, if the initial data are strictly positive, the solutions remain strictly positive for all $t \in I_{max}$.

Proof. The proof is similar to the proof of Proposition 7.1. In particular, we can show that the positive u-axis and the positive v-axis are covered by trajectories of the system (7.4) connecting steady states. □

Coexistence of two species

We now assume that

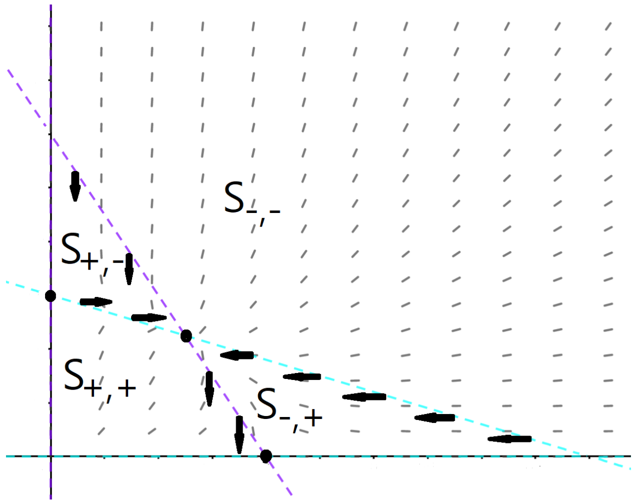
$$\frac{\alpha_1}{\beta_1} < \frac{\alpha_2}{\gamma_2}, \qquad \frac{\alpha_2}{\beta_2} < \frac{\alpha_1}{\gamma_1}, \tag{7.5}$$

and will show that these conditions imply the coexistence of the two species.

To study the qualitative behavior of solutions, we consider the null-isoclines for u and v, i.e. the curves along which the derivatives u' or v' vanish,

$$G_1 = \{(u,v) \in \mathbb{R}^2 : \alpha_1 - \beta_1 u - \gamma_1 v = 0\}, \quad G_2 = \{(u,v) \in \mathbb{R}^2 : \alpha_2 - \beta_2 v - \gamma_2 u = 0\}.$$

In fact, on G_1 as well as along the v-axis, we have $u'(t) = 0$, and on G_2 as well as along the u-axis, we have $v'(t) = 0$. In the figure below, the null-isoclines for u are the dashed purple lines and the null-isoclines for v are the turquoise dashed lines. The intersections of the purple and turquoise lines are the equilibria of the competition model (7.4).



The null-isoclines for u and v divide the first quadrant into four regions:

$$\begin{aligned} S_{+,+} &= \{(u,v) \in \mathbb{R}^2 : u' > 0, v' > 0\}, \\ S_{+,-} &= \{(u,v) \in \mathbb{R}^2 : u' > 0, v' < 0\}, \\ S_{-,+} &= \{(u,v) \in \mathbb{R}^2 : u' < 0, v' > 0\}, \\ S_{-,-} &= \{(u,v) \in \mathbb{R}^2 : u' < 0, v' < 0\}. \end{aligned}$$

In $S_{+,+}$, trajectories move in the upward right direction, in $S_{+,-}$, in the downward right direction, in $S_{-,+}$, in the upward left direction and $S_{-,-}$, trajectories move in the downward left direction. This gives a rough indication how the phase portrait looks like.

We observe that there are four equilibria, $(0,0)$, $(\frac{\alpha_1}{\beta_1}, 0)$, $(0, \frac{\alpha_2}{\beta_2})$ and one strictly positive equilibrium (u^*, v^*). These are the intersections of the dashed red and dashed green lines. The positive equilibrium (u^*, v^*) is the intersection of G_1 and G_2 and represents the *coexistence* of both species. Using Theorem 5.9, we can analyze the stability of the steady states. Denoting by f the right hand side of (7.4), the Jacobian matrix of f is

$$f'(u,v) = \begin{pmatrix} \alpha_1 - 2\beta_1 u - \gamma_1 v & -\gamma_1 u \\ -\gamma_2 u & \alpha_2 - 2\beta_2 v - \gamma_2 u \end{pmatrix}.$$

Computing the linearizations in the equilibria and corresponding eigenvalues we find that the origin $(0,0)$ and the equilibria $(\frac{\alpha_1}{\beta_1}, 0)$ and $(0, \frac{\alpha_2}{\beta_2})$ are unstable, while the strictly positive equilibrium (u^*, v^*) is asymptotically stable.

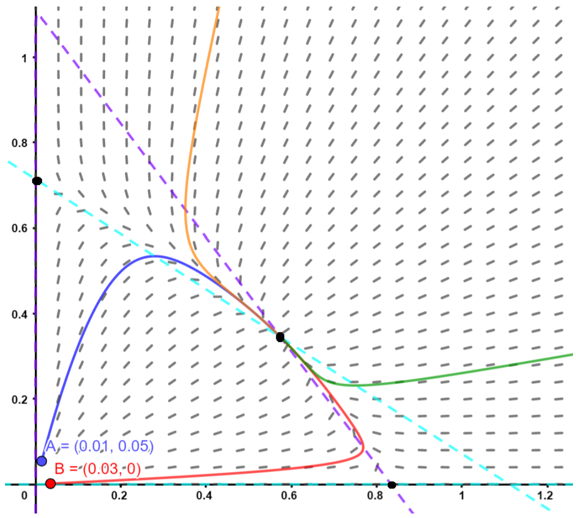
Proposition 7.5. Let the condition (7.5) be satisfied. Then, all solutions of (7.4) with a strictly positive initial value, $u_0 > 0, v_0 > 0$, exists for all times $t \geq 0$ and converge to the coexistence equilibrium (u^*, v^*) as $t \to \infty$.

Proof. We only give a sketch of the proof. First, we assume that $(u_0, v_0) \in S_{+,+}$, which implies that the derivative $(u + v)' > 0$. Consequently, the solution moves upwards right and it either tends to (u^*, v^*) or it enters the region $S_{+,-}$ or $S_{-,+}$. If $(u_0, v_0) \in S_{-,-}$, then the solution moves downwards left and the same statement holds. Hence, it remains to analyze the behavior of trajectories in $S_{+,-}$ or $S_{-,+}$.

In $S_{-,+}$, we have $(-u+v)' > 0$ and hence, solutions move upwards left. Moreover, they remain in $S_{-,+}$ for all times. Indeed, solutions cannot hit the u-axis as $v' > 0$ in this region. Moreover, assume that the solution hits a point on G_1 at time t_1, then $u'(t_1) = 0$ and $v'(t_1) > 0$. This implies that the solution intersects G_1 vertically, which is impossible when coming from $S_{-,+}$. In a similar way, one can show that solutions coming from $S_{+,-}$ cannot hit the line G_2 and remain in $S_{+,-}$ for all times.

We can conclude that the solutions are bounded, exist for all times and converge to the coexistence equilibrium (u^*, v^*) as $t \to \infty$. □

In the figure below, several trajectories for the model (7.4) with parameters satisfying (7.5) are plotted that illustrate the qualitative behavior of solutions.



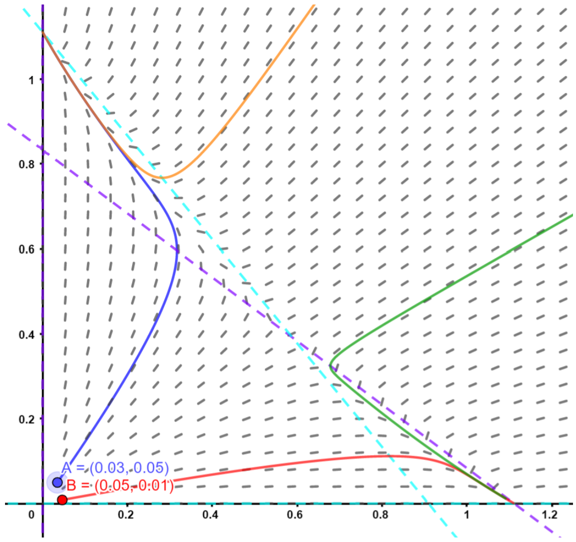
Extinction of one species

We again consider the competition model (7.4), but now assume that

$$\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\gamma_2}, \qquad \frac{\alpha_2}{\beta_2} > \frac{\alpha_1}{\gamma_1}. \tag{7.6}$$

As before, we have the four equilibria $(0,0)$, $(\frac{\alpha_1}{\beta_1}, 0)$, $(0, \frac{\alpha_2}{\beta_2})$ and the coexistence equilibrium (u^*, v^*). However, assuming (7.6) their stability properties are different compared to the case (7.5). In fact, the origin $(0,0)$ is an unstable node, the strictly positive equilibrium (u^*, v^*) is a saddle (i.e. unstable) and the *extinction equilibria* $(\frac{\alpha_1}{\beta_1}, 0)$ and $(0, \frac{\alpha_2}{\beta_2})$ are asymptotically stable. This can be shown by Theorem 5.9 and determining the linearization in each equilibrium.

Similarly as in the coexistence case we can divide the positive quadrant into four regions $\widetilde{S}_{\pm,\pm}$. Then, one can show that all solutions starting from an initial value (u_0, v_0) with $u_0 > 0, v_0 > 0$ converge to one of the asymptotically stable equilibria as $t \to \infty$, except for those solutions that lie on the trajectory connecting the origin with (u^*, v^*) or the saddle point (u^*, v^*) with infinity. This means, for almost all initial conditions $u_0 > 0, v_0 > 0$ one of the two species will eventually become extinct while the other one survives. The nullclines for u and v, the equilibria and several trajectories of the model with parameters satisfying (7.6) are shown in the figure below.



7.3 Exercises

E7.1 **Lotka-Volterra model**

Consider the modified Lotka-Volterra model (7.3) with the parameter values $\alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = 1$ and $\kappa_2 = 0$. This is a modification of the classical Lotka-Volterra model assuming that the growth of the prey population is described according to the Verhulst model. We aim to study the behavior of the model depending on the value of the parameter $\kappa_1 > 0$.

(a) Determine the equilibria of the model and their stability, distinguishing the cases $\kappa_1 \in (0, 1)$ and $\kappa_1 > 1$.

(b) Draw the null-isoclines for u and v and determine the regions $S_{\pm,\pm}$ (as done in Section 7.2 in the lecture notes).

(c) Sketch qualitatively several trajectories of the ODE.

Remark: One can show that the predator population v will eventually become extinct if $\kappa_1 > 1$.

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