Mimicking automorphisms over finite fields by tame automorphisms

Stefan Maubach

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Notation: If $F \in \mathrm{GA}_n(\mathbb{F}_q[Z])$ and $c \in \mathbb{F}_{q^m}$, then $F_c := F|_{Z=c} \in \mathrm{GA}_n(\mathbb{F}_q(c))$.

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Corollary

Let $F \in \mathrm{GA}_2(\mathbb{F}_q[Z])$. Then F is tamely mimickable.

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Example: Nagata's automorphism

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How to mimick this one? Idea: replace Z^{-1} by Z^{q^m-2}.

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Then MN_{fake} mimicks N over $\mathbb{F}_{q^m}$.

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Let D_i diagonal such that $\det(D_i) = \det(A_i)$. Replace A_i by $D_i^{-1} A_i$ and push D_i to the left:

$E_{i-1} D_i = D_i E'_{i-1}$.

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Use Gaussian elimination: $A_i = DE_1E_2 \ldots E_nP$.

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D diagonal: composition of maps of the form
$(\lambda^{-1}X_1, \lambda X_2, X_3, \ldots, X_n)$.

$$\begin{pmatrix} f^{-1} & 0 \\ 0 & f \end{pmatrix} =$$
$$\begin{pmatrix} 1 & f^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1-f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1-f^{-1} \\ 0 & 1 \end{pmatrix}.$$

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$\tilde{E} = I + \tilde{H}$. Then we are left with $G := \tilde{E}^{-1}E$.

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Result said differently:

$$\mathsf{GA}_2(\mathbb{F}_q[Z]) \subseteq \varinjlim \pi_{q^m}(\mathsf{TA}_2(\mathbb{F}_q[Z]))$$

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Start with $T := E_1A_1 \cdots E_sA_s$. If for all high m, $\pi_{q^m}(T)$ is "nice" like Nagata, then T is of length one.

To see that there exist non-tame automorphisms over $\mathbb{F}_p$ we need a new approach.

For each extension $\mathbb{F}_{q^m}$ there's a mimicking tame map T_m.

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THANK YOU