The Makar-Limanov invariant and related topics

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Linear algebra v.s. affine algebraic geometry

Linear algebra is (in my opinion) the motivating factor for affine algebraic geometry. Perhaps, one day, we will use polynomial automorphisms in many cases where we use linear maps. Linear algebra is one of the dominating factors in spawning conjectures.

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Affine algebraic geometry:

Question 1: Let k be a field. Let U, V, W be k-varieties. Suppose

$U \times W \cong V \times W$. Does this imply $U \cong V$?

(Later today)

Two Basic Questions in Affine Algebraic Geometry

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Most important case: when is a variety k^n? (k a field.) When is a ring a polynomial ring?

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- (Basic) algebraic or geometric properties (singularities, UFD, etc. etc.)
- Relatively new: certain group actions ($\mathcal{G}_a$-actions, derivations, etc.)

A motivating example: Koras-Russell 3-folds

1993: M. Koras and P. Russell tracked down a class of 3-folds on $\mathbb{C}$ which were:

affine, smooth, diffeomorphic to $\mathbb{R}^6$, + something extra.

Were they isomorphic to $\mathbb{C}^3$?

Simplest example: $X + X^2Y + Z^2 + T^3$.

Topological arguments do not work, basic algebraic properties do not work to distinguish this from $\mathbb{C}^3$.

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LND equals $k+$ action ($\mathcal{G}_a$-action)

Define $A \longrightarrow A[T]$ by

$a \longrightarrow \exp(TD)(a) = a + TD(a) + \frac{T^2}{2!}D^2(a) + \frac{T^3}{3!}D^3(a) + \dots$

In case $A = \mathcal{O}(V)$ then this gives an algebraic $k+$ action on V:

$$\begin{aligned} \mathcal{G}_a \times V &\longrightarrow V \\ t \times v &\longrightarrow exp(tD)(v) \end{aligned}$$

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If D has a slice, an element s such that $D(s) = 1$, (think of ∂_X) then $A = A^D[s]$. ($\mathbb{C}[X, Y, Z]^{\partial_X}[X] = \mathbb{C}[X, Y, Z]$.)

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Notice:

$$\begin{aligned} ML(\mathbb{C}[X,Y,Z]) \subseteq\ & \mathbb{C}[X,Y,Z]^{\partial_X} \cap \mathbb{C}[X,Y,Z]^{\partial_Y} \cap \mathbb{C}[X,Y,Z]^{\partial_Z} \\ & \mathbb{C}[Y,Z] \cap \mathbb{C}[X,Z] \cap \mathbb{C}[X,Y] = \mathbb{C}. \end{aligned}$$

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Hence $X^2Y - P(Z) = 0$ is not isomorphic to $\mathbb{C}^3$.

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Let me give an indication of how it works.

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Then, $\tilde{D} := gr_2gr_1(D)$ is an LND on $\tilde{A}$ which is doubly homogeneous, and it has $\tilde{D}(X) = \tilde{f} \neq 0$. Also $\tilde{f}$ is homogeneous.
Since $\tilde{D}(f) = 0$, ...calculatecalculate... $\tilde{D} = 0$, which is not possible.

Makar-Limanov techniques

The strength of ML invariant comes because of the techniques to compute it. Sometimes one can use these techniques, sometimes not. But - there are cases where the ML invariant will fail!

Non-omnipotency of ML invariant

Example: Let $A_1 := \mathbb{C}[X, Y, Z, T]/(XY - ZT - 1), A_2 := \mathbb{C}[X, Y, Z, T]/(XY - Z^2 - T^3)$.

Non-omnipotency of ML invariant

Example: Let $A_1 := \mathbb{C}[X, Y, Z, T]/(XY - ZT - 1), A_2 := \mathbb{C}[X, Y, Z, T]/(XY - Z^2 - T^3)$. Both have many LNDs, for example $z\partial_y + x\partial_t$ on A_1. It turns out that $ML(A_1) = ML(A_2) = \mathbb{C}$.

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(ii). $D_i = \frac{\partial}{\partial s_i}$.

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Contradiction, so $A \ncong \mathbb{C}^{[3]}$!

Motivating example:

$A := \mathbb{C}[x, y, z, t] = \mathbb{C}[X, Y, Z, T]/(p(X)Y + q(X, Z, T))$,
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Even more general: for any

$$p(X)Y + q(X, Z, T)$$

we know exactly when it is isomorphic to $\mathbb{C}^3$, using this theorem !!

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2. . $D_1 \mod (f-\alpha), \ldots, D_n \mod (f-\alpha)$ independent over $A/(f-\alpha)$ for all $\alpha \in k$
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Ring theoretic version:

Suppose A, B are finitely generated k-algebras. Suppose $A[X] \cong B[X]$. Is $A \cong B$?

Some positive results on "The" cancellation problem

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(Fujita, Miyanishi, Sugie (1980), Russell (1981)): V affine surface over field of char=0 such that $V \times k^n \cong k^{n+2}$, then $V \cong k^2$. (Recent purely algebraic proof by Makar-Limanov and Crachiola.)

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Danielewski surfaces

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Idea of proof:

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If V, W $\mathbb{C}$-algebras of dim=2, then

$V \times_{\mathbb{C}} \mathbb{C} \cong W \times_{\mathbb{C}} \mathbb{C} \longrightarrow V \cong W$.

(Due to Miyanishi)

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Amongst others - use $ML(A_{12}) = ML(A_{34}) = R$!

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\begin{array}{ccccc}
A_{12}[X] \cong & & A_{12} \otimes_R A_{34} & & \cong A_{34}[X] \\
\nearrow & & & & \nwarrow \\
A_{12} & & & & A_{34} \\
\nwarrow & & & & \nearrow \\
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******THANK YOU******