Locally finite polynomial endomorphisms

Stefan Maubach

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Notations:

	Linear	Polynomial
All	$ML_n(\mathbb{C})$	$MA_n(\mathbb{C})$
Invertible	$GL_n(\mathbb{C})$	$GA_n(\mathbb{C})$

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Jacobian Conjecture in dimension n (JC(n)):

Let $F \in MA_n(\mathbb{C})$. Then

$$\det(Jac(F)) \in \mathbb{C}^* \Rightarrow F \text{ is invertible.}$$

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Elementary map: $(X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)$, invertible with inverse

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$$F = \begin{pmatrix} X - 2(XZ + Y^2)Y - (XZ + Y^2)^2 Z, \\ Y + (XZ + Y^2)Z, \\ Z \end{pmatrix}$$

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$n = 3$:(Shestakov-Umirbaev, 2004)

Nagata's map not tame, i.e. $GA_3(\mathbb{C}) \neq TA_3(\mathbb{C})$

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Definition: If F is a zero of some $P(T) \in \mathbb{C}[T]\backslash\{0\}$, then we will call F a Locally Finite Polynomial Endomorphism (short LFPE).

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But: the minimum polynomial may change if G is not linear!

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$$P_F(T) := \prod_{\substack{0 \leq k \leq d-1 \\ 0 \leq m \leq d \\ (k,m) \neq (0,0)}} (T^2 - (detL^k)(TrL^m)T + det(L^{2k+m})).$$

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Conjecture: in dimension n,
F is LFPE $\iff deg(F^m) \leq deg(F)^{n-1}$ for all $m \in \mathbb{N}$.

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If $F^i = (F_1^{(i)}, \ldots, F_n^{(i)})$ and $F_j^{(i)} = \sum F_{j,\alpha}^{(i)} X^\alpha$, then $\sum a_i F^i = 0 \iff \sum a_i F_{j,\alpha}^{(i)} = 0 \forall j, \alpha$. If $\{F_{j,\alpha}^{(i)}\}_{i \in \mathbb{N}}$ is such a sequence, then it is a linear recurrent sequence belonging to $\sum a_i T^i$, etc....

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(2) $D(fg) = D(f)g + fD(g)$ for all $f, g \in \mathbb{C}[X_1, \ldots, X_n]$.

A derivation will have the form:

$a_1 \frac{\partial}{\partial X_1} + \ldots + a_n \frac{\partial}{\partial X_n}$ for some $a_i \in \mathbb{C}[X_1, \ldots, X_n]$.

D is called **locally nilpotent** if:

For all $g \in \mathbb{C}[X_1, \ldots, X_n]$ there exists $m \in \mathbb{N}$ such that $D^m(g) = 0$.

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So: we can make many examples of LFPEs!

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?$\Leftarrow$?

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Don't know how to make D_s, given F_s.

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if $c \in \mathbb{C}$, then no natural choice $\log(c)$.

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(Note: can prove that this work if eigenvalues are "generic", to be precise:

$\lambda_1^{d_1} \cdots \lambda_n^{d_n} = 1$ then all $d_i = 0$.)

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(Me & Han Peters:) allow power series $\mathbb{C}[[T]]$ in stead of $\mathbb{C}[T]$. Nice, but allows way too many maps. Interestingly: automorphisms which are zeroes of power series generate automorphism group! (Okay. . . we have a generating set of $GA_n(\mathbb{C})$. . .)

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$F^n = a_{n-1}F^{n-1} + \ldots + a_1 F + a_0 I$

give a flow by

$$F_t = (a_{n-1}, a_{n-2}, \ldots, a_0) \begin{pmatrix} a_{n-1} & 1 & 0 & \ldots & 0 \\ a_{n-2} & 0 & 1 & \ldots & 0 \\ \vdots & & & \vdots & \\ a_0 & 0 & 0 & \ldots & 0 \end{pmatrix}^t \begin{pmatrix} F^{-1} \\ F^{-2} \\ \vdots \\ F^{-n} \end{pmatrix}$$

Finally... last slide ... phew...

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THANK YOU