# Locally Finite Polynomial Endomorphisms

Stefan Maubach

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A polynomial map $F$ is invertible if there is a polynomial map $G$ such that $F(G) = (X_1, \dots, X_n)$.

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Why this bold claim? Polynomial maps seem to have similar properties as linear maps (much more so than holomorphic maps for example). Well. . . to be honest, most are conjectures. . . Let's look at a few of these conjectures!

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**Jacobian Conjecture** in dimension $n$ (JC(n)):

Let $F \in MA_n(\mathbb{C})$. Then

$$\det(Jac(F)) \in \mathbb{C}^* \Rightarrow F \text{ is invertible.}$$

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**Cancelation Problem:**

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$GA_n(\mathbb{K})$ is generated by ??? (Sometimes called "the automorphism problem", which means: "we don't understand the automorphism group, whatever understanding means".)

| | | |
|---|---|---|
| | $\underline{\mathrm{GA}_n(\mathbb{R})}$ | : (Dynamical systems, flows<br>Markus-Yamabe Conjecture) |
| $\underline{\mathrm{GA}_n(\mathbb{C})}$ | | : (Complex Analysis) |
| | $\underline{\mathcal{O}(V)}$ | : (Algebraic Geometry, |
| $\underline{\mathrm{GA}_n(k)}$ | | : Ring theory) |
| | $\underline{\mathrm{GA}_n(R)}$ | |
| $\underline{\mathrm{GA}_n(\mathbb{F}_q)}$ | | (Group theory, number theory,<br>Secret-sharing cryptography) |

Let us make some non-trivial polynomial automorphisms!

**Elementary map:** $(X_1 + f(X_2, \dots, X_n), X_2, \dots, X_n)$,

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$\mathrm{TA}_n(\mathbb{K}) := < J_n(\mathbb{K}), \mathrm{Aff}_n(\mathbb{K}) >$

In dimension 1: we understand the automorphism group.
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In dimension 2: famous Jung-van der Kulk-theorem:

$$\mathsf{GA}_2(\mathbb{K}) = \mathsf{TA}_2(\mathbb{K}) = \mathsf{Aff}_2(\mathbb{K}) \ltimes \mathsf{J}_2(\mathbb{K})$$

Jung-van der Kulk is the reason that we can do a lot in dimension 2 !!!!

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(Difficult and technical proof. ) (2007 AMS Moore paper award.) So now it is official. Nagata is complicated.

# AMS E.H. Moore Research Article Prize


Ivan Shestakov (center) and Ualbai Umirbaev (right) with Jim Arthur.

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If $D$ is LND(locally nilpotent derivation) then $\exp(D)$ is automorphism !! We have a *non-trivial* way of making automorphisms! In fact: Nagata $= \exp(D)$ !

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**Conjecture 1:**

$$\mathsf{GA}_n(\mathbb{C}) = < \mathsf{Aff}_n(\mathbb{C}), \mathsf{ELND}_n(\mathbb{C}) > .$$

…candidate counterexamples start to emerge …

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$\exp(X\frac{\partial}{\partial X}) = X + X + \frac{1}{2!}X + \frac{1}{6!}X + \ldots = eX$.

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**Conjecture 2:**

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$$\mathrm{GA}_n(\mathbb{C}) = \mathrm{GLIN}_n(\mathbb{C}).$$

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$\cup|$

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$\mathsf{LF}_n(k)$ I will talk about this!

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If we want to have any hope of applying polynomial maps like linear maps, then we need to strengthen the theoretical foundation of polynomial maps.

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Now, let's try to make a Cayley-Hamilton theorem for polynomial maps! (Perhaps the constant term can replace that stupid $\det(Jac(F)) = 1$ requirement!)

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EXAMPLE:

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But... **Definition:** If $F$ is a zero of some $P(T) \in \mathbb{C}[T] \backslash \{0\}$, then we will call $F$ a Locally Finite Polynomial Endomorphism (short LFPE).

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Let's be a little less ambitious and study this set. LFPE's should resemble linear maps more than general polynomial maps!

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But: the minimum polynomial may change if $G$ is not linear!

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**Conjecture:** in dimension $n$,
$F$ is LFPE $\Longleftrightarrow deg(F^m) \leq deg(F)^{n-1}$ for all $m \in \mathbb{N}$.

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$$P_F(T) := \prod_{\substack{0 \leq k \leq d-1 \\ 0 \leq m \leq d \\ (k,m) \neq (0,0)}} (T^2 - (detL^k)(TrL^m)T + det(L^{2k+m})).$$

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If $\{F_{j,\alpha}^{(i)}\}_{i \in \mathbb{N}}$ is such a sequence, then it is a linear recurrent sequence belonging to $\sum a_i T^i$, etc. . .

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So: we can make many examples of LFPEs!

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$F = (2X + 2Y^2, 3Y) = (2X, 3Y) \circ (X + Y^2, Y)$

$(2X, 3Y) = \exp(\lambda X \partial_X + \mu Y \partial_Y)$,

$D$ locally finite automorphism, then unique decomposition $D = D_n + D_s$ where $D_n$ is locally nilpotent, $D_s$ is semisimple, and $D_n D_s = D_s D_n$.

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Don't know how to make $D_s$, given $F_s$.

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$F^2 - 2F + I = 0$ i.e. zero of $(T-1)^2$.

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THANK YOU