Locally Finite Polynomial Endomorphisms

Stefan Maubach

June 2008
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- A list of \( n \) polynomials: \( F \in (k[X_1, \ldots, X_n])^n \).
- A ring automorphism of \( k[X_1, \ldots, X_n] \) sending \( g(X_1, \ldots, X_n) \) to \( g(F_1, \ldots, F_n) \).
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A polynomial map $F$ is invertible if there is a polynomial map $G$ such that $F(G) = (X_1, \ldots, X_n)$. 
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Why this bold claim? Polynomial maps seem to have similar properties as linear maps (much more so than holomorphic maps for example). Well... to be honest, most are conjectures... Let's look at a few of these conjectures!
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\det \begin{pmatrix} \frac{\partial F_1}{\partial X} & \frac{\partial F_1}{\partial Y} \\ \frac{\partial F_2}{\partial X} & \frac{\partial F_2}{\partial Y} \end{pmatrix} \in \mathbb{C}^* \iff F \in GA_2(\mathbb{C})
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**Jacobian Conjecture** in dimension $n$ ($JC(n)$):

Let $F \in MA_n(\mathbb{C})$. Then

$$\det(Jac(F)) \in \mathbb{C}^* \Rightarrow F \text{ is invertible.}$$
Let $V$ be a vector space. Then

$$V \times \mathbb{C} \cong \mathbb{C}^{n+1} \implies V \cong \mathbb{C}^n.$$
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**Cancelation Problem:**

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$GA_n(\mathbb{K})$ is generated by ??? (Sometimes called “the automorphism problem”, which means: “we don’t understand the automorphism group, whatever understanding means”.)
| \( GA_n(\mathbb{R}) \) | (Dynamical systems, flows Markus-Yamabe Conjecture) |
| \( GA_n(\mathbb{C}) \) | (Complex Analysis) |
| \( O(V) \) | (Algebraic Geometry, Ring theory) |
| \( GA_n(k) \) | |
| \( GA_n(R) \) | |
| \( GA_n(\mathbb{F}_q) \) | (Group theory, number theory, Secret-sharing cryptography) |
Let us make some non-trivial polynomial automorphisms!

**Elementary map:** \((X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)\), invertible with inverse

\((X_1 - f(X_2, \ldots, X_n), X_2, \ldots, X_n)\).
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**Triangular map:** \((X + f(Y, Z), Y + g(Z), Z + c)\)

\[= (X, Y, Z + c)(X, Y + g(Z), Z)(X + f(Y, Z), Y, Z)\]
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\(J_n(\mathbb{K}) := \text{set of triangular maps.}\)

\(\text{Aff}_n(\mathbb{K}) := \text{set of compositions of invertible linear maps and translations.}\)
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\(\text{TA}_n(\mathbb{K}) := \langle J_n(\mathbb{K}), \text{Aff}_n(\mathbb{K}) \rangle\)
In dimension 1: we understand the automorphism group. (They are linear.)
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In dimension 2: famous Jung-van der Kulk-theorem:

\[ GA_2(\mathbb{K}) = TA_2(\mathbb{K}) = \text{Aff}_2(\mathbb{K}) \times J_2(\mathbb{K}) \]

Jung-van der Kulk is the reason that we can do a lot in
dimension 2 !!!!
What about dimension 3?
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(What is a difficult and technical proof. ) (2007 AMS Moore paper award.) So now it is official. Nagata is complicated.
AMS E.H. Moore Research Article Prize

Ivan Shestakov (center) and Ualbai Umirbaev (right) with Jim Arthur.
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  $$D^n(g) = 0.$$

If $D$ is LND (locally nilpotent derivation) then $\exp(D)$ is automorphism !! We have a *non-trivial* way of making automorphisms! In fact: Nagata $= \exp(D)$!
Let

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**Conjecture 1:**
\[ \text{GA}_n(\mathbb{C}) = \langle \text{Aff}_n(\mathbb{C}), \text{ELND}_n(\mathbb{C}) \rangle. \]

... candidate counterexamples start to emerge ...
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for all $g \in \mathbb{C}^n$:  $g, D(g), D^2(g), \ldots$ span a finite dimensional \(\mathbb{C}\)-vector space.
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$D$ locally finite $\longrightarrow$ $\exp(D)$ automorphism.

$\exp(X \frac{\partial}{\partial X}) = X + X + \frac{1}{2!}X + \frac{1}{6!}X + \ldots = e^X.$
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Define: $\text{LFD}_n(\mathbb{C}) = \text{set of Locally Finite Derivations.}$
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**Conjecture 2:**

$$\text{GA}_n(\mathbb{C}) = \text{ELFD}_n(\mathbb{C}).$$
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KNOWN: Nagata is not linearizable.

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Now compute: $N^{-\frac{4}{3}}(2N)N^{\frac{4}{3}} = (2X, 2Y, 2Z)$!!!
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Define $\text{GLIN}_n(\mathbb{C})$ as the group generated by the linearizable automorphisms. (I.e. $\text{GLIN}_n(\mathbb{C})$ is smallest normal subgroup of $\text{GA}_n(\mathbb{C})$ containing $\text{GL}_n(\mathbb{C})$.)
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$$\text{GA}_n(\mathbb{C}) = \text{GLIN}_n(\mathbb{C}).$$
$GA_n(k)$

$TA_n(k)$
\[ \mathcal{G}A_n(k) \]

\[ \mathcal{U}| \]

\[ \mathcal{E}L\mathcal{N}D_n(k) := \langle \mathcal{A}ff_n(k), \exp(D) \mid D \text{ locally nilpotent derivation} \rangle \]

\[ \mathcal{U}| \]

\[ \mathcal{T}A_n(k) \]
\[ \text{GA}_n(k) \]

\[ \bigcup | \]

\[ \text{ELFD}_n(k) := \langle \exp(D) \mid D \text{ locally finite derivation} \rangle \]

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\[ \text{ELND}_n(k) := \langle \text{Aff}_n(k), \exp(D) \mid D \text{ locally nilpotent derivation} \rangle \]

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\[ \text{TA}_n(k) \]
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∪

$GLIN_n(k) := \text{normalizer of } GL_n(k)$

∪

not equal if char$(k) \neq 0$.

$TA_n(k)$
GA_n(k)

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GTAM_n(k) := \text{normalizer of } TA_n(k)

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TA_n(k)
$G\!A_n(k)$
∪
$LF_n(k)$ I will talk about this!
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Let us step back for a moment . . .

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If we want to have any hope of applying polynomial maps like linear maps, then we need to strengthen the theoretical foundation of polynomial maps.
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Very good: the Cayley-Hamilton theorem (characteristic polynomials of linear maps etc.).
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Very good: the Cayley-Hamilton theorem (characteristic polynomials of linear maps etc.).

Now, let’s try to make a Cayley-Hamilton theorem for polynomial maps! (Perhaps the constant term can replace that stupid \( \det(Jac(F)) = 1 \) requirement!)
Cayley-Hamilton:

Let $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear map. Then $L$ is a zero of

$$P_L(T) := \det(TI - L).$$
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What about generalizing $ML_n(\mathbb{C}) \rightarrow MA_n(\mathbb{C})$?
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But... **Definition:** If \( F \) is a zero of some \( P(T) \in \mathbb{C}[T]\setminus\{0\} \), then we will call \( F \) a Locally Finite Polynomial Endomorphism (short LFPE).
Example:

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F^2 - 2F + I = 0, \text{ so } F \text{ is “zero of } T^2 - 2T + 1 = (T - 1)^2 \text{”}.
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Let’s be a little less ambitious and study this set. LFPE’s should resemble linear maps more than general polynomial maps!
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Proof: due to the first remark.
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But: the minimum polynomial may change if $G$ is not linear!
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We’ll get back on that... First some results!
“Cayley-Hamilton” in $n$ variables
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$$0 < |\alpha| \leq D$$

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Equivalent are:

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**Conjecture:** in dimension $n$, $F$ is LFPE $\iff \deg(F^m) \leq \deg(F)^{n-1}$ for all $m \in \mathbb{N}$. 
$n = 2$: Classification of LFPE
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Two essential cases:
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$F = (aX + P(Y), bY)$ \hspace{1em} (F invertible)

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Then $F$ is a zero of

$$P_F(T) := \prod_{0 \leq k \leq d - 1} \left( T^2 - (\det L^k)(\text{Tr} L^m) T + \det(L^{2k+m}) \right).$$

$$0 \leq m \leq d$$

$(k, m) \neq (0, 0)$
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If $F^i = (F_1^{(i)}, \ldots, F_n^{(i)})$ and $F_j^{(i)} = \sum F_{j,\alpha} X^{\alpha}$, then $\sum a_i F^i = 0 \iff \sum a_i F_{j,\alpha}^{(i)} = 0 \forall j, \alpha.$
How did we prove that?

If \( F^i = (F_1^{(i)}, \ldots, F_n^{(i)}) \) and \( F_j^{(i)} = \sum F_{j,\alpha} X^\alpha \),
then \( \sum a_i F^i = 0 \iff \sum a_i F_{j,\alpha}^{(i)} = 0 \forall j, \alpha \).

If \( \{F_{j,\alpha}^{(i)}\}_{i \in \mathbb{N}} \) is such a sequence, then it is a **linear recurrent sequence** belonging to \( \sum a_i T^i \), etc....
Exponents of derivations
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If $D$ is a locally finite derivation, then

$$\exp(D)(g) := g + D(g) + \frac{1}{2!} D^2(g) + \frac{1}{3!} D^3(g) + \ldots$$

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\[ \exp(D)^2 = \exp(D) \circ \exp(D) = \exp(2D) \]
\[ F^n = \exp(nD) = (X + nY^2 + n^2YZ + \frac{n^3}{6}Z^2, Y + nZ, Z) \]
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i.e. \( \{ \deg(\exp(nD)) \} \) \( n \in \mathbb{N} \) is bounded sequence
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So: \( F = \exp(D) \longrightarrow F \text{ is LFPE.} \)
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So: we can make many examples of LFPEs!
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(A flow of \( F \) is:

- \( F_t \) for each \( t \in \mathbb{C} \)
- \( F_1 = F, F_0 = I, F_t F_u = F_{t+u} \).)
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$D$ locally finite automorphism, then unique decomposition $D = D_n + D_s$ where $D_n$ is locally \textit{nilpotent}, $D_s$ is \textit{semisimple}, and $D_n D_s = D_s D_n$. 
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Given $F$ LFPE, then we find unique decomposition $F = F_nF_s = F_sF_n$. 
A locally finite automorphism, then unique decomposition
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Don’t know how to make $D_s$, given $F_s$. 
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\[
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Example: $F = \exp(Y^2 \partial_X) = (X + Y^2, Y)$

$F^2 - 2F + I = 0$ i.e. zero of $(T - 1)^2$.\)


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if $c \in \mathbb{C}$, then no natural choice $\log(c)$. So, to repeat: QUESTION: if $F$ is L.F., is $F = \exp(D)$?

THANK YOU