

$k[z]$ -automorphisms and coordinates in two variables

Stefan Maubach

Arjeh Cohen 60, 27 May 2009

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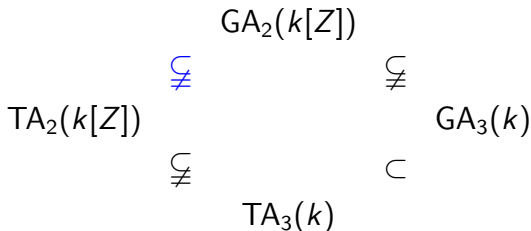
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(2004) Umirbaev-Shestakov: $TA_3(k) \neq GA_3(k)$.

In fact: $GA_2(k[Z]) \not\subseteq TA_3(k)$.

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(2003) Berson: result improved in:

(2008) Berson-v/d Essen-Wright: $F \in GA_2(k[Z])$ then
 $F \in TA_5(k[Z])$.

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Today: a special case of $n = 3$ is proven.

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1. $k[X, Y, Z]/(f) \cong k[T_1, T_2]$
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$f(X, Y, Z)$ is a $k[Z]$ -coordinate (i.e. exists $g(X, Y, Z)$ such that $(f, g) \in GA_2(k[Z])$. Or $k[Z][f, g] = k[Z][X, Y]$.)

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Proof is surprisingly simple! (Joint work with E.Edo)

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Theorem: Let R be a \mathbb{Q} -algebra and $f \in R[X, Y]$. Let D be the derivation

$$D_f := f_y \partial_x - f_x \partial_y.$$

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- i) f is a coordinate in $R[X, Y]$.
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- $$1 \in (f_x, f_y) = R[X, Y]f_x + R[X, Y]f_y.$$

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Hence, to prove: $1 \in (f_x, f_y)$ and $D_f := f_y \partial_x - f_x \partial_y$ locally nilpotent.

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There is no equivalent of this theorem in higher dimensions!

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With examples it seems like there might be a (heuristic) *algorithm* for doing this. The difficulty will be to prove that such an algorithm always works! The finding of such a φ is equivalent to finding a (candidate) “mate” of f , i.e. a coordinate p such that $f \bmod p$ is a coordinate... still quite difficult?

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Thank you for your attention!