$k[z]$-automorphisms and coordinates in two variables

Stefan Maubach

Arjeh Cohen 60, 27 May 2009

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Today:

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Known				No idea!

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Known		...we're getting there!		No idea!

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In fact: $\mathrm{GA}_2(k[Z]) \not\subseteq \mathrm{TA}_3(k)$.

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In fact: algorithm to decide: $F \in GA_2(k[Z])$, when is $F \in TA_3(k)$.

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(2003) Berson: result improved in:

(2008) Berson-v/d Essen-Wright: $F \in \mathrm{GA}_2(k[Z])$ then $F \in \mathrm{TA}_5(k[Z])$.

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Today: a special case of $n = 3$ is proven.

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$f(X, Y, Z)$ is a $k[Z]$-coordinate (i.e. exists $g(X, Y, Z)$ such that $(f, g) \in GA_2(k[Z])$. Or $k[Z][f, g] = k[Z][X, Y]$.)

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Proof is surprisingly simple! (Joint work with E.Edo)

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Theorem: Let R be a $\mathbb{Q}$-algebra and $f \in R[X,Y]$. Let D be the derivation

$$D_f := f_y \partial_x - f_x \partial_y.$$

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i) f is a coordinate in $R[X,Y]$.

ii) D_f is locally nilpotent and

$1 \in (f_x, f_y) = R[X,Y]f_x + R[X,Y]f_y$.

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Hence, to prove: $1 \in (f_x, f_y)$ and $D_f := f_y \partial_x - f_x \partial_y$ locally nilpotent.

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There is no equivalent of this theorem in higher dimensions!

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With examples it seems like there might be a (heuristic) *algorithm* for doing this. The difficulty will be to prove that such an algorithm always works! The finding of such a φ is equivalent to finding a (candidate) "mate" of f, i.e. a coordinate p such that $f \mod p$ is a coordinate... still quite difficult?

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Thank you for your attention!