The Nagata Automorphism is Shifted Linearizable

. . . and a conjecture that interests discrete mathematicians

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...we'll get back to this...

Nagata's automorphism:

$N := (X - Y\Delta - Z\Delta^2, Y + Z\Delta, Z)$ where $\Delta = XZ + Y^2$.

In fact:

$N = \exp(\Delta\partial)$ where $\partial = -2Y\frac{\partial}{\partial X} + Z\frac{\partial}{\partial Y}$.

Let's define:

$N^\lambda = \exp(\lambda\Delta\partial)$ where $\lambda \in \mathbb{C}$.

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What is going on?

General shifted theorems

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(2) $L_{(a,b,c)}\partial = \frac{c}{b}\partial L_{(a,b,c)}$

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$$L_{(a,b,c)}^{-1}DL_{(a,b,c)} = abD.$$

D is a *homogeneous* derivation (of degree 1 and 2), with respect to a 2-dimensional "grading" space (i.e. 2 completely different gradings).

$$L_{(a,b,c)}^{-1} D L_{(a,b,c)} = abD.$$

So:

$$\begin{aligned} L_{(a,b,c)}^{-1} N^{\lambda} L_{(a,b,c)} &= L_{(a,b,c)}^{-1} \exp(\lambda D) L_{(a,b,c)} \\ = \exp(\lambda L_{(a,b,c)}^{-1} D L_{(a,b,c)}) &= \exp(\lambda abD) = N^{ab\lambda} \end{aligned}$$

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Thus $L_{(a,b,c)} N^{\lambda}$ is shifted linearizable if $ab \neq 1$.

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Define eigenspaces $E_\lambda(\varphi) = \{p \mid \varphi(p) = \lambda p\}$.
Example: $E_\lambda(N) = \mathbb{C}$ if $\lambda \neq 1$ and $E_1 = ker(D) = \mathbb{C}[Z, \Delta]$.

Lemma

If b is not a root of unity, then $E_1(L_b N^\lambda) = \mathbb{C}[Z^2\Delta]$.

Corollary

If b is not a root of unity, then $L_b N^\lambda$ *is not linearizable.*

Follows from the fact that then $E_1(L_b N^\lambda)$ and $E_1(L_b) = \mathbb{C}[XZ^3, YZ]$ have to be isomorphic.

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Suppose $b^n = 1$. Then $(L_b N^\lambda)^n = N^{n\lambda}$ which is not linearizable. Hence $L_b N^\lambda$ cannot be linearizable.

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Note: $T_n(\mathbb{K}) \subset< \mathrm{Linzble}_n(\mathbb{K}) >$. (Example: $(X + f(Y), Y) = (1/2X, Y) \circ (2X + 2f(Y), Y)$ and $(2X + 2f(Y), Y) = (X - 2f(Y), Y)(2X, Y)(X + 2f(Y), Y)$)

Second subject: . . . and a conjecture that interests discrete mathematicians

Consider $\varphi \in \mathrm{GA}_n(\mathbb{F}_q)$. Induces bijection $\mathcal{E}(\varphi) : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$,
i.e. $\mathcal{E}(\varphi) \in \mathrm{Sym}(q^n)$.
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Hence if $q = 4, 8, 16, \ldots$ then $\mathcal{E}(T_n(\mathbb{F}_q))$ is either $\mathrm{Alt}(m)$ or $\mathrm{Sym}(m)$.
But - you can check that all elementary maps will be even - hence $\mathcal{E}(T_n(\mathbb{F}_q)) = \mathrm{Alt}(m)$!

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Problem: Do there exist "odd" polynomial automorphisms over $\mathbb{F}_4$?

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******THANK YOU******