# Locally finite polynomial endomorphisms

Stefan Maubach

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Notations:

| | Linear | Polynomial |
|---|---|---|
| All | $ML_n(\mathbb{C})$ | $MA_n(\mathbb{C})$ |
| Invertible | $GL_n(\mathbb{C})$ | $GA_n(\mathbb{C})$ |

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Why this bold claim?Polynomial maps seem to have similar properties as linear maps (much more so than holomorphic maps for example). Well. . . to be honest, most are conjectures. . . Let's look at a few of these conjectures!

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**Jacobian Conjecture** in dimension $n$ (JC(n)):

Let $F \in MA_n(\mathbb{C})$. Then

$$\det(Jac(F)) \in \mathbb{C}^* \Rightarrow F \text{ is invertible.}$$

Let $V$ be a vector space. Then

$$V \times \mathbb{C} \cong \mathbb{C}^{n+1} \Longrightarrow V \cong \mathbb{C}^n.$$

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$GA_n(\mathbb{C})$ is generated by ???

**Elementary map:** $(X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)$, invertible with inverse

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Nagata's map:

$$F = \begin{pmatrix} X - 2(XZ + Y^2)Y - (XZ + Y^2)^2 Z, \\ Y + (XZ + Y^2)Z, \\ Z \end{pmatrix}$$

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$n = 3$:(Shestakov-Umirbaev, 2004)

Nagata's map not tame, i.e. $GA_3(\mathbb{C}) \neq TA_3(\mathbb{C})$

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Very good: the Cayley-Hamilton theorem (characteristic polynomials of linear maps etc.).

Now, let's try to make a Cayley-Hamilton theorem for polynomial maps! (Perhaps the constant term can replace that stupid $\det(Jac(F)) = 1$ requirement!)

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Let's be a little less ambitious and study this set. LFPE's should resemble linear maps more than general polynomial maps!

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But: the minimum polynomial may change if $G$ is not linear!

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We'll get back on that... First some results!

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$$
\begin{array}{lcl}
F \text{ invertible} & \iff & F \text{ is conjugate of} \\
 & & (aX + P(Y), bY) \\
 & & a, b \in \mathbb{C}^*, P(Y) \in \mathbb{C}[Y].
\end{array}
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| | | |
|---|---|---|
| $F$ invertible | $\iff$ | $F$ is conjugate of $(aX + P(Y), bY)$ $a, b \in \mathbb{C}^*, P(Y) \in \mathbb{C}[Y]$. |
| $F$ not invertible | $\iff$ | $F$ is conjugate of $(aX + YP(X, Y), 0)$ $a, \in \mathbb{C}, P(X, Y) \in \mathbb{C}[X, Y]$. |

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$$P_F(T) := \prod_{\substack{0 \le k \le d-1 \\ 0 \le m \le d \\ (k,m) \ne (0,0)}} (T^2 - (detL^k)(TrL^m)T + det(L^{2k+m})).$$

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**Conjecture:** in dimension $n$,
$F$ is LFPE $\iff deg(F^m) \leq deg(F)^{n-1}$ for all $m \in \mathbb{N}$.

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(where $\lambda^\alpha = \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$)

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Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the linear part of $F$.

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then $\sum a_i F^i = 0 \iff \sum a_i F_{j,\alpha}^{(i)} = 0 \forall j, \alpha$.
If $\{F_{j,\alpha}^{(i)}\}_{i \in \mathbb{N}}$ is such a sequence, then it is a linear recurrent sequence belonging to $\sum a_i T^i$, etc. . .

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So: we can make many examples of LFPEs!

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?$\Leftarrow$?

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Don't know how to make $D_s$, given $F_s$.

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if $c \in \mathbb{C}$, then no natural choice $\log(c)$.

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as exponential map. So: $F$ LFPE then you can make $F_t$.

$F^2 = aF + bI.$

$$
\begin{aligned}
F^2 &= aF + bI &&= (a, b)\binom{F}{I} \\
F^n & &&= (a, b)\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}^n \binom{I}{F^{-1}} \\
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where $t \in \mathbb{C}$. QUESTION: Does that work?? Is $F_t$ flow?

(Note: can prove that this work if eigenvalues are "generic",

to be precise:

$\lambda_1^{d_1} \cdots \lambda_n^{d_n} = 1$ then all $d_i = 0$.)

# last slide . . .

# Finally. . . last slide . . . phew. . .

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Funny detail: true for linear $F$, but not trivial.

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THANK YOU