Polynomial automorphisms, especially over finite fields

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A polynomial map F is invertible if there is a polynomial map G such that $F(G) = (X_1, \ldots, X_n)$.

Notations:

	Linear	Polynomial
All	$ML_n(k)$	$MA_n(k)$
Invertible	$GL_n(k)$	$GA_n(k)$

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Why this bold claim? Polynomial maps seem to have similar properties as linear maps (much more so than holomorphic maps for example). Today I will not really go into these similarities, but they are there. Today: I will talk about $\mathrm{GA}_n(k)$, especially if k is a finite field.

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REASON 2: Polynomial maps over finite fields may have applications in discrete-mathematics like settings!

RE-MOTIVATION: Why **NOT** study polynomial maps over finite fields! In fact, why didn't anyone fill that **gaping hole** yet!

REASON 1: Reduction-mod-p techniques to solve problems over $\mathbb{C}$. Classical example: an injective polynomial map is surjective. Reason: an injective map from a finite set to a finite. Very recent: Belov-Kontsjevich (yes, that guy) proved equivalence of two already long-standing conjectures: the Dixmier Conjecture ('68) and the Jacobian Conjecture ('39).

REASON 2: Polynomial maps over finite fields may have applications in discrete-mathematics like settings! (In fact, one of the reasons for this talk is the hope that there may be one or two of you in the audience who may see such a possible application!)

The Automorphism Group

(This whole talk: $n \geq 2$)

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$GA_n(k)$ is generated by ???

Elementary map: $(X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)$, invertible with inverse

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Triangular map: $(X + f(Y, Z), Y + g(Z), Z + c)$

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$TA_n(k) := < J_n(k), Aff_n(k) >$

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In dimension 2: famous Jung-van der Kulk-theorem:

$$\mathrm{GA}_2(\mathbb{K}) = \mathrm{TA}_2(\mathbb{K}) = \mathit{Aff}_2(\mathbb{K}) \ltimes \mathrm{J}_2(\mathbb{K})$$

Jung-van der Kulk is the reason that we can do a lot in dimension 2 !!!!

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(Difficult and technical proof.) (2007 AMS Moore paper award.)

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Nagata proved: N is NOT tame over $k[z]$, i.e. N not in $\mathrm{TA}_2(k[z])$.

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Hence: Nagata map is in $\mathrm{GLIN}_3(k)$! - If $k \neq \mathbb{F}_2, \mathbb{F}_3$, that is !!

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Simpler question: what is $\pi(\mathsf{TA}_n(\mathbb{F}_q))$?

Why simpler? Because we have a set of generators!

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$T_n(\mathbb{F}_q)$ is generated by $\mathrm{GL_n(\mathbb{F}_q)}$ (for which we have a finite set of generators) and maps of the form

$$\sigma_f := (X_1 + f, X_2, \ldots, X_n)$$

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Hence, take the following set:

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where $< a >= \mathbb{F}_q^*$.

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So: Start looking for an odd automorphism!!! (Or prove they don't exist)

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So far: we did not find an odd automorphism. Perhaps we didn't look hard enough! Perhaps all polynomial automorphisms are even - but why?

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Corollary

(of some theorem I proved) Let $F \in \mathrm{GA}_2(\mathbb{F}_q[Z])$. Then F is tamely mimickable.

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This almost works - a bit more wiggling necessary (And for the general case, even more work.)

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Fix a tame map F. Consider it modulo all p, where p are big primes. Then F does not behave like Nagata modulo p.

Another “characteristic 2” anomaly: compare

$\mathsf{GTAM}_n(\mathbb{F}_2) :=$ normalizer of $\mathsf{TA}_n(\mathbb{F}_2)$

$\cup|$

$\mathsf{GLIN}_n(\mathbb{F}_2) :=$ normalizer of $\mathrm{GL_n}(\mathbb{F}_2)$

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After some trial-and-error: $f(Y) \in \mathbb{F}_2[Y^2 + Y] + \mathbb{F}_2 Y + \mathbb{F}_2$.

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In particular - we couldn't make $(X + Y^3, Y)$.

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YES! We can make all affine ones (not that hard).

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Now $(X + Y^i Z, Y, Z)(X, Y, Z + 1)(X + Y^i Z, Y, Z) =$
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So: $\mathrm{GTAM}_n(\mathbb{F}_2) \subset \mathrm{GLIN}_{n+1}(\mathbb{F}_2)$.

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But - we run into other monomials that we cannot make:

$(X + YZ, Y, Z)$

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$$\frac{\#\pi_4(\mathsf{GLIN}_2(\mathbb{F}_2))}{\#\pi_4(\mathsf{GTAM}_2(\mathbb{F}_2))} = 2.$$

End proof.

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***** THANK YOU *****

(for watching 175 slides...)