Polynomial automorphisms over finite fields and Locally Finite Polynomial Maps

Stefan Maubach

Arpil 2008

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Notations:

	Linear	Polynomial
All	$ML_n(\mathbb{C})$	$MA_n(\mathbb{C})$
Invertible	$GL_n(\mathbb{C})$	$GA_n(\mathbb{C})$

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Why this bold claim?Polynomial maps seem to have similar properties as linear maps (much more so than holomorphic maps for example). Well. . . to be honest, most are conjectures. . . Let's look at a few of these conjectures!

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Jacobian Conjecture in dimension n (JC(n)):
Let $F \in MA_n(\mathbb{C})$. Then

$$\det(Jac(F)) \in \mathbb{C}^* \Rightarrow F \text{ is invertible.}$$

Let V be a vector space. Then

$$V \times \mathbb{C} \cong \mathbb{C}^{n+1} \implies V \cong \mathbb{C}^n.$$

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$n = 3$:(Shestakov-Umirbaev, 2004)

Nagata's map not tame, i.e. $GA_3(\mathbb{C}) \neq TA_3(\mathbb{C})$

First subject: Polynomial maps over finite fields

Consider $\varphi \in \mathrm{GA}_n(\mathbb{F}_q)$. Induces bijection $\mathcal{E}(\varphi) : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$, i.e. $\mathcal{E}(\varphi) \in \mathrm{Sym}(q^n)$.

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$$\sigma_\alpha := \sigma_{f_\alpha}.$$

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$$\sigma_\alpha := (X_1 + f_\alpha, X_2, \ldots, X_n)$$
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$$\tau := (aX_1, X_2, \ldots, X_n)$$

where $< a >= \mathbb{F}_q^*$.

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Hence if $q = 2$ or $q = \text{odd}$, then $\mathcal{E}(T_n(\mathbb{F}_q)) = \mathrm{Sym}(q^n)$.

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Theorem: $H < \mathrm{Sym}(m)$ Primitive + 3-cycle $\longrightarrow H = \mathrm{Alt}(m)$ or $H = \mathrm{Sym}(m)$.

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But - $\mathcal{E}(\sigma_\alpha, \mathcal{E}(\sigma_i), \mathcal{E}(\tau)$ are all even ! Hence $\mathcal{E}(T_n(\mathbb{F}_q)) = \mathrm{Alt}(m)$!

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Problem: Do there exist "odd" polynomial automorphisms over $\mathbb{F}_4$?

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(3) (if $n = 3$:) $\mathsf{GA}_3(\mathbb{K}) \neq < \mathsf{Aff}_3(\mathbb{K}), \mathsf{GA}_2(\mathbb{K}[Z]) >$.

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So: Start looking for an odd automorphism!!! (Or prove they don't exist)

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Now, let's try to make a Cayley-Hamilton theorem for polynomial maps! (Perhaps the constant term can replace that stupid $\det(Jac(F)) = 1$ requirement!)

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Corollary: if R is a field, there is a unique minimum polynomial

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But: the minimum polynomial may change if G is not linear!

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Conjecture: in dimension n,

F is LFPE $\iff deg(F^m) \leq deg(F)^{n-1}$ for all $m \in \mathbb{N}$.

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If $\{F_{j,\alpha}^{(i)}\}_{i \in \mathbb{N}}$ is such a sequence, then it is a linear recurrent sequence belonging to $\sum a_i T^i$, etc.. . .

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EXAMPLE: $D = \frac{\partial}{\partial X_1}$.

D is called **locally finite** if:
For all $g \in \mathbb{C}[X_1, \ldots, X_n]$, the vector space $\mathbb{C}g + \mathbb{C}D(g) + \mathbb{C}D^2(g) + \ldots$ is finite dimensional.
EXAMPLE: $D = X_1 \frac{\partial}{\partial X_1}$.
Locally nilpotent $\Rightarrow$ Locally finite

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So: we can make many examples of LFPEs!

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Don't know how to make D_s, given F_s.

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So there's some funny stuff you might be able to read off $\mathfrak{m}_F$!

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***** THANK YOU *****