The automorphism group of affine spaces (especially $\mathbb{A}^n$)

Stefan Maubach

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Why this bold claim? Polynomial maps seem to have similar properties as linear maps (much more so than holomorphic maps for example). Well. . . to be honest, most are conjectures. . . Let's look at a few of these conjectures!

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Jacobian Conjecture in dimension n (JC(n)):

Let $F \in MA_n(\mathbb{C})$. Then

$$\det(Jac(F)) \in \mathbb{C}^* \Rightarrow F \text{ is invertible.}$$

Let V be a vector space. Then

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$\mathsf{J}_n(\mathbb{K})$:= set of triangular maps.
$\mathsf{Aff}_n(\mathbb{K})$:= set of compositions of invertible linear maps and translations.
$\mathsf{TA}_n(\mathbb{K}) := < \mathsf{J}_n(\mathbb{K}), \mathsf{Aff}_n(\mathbb{K}) >$

In dimension 1: we understand the automorphism group.
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In dimension 2: famous Jung-van der Kulk-theorem:

$$\mathsf{GA}_2(\mathbb{K}) = \mathsf{TA}_2(\mathbb{K}) = \mathsf{Aff}_2(\mathbb{K}) \ltimes \mathsf{J}_2(\mathbb{K})$$

Jung-van der Kulk is the reason that we can do a lot in dimension 2 !!!!

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(Difficult and technical proof.) (2007 AMS Moore paper award.) So now it is official. Nagata is complicated.

AMS E.H. Moore Research Article Prize


Ivan Shestakov (center) and Ualbai Umirbaev (right) with Jim Arthur.

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If D is LND(locally nilpotent derivation) then $\exp(D)$ is automorphism !! We have a *non-trivial* way of making automorphisms! In fact: Nagata $= \exp(D)$!

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Conjecture 1:

$$\mathsf{GA}_n(\mathbb{C}) = < \mathsf{Aff}_n(\mathbb{C}), e^{\mathsf{LND}_n(\mathbb{C})} > .$$

...candidate counterexamples start to emerge ...

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D locally finite $\longrightarrow \exp(D)$ automorphism.

$\exp(X\frac{\partial}{\partial X}) = X + X + \frac{1}{2!}X + \frac{1}{6!}X + \ldots = eX$.

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Conjecture 2:

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Conjecture 3:

$$\mathrm{GA}_3(\mathbb{C}) =< \mathrm{Aff}_3(\mathbb{C}), \mathrm{GA}_2(\mathbb{C}[Z]) >$$

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KNOWN: Nagata is not linearizable.

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Conjecture 4:

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If we want to have any hope of applying polynomial maps like linear maps, then we need to strengthen the theoretical foundation of polynomial maps.

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Now, let's try to make a Cayley-Hamilton theorem for polynomial maps! (Perhaps the constant term can replace that stupid $\det(Jac(F)) = 1$ requirement!)

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But... **Definition:** If F is a zero of some $P(T) \in \mathbb{C}[T]\backslash\{0\}$, then we will call F a Locally Finite Polynomial Endomorphism (short LFPE).

Example:

$$F := (X + Y^2, Y)$$

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$F^0 := (X, Y)$

$F := (X + Y^2, Y)$

$F^2 := (X + 2Y^2, Y)$

$F^2 - 2F + I = 0$, so F is "zero of $T^2 - 2T + 1 = (T-1)^2$".

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- $I_F := \{P(T) \in \mathbb{C}[T] \mid P(F) = 0\}$ is an ideal of $\mathbb{C}[T]$
- $F \in \mathrm{GA}_n(\mathbb{C})$ is "zero of $(T-1)^m$ some $m \in \mathbb{N}$" $\iff$ $F = \exp(D)$ where $D \in \mathrm{LND}_n(\mathbb{C}) \iff F$ is unipotent.

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- $F \in \mathrm{GA}_n(\mathbb{C})$ is semisimple $\iff F$ zero of $Q(T)$ where Q is radical, $\impliedby F = \exp(D)$ where D is semisimple

Conjecture 5:

$$\mathrm{GA}_n(\mathbb{C}) =< LFPE >$$

. . . and a conjecture that interests discrete mathematicians

Consider $\varphi \in \mathsf{GA}_n(\mathbb{F}_q)$. Induces bijection $\mathcal{E}(\varphi) : \mathbb{F}_q^n \longrightarrow \mathbb{F}_q^n$, i.e. $\mathcal{E}(\varphi) \in \mathrm{Sym}(q^n)$.

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Answer: if $q = 2$ or $q =$ odd, then $\mathcal{E}(T_n(\mathbb{F}_q)) = \mathrm{Sym}(q^n)$.

Answer: if $q = 4, 8, 16, 32, \ldots$ then $\mathcal{E}(T_n(\mathbb{F}_q)) = \mathrm{Alt}(q^n)$.

Question: what is $\mathcal{E}(T_n(\mathbb{F}_q))$?

Answer: if $q = 2$ or $q =$ odd, then $\mathcal{E}(T_n(\mathbb{F}_q)) = \mathrm{Sym}(q^n)$.

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Problem: Do there exist "odd" polynomial automorphisms over $\mathbb{F}_4$?

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So: Start looking for an odd automorphism!!! (Or prove they don't exist)

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- Relatively new: certain group actions ($\mathcal{G}_a$-actions, derivations, etc.)

A new method: Makar-Limanov invariant

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Introduction of the ~~AK-invariant~~ ML-invariant.

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Notice:

$$\begin{aligned} ML(\mathbb{C}[X,Y,Z]) \subseteq \;& \mathbb{C}[X,Y,Z]^{\partial_X} \cap \mathbb{C}[X,Y,Z]^{\partial_Y} \cap \mathbb{C}[X,Y,Z]^{\partial_Z} \\ & \mathbb{C}[Y,Z] \cap \mathbb{C}[X,Z] \cap \mathbb{C}[X,Y] = \mathbb{C}. \end{aligned}$$

The Makar-Limanov invariant

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Hence $X^2Y - P(Z) = 0$ is not isomorphic to $\mathbb{C}^3$.

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The Makar-Limanov invariant

In '93 Russel and Koras constructed surfaces which were topologically the same as $\mathbb{C}^3$, but of which they didn't know if they were $\mathbb{C}^3$.

Simplest example: $V := X^2Y + X + Z^2 + T^3$. Breakthrough by Makar-Limanov:

$ML(\mathcal{O}(V)) = \mathbb{C}[X]$.

Proof is quite elaborate - using smart gradings, filtrations, etc. etc.

Makar-Limanov techniques

The strength of ML invariant comes because of the techniques to compute it. Sometimes one can use these techniques, sometimes not. But - there are cases where the ML invariant will fail. Example: $\mathbb{C}[X, Y, Z, T]/(XY + X + Z^2 + T^3)$. (You can see exactly when $p(X)Y + q(X, Z, T)$ is $\mathbb{C}^3$ (M. 2003) by studying commuting derivations)

(Biregular) cancellation problems

Let k be a field. Let U, V, W be k-varieties.

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Let k be a field. Let U, V, W be k-varieties. Suppose $U \times k \cong V \times k$. Is $U \cong V$?

Ring theoretic version:

Suppose A, B are finitely generated k-algebras. Suppose $A[X] \cong B[X]$. Is $A \cong B$?

First counterexamples over $\mathbb{R}$

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(Hoechster (1972):) Let $R := \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$

etc. . .

Danielewski surfaces

Preprint of Danielewski(83?): Examples over $\mathbb{C}$!

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Preprint of Danielewski(83?): Examples over $\mathbb{C}$!

Let $V_1 := \{xy - z^2 + 1 = 0\}, V_2 = \{x^2y - z^2 + 1\}$. Then $V_1 \times \mathbb{C} \cong V_2 \times \mathbb{C}$ but $V_1 \neq V_2$.

Danielewski surfaces are not UFDs.

Danielewski surfaces are not UFDs. In fact:

If V, W $\mathbb{C}$-algebras of dim$=$2, then

$V \times_{\mathbb{C}} \mathbb{C} \cong W \times_{\mathbb{C}} \mathbb{C} \longrightarrow V \cong W$.

(Due to Miyanishi)

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THANK YOU