INTUITIONISTIC LOGIC AND MUCHNIK DEGREES

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Abstract. We prove that there is a factor of the Muchnik lattice that captures intuitionistic propositional logic. This complements a now classic result of Skvortsova for the Medvedev lattice.

1. Introduction

Among the structures arising from computability theory, the lattices introduced by Medvedev and Muchnik stand out for several distinguished features and a broad range of applications. In particular these lattices have additional structure that makes them suitable as models of certain propositional calculi. The structure of the Medvedev lattice as a Brouwer algebra, and thus as a model for propositional logics, has been extensively studied in several papers, see e.g. [10], [15], [17], [20], [22]. Originally motivated in [10] as a formalization of Kolmogorov's calculus of problems [7], the Medvedev lattice fails to provide an exact interpretation of the intuitionistic propositional calculus IPC; however, as shown by Skvortsova [15], there are initial segments of the Medvedev lattice that model exactly IPC. On the other hand, little is known about the structure of the Muchnik lattice, and of its dual, as Brouwer algebras. The goal of this paper is to show that there are initial segments (equivalently: factors obtained dividing the lattice by principal filters) of the Muchnik lattice, in which the set of valid propositional sentences coincides with IPC. This shows that the analogue of Skvortsova's theorem also holds for the Muchnik lattice. From this, it readily follows that the propositional sentences that are valid in the Muchnik lattice are exactly the sentences of the so-called logic of the weak law of the excluded middle ([17]). Similar results (as announced, with outlined proofs, in [18]) hold of the dual of the Muchnik lattice: detailed proofs are provided in Section 5.

For all unexplained notions from computability theory, the reader is referred to Rogers [14]; our main source for Brouwer algebras and the algebraic semantics of propositional calculi is Rasiowa-Sikorski [13]. A comprehensive survey on the Medvedev and Muchnik lattices, and their mutual relationships, can be found in [19]. Throughout the paper we use the symbols $+$ and $\times$ to denote the join and meet operations, respectively, in any lattice.

1.1. **The Medvedev and the Muchnik lattices.**

Although our main object of study is the Muchnik lattice, reference to the Medvedev lattice will be sometimes useful. Therefore, we start by reviewing some basic definitions and facts concerning both lattices. Following Medvedev [10], a *mass problem* is a set of functions from the set of natural numbers ω to ω. There are two natural ways to extend Turing reducibility to mass problems: following [10],

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a mass problem $\mathcal{A}$ is *Medvedev-reducible* to a mass problem $\mathcal{B}$ (denoted by $\mathcal{A} \leqslant_M \mathcal{B}$), if there is an oracle Turing machine by means of which every function of $\mathcal{B}$, when supplied to the machine as an oracle, computes some function of $\mathcal{A}$. (Any oracle Turing machine defines in this sense a partial mapping from ω^ω to ω^ω, called a *partial computable functional*.) A different approach, which consists in dropping uniformity, leads to *Muchnik reducibility*, [12], denoted by $\leqslant_w$: here $\mathcal{A} \leqslant_w \mathcal{B}$, if for every $g \in \mathcal{B}$ there is an oracle Turing machine which computes some $f \in \mathcal{A}$ when given g as an oracle. This amounts to saying that $\mathcal{A} \leqslant_w \mathcal{B}$ if and only if for every $g \in \mathcal{B}$ there is some $f \in \mathcal{A}$ such that $f \leqslant_T g$. Both definitions may be viewed as attempts at formalizing Kolmogorov's idea of a calculus of problems: Kolmogorov's informal problems are now identified with mass problems; to "solve" a mass problem means to find a computable member in it; $\mathcal{A} \leqslant_M \mathcal{B}$ and $\mathcal{A} \leqslant_w \mathcal{B}$ are then formalizations of "$\mathcal{A}$ is less difficult than $\mathcal{B}$", as one can solve $\mathcal{A}$ given any solution to $\mathcal{B}$. In the same vein, one can introduce a formal "calculus" of mass problems, by defining $\mathcal{A} + \mathcal{B} = \{f \oplus g : f \in \mathcal{A} \text{ and } g \in \mathcal{B}\}$, where

$$f \oplus g(x) = \begin{cases} f(y), & \text{if } x = 2y, \\ g(y), & \text{if } x = 2y+1; \end{cases}$$

and $\mathcal{A} \times \mathcal{B} = \langle 0\rangle^\frown\mathcal{A} \cup \langle 1\rangle^\frown\mathcal{B}$, where in general, for $i \in \omega$ and a given mass problem $\mathcal{C}$, $\langle i\rangle^\frown\mathcal{C} = \{\langle i\rangle^\frown f : f \in \mathcal{C}\}$, and $\langle i\rangle^\frown f$ denotes the concatenation of the string $\langle i\rangle$ with the function f. We see that $\mathcal{A} + \mathcal{B}$ has a solution if and only if both $\mathcal{A}$ and $\mathcal{B}$ have solutions; and $\mathcal{A} \times \mathcal{B}$ has a solution if and only if at least one of them has. Being preordering relations, both $\leqslant_M$ and $\leqslant_w$ give rise to degree structures: the equivalence class $\deg_M(\mathcal{A})$ of a mass problem $\mathcal{A}$, under the equivalence relation $\equiv_M$ generated by $\leqslant_M$, is called the *Medvedev degree* of $\mathcal{A}$; the equivalence class $\deg_w(\mathcal{A})$ of a mass problem $\mathcal{A}$, under the equivalence relation $\equiv_w$ generated by $\leqslant_w$ is called the *Muchnik degree* of $\mathcal{A}$. The corresponding degree structures are not only partial orders, but in fact bounded distributive lattices, with operations of join and meet (still denoted by $+$ and $\times$) defined through the corresponding operations on mass problems. It is easily seen that both lattices are distributive. The lattice of Medvedev degrees is called the *Medvedev lattice*, denoted by $\mathfrak{M}$; the lattice of Muchnik degrees is called the *Muchnik lattice*, denoted by $\mathfrak{M}_w$. Finally the least element 0 in both lattices is the degree of any mass problem containing some computable function; and the greatest element 1 is the degree of the mass problem $\emptyset$.

A *Muchnik mass problem* $\mathcal{A}$ is a mass problem satisfying: If $f \in \mathcal{A}$ and $f \leqslant_T g$ then $g \in \mathcal{A}$.

Lemma 1.1. *The following hold:*

1. *for every mass problem $\mathcal{A}$, there is a unique Muchnik mass problem $C(\mathcal{A})$ such that $\mathcal{A} \equiv_w C(\mathcal{A})$;*
2. *$\mathfrak{M}_w$ is a completely distributive complete lattice, with $\mathcal{A}\times\mathcal{B} \equiv_w \mathcal{A}\cup\mathcal{B}$, and if $\mathcal{A}$ and $\mathcal{B}$ are Muchnik mass problems then $\mathcal{A}+\mathcal{B} \equiv_w \mathcal{A}\cap\mathcal{B}$.*

Proof. Define $C(\mathcal{A}) = \{f : (\exists g \in \mathcal{A})[g \leqslant_T f]\}$. The lattice $\mathfrak{M}_w$ is complete: if $\{\mathcal{A}_i : i \in I\}$ is any collection of mass problems, then the infimum and the supremum of the corresponding Muchnik degrees are given by

$$\prod\{\deg_w(\mathcal{A}_i) : i \in I\} = \deg_w(\bigcup\{\mathcal{A}_i : i \in I\}),$$
$$\sum\{\deg_w(\mathcal{A}_i) : i \in I\} = \deg_w(\bigcap\{C(\mathcal{A}_i) : i \in I\}).$$

We will often extend the $\prod$ and $\sum$ operations to mass problems by defining:

$$\prod\{\mathcal{A}_i : i \in I\} = \bigcup\{\mathcal{A}_i : i \in I\}$$
$$\sum\{\mathcal{A}_i : i \in I\} = \bigcap\{C(\mathcal{A}_i) : i \in I\}.$$

Complete distributivity follows from the fact that infima and suprema are essentially given by set theoretic unions and intersections. $\square$

Both in $\mathfrak{M}$ and in $\mathfrak{M}_w$, a degree $\mathbf{S}$ is called a *degree of solvability* if it contains a singleton. The following considerations concerning degrees of solvability apply to both $\mathfrak{M}$ and $\mathfrak{M}_w$: it is easy to see that the degrees of solvability form an upper semilattice, with least element, which is isomorphic to the upper semilattice, with least element, of the Turing degrees; for every degree of solvability $\mathbf{S}$ there is a unique minimal degree $> \mathbf{S}$ that is denoted by $\mathbf{S}'$ (cf. Medvedev [10]): If $\mathbf{S} = \deg_M(\{f\})$ then $\mathbf{S}'$ is the degree of the mass problem

$$\{f\}' = \{\langle n\rangle^\frown g : g \not\leq_T f \wedge \Phi_n(g) = f\},$$

where $\{\Phi_n\}_{n\in\omega}$ is an effective list of all partial computable functionals; note further that for any f we have $\{f\}' \equiv_w \{g \in \omega^\omega : f <_T g\}$ so that in $\mathfrak{M}_w$ we can use this simplified version of $\{f\}'$. In particular, $0' = \{g : g >_T \emptyset\}$ is the unique minimal nonzero Muchnik degree.

2. Brouwer algebras and intermediate propositional calculi

We now recall the basic definitions and facts about Brouwer and Heyting algebras, and their relation with propositional logics.

Definition 2.1. A distributive lattice L with least and largest elements 0 and 1, and with operations of join and meet denoted by $+$ and $\times$ respectively, is a *Brouwer algebra* if for every pair of elements a and b there is a smallest element, denoted by $a \to b$, such that $a + (a \to b) \geqslant b$. Thus a Brouwer algebra can be viewed as an algebraic structure with three binary operations $+, \times, \to$, together with the nullary operations $0, 1$. For applications to propositional logic, it is also convenient to enrich the signature of a Brouwer algebra with a further unary operation $\neg$, given by $\neg a = a \to 1$.

Given a Brouwer algebra L, we can identify a propositional formula φ, having n variables, with an n-ary polynomial p_φ of L, in the restricted signature $\langle +, \times, \to, \neg\rangle$: the identification makes the propositional connectives $\vee, \wedge, \to, \neg$ correspond to the operations $\times, +, \to, \neg$ of L, respectively. Note that $\vee$ corresponds to $\times$, not $+$, and dually $\wedge$ corresponds to $+$, not $\times$. (For polynomials in the sense of universal algebra, we refer to [3].) The polynomial p_φ is a function $p_\varphi : L^n \longrightarrow L$.

Definition 2.2. Let L be a Brouwer algebra. A propositional formula φ having n variables is *true* in L if $p_\varphi(a_0, \dots, a_{n-1}) = 0$ for all $(a_0, \dots, a_{n-1}) \in L^n$. The set of all propositional formulas that are true in L is denoted by $\mathrm{Th}(L)$.

The propositional formulas lying in $\mathrm{Th}(L)$ are called in [14] the *identities* of L. This is consistent with the way the term "identity" is commonly used in universal algebra: indeed, $\varphi \in \mathrm{Th}(L)$ if and only if $p_\varphi \approx 0$ is an identity of L (with p_φ and 0 regarded as terms of the type of Brouwer algebras: terminology and notations are here as in [2]).

The dual notion is studied as well.

Definition 2.3. A distributive lattice L with least and largest elements 0 and 1 is a *Heyting algebra* if its dual L^{op} is a Brouwer algebra. That is, $a \to b$ is the largest element of L such that $a \times (a \to b) \leqslant b$. A propositional formula is *true* in the Heyting algebra L (or, an *identity* of L) if the polynomial p_φ^{op}, obtained from p_φ by interchanging $\times$ and $+$, evaluates to 1 under every valuation of its variables with elements from L. The set of all formulas that are true in L as a Heyting algebra is denoted by $\mathrm{Th}_H(L)$. Note that $\mathrm{Th}_H(L) = \mathrm{Th}(L^{\mathrm{op}})$.

Lemma 2.4. *Suppose that L_0 and L_1 are Brouwer algebras, and suppose that $F : L_0 \longrightarrow L_1$ is a Brouwer homomorphism (i.e. a homomorphism of bounded lattices, which also preserves $\to$).*

(1) If F is injective then $\mathrm{Th}(L_1) \subseteq \mathrm{Th}(L_0)$*;*
(2) If F is surjective then $\mathrm{Th}(L_0) \subseteq \mathrm{Th}(L_1)$*.*

Proof. See [13]. □

Given $a \leqslant b$ in a Brouwer algebra L, $L[a, b]$ denotes the interval

$$[a, b] = \{x \in L : a \leqslant x \leqslant b\} .$$

We abbreviate $L[0, b]$ by $L(\leqslant b)$, and we abbreviate $L[a, 1]$ by $L(\geqslant a)$.

Lemma 2.5. *Suppose that L is a Brouwer algebra, and let $a, b \in L$ be such that $a < b$. Then $L[a, b]$ is again a Brouwer algebra.*

Proof. Let $\to$ be the arrow operation in L. Then the arrow operation $\to_{[a,b]}$ in $L[a, b]$ is given by

$$x \to_{[a,b]} y = a + (x \to y).$$

□

Lemma 2.6. *Let L be a Brouwer algebra and let $a, b, c \in L$ be such that $a < b$ and $c + a = b$. Then the mapping $f(x) = x + a$ is a Brouwer homomorphism of $L(\leqslant c)$ onto $L[a, b]$. As a consequence,* $\mathrm{Th}(L(\leqslant c)) \subseteq \mathrm{Th}(L[a, b])$.

Proof. See [15, Lemma 4]. □

Lemma 2.7. *Let L be a distributive lattice, and suppose that $x \leqslant y$ and z are arbitrary. Then the mapping $c \mapsto c \times z$ is a surjective lattice-theoretic homomorphism from the interval $[x, y]$ onto the interval $[x \times z, y \times z]$.*

Proof. It is obvious that the mapping is a lattice-theoretic homomorphism. Surjectivity follows from the fact that if $x \times z \leqslant u \leqslant y \times z$ then u is the image of $x + (u \times y)$. □

2.1. **The Medvedev and the Muchnik lattices as Brouwer algebras.** Examples of Brouwer algebras are provided by $\mathfrak{M}$ (Medvedev [10]), $\mathfrak{M}_w$ (Muchnik [12]), and the dual $\mathfrak{M}_w^{\mathrm{op}}$ (Sorbi [16]):

Proposition 2.8. *The Muchnik lattice $\mathfrak{M}_w$ is both a Brouwer algebra and a Heyting algebra. The Medvedev lattice $\mathfrak{M}$ is a Brouwer algebra, but not a Heyting algebra.*

Proof. $\mathfrak{M}_w$ is a Brouwer algebra ([12]), and a Heyting algebra ([16]) since it is a completely distributive complete lattice by Lemma 1.1. To show that $\mathfrak{M}_w$ is a Brouwer algebra, take for instance, on mass problems,

$$\mathcal{A} \to \mathcal{B} = \prod\{\mathcal{C} : \mathcal{B} \leqslant_w \mathcal{A} + \mathcal{C}\}.$$

To show that $\mathfrak{M}$ is a Brouwer algebra ([10]), on mass problems $\mathcal{A}, \mathcal{B}$, define

$$\mathcal{A} \to \mathcal{B} = \{\langle n\rangle^\frown f : (\forall g \in \mathcal{A})[\Phi_n(g \oplus f) \in \mathcal{B}]\} :$$

it is immediate that $\mathcal{B} \leqslant_M \mathcal{A} + (\mathcal{A} \to \mathcal{B})$, and

$$\mathcal{B} \leqslant_M \mathcal{A} + \mathcal{C} \Leftrightarrow \mathcal{A} \to \mathcal{B} \leqslant_M \mathcal{C}.$$

Since Muchnik reducibility is a nonuniform version of Medvedev reducibility, we can also notice that for the $\to$ operation in the Muchnik lattice as a Brouwer algebra, one can take

$$\mathcal{A} \to \mathcal{B} = \{f : (\forall g \in \mathcal{A})(\exists h \in \mathcal{B})[h \leqslant_T g \oplus f]\}.$$

In terms of the calculus of problems, we observe that with these definitions of $\to$, for both Medvedev and Muchnik reducibility one has that $\mathcal{A} \to \mathcal{B}$ is a mass problem such that any solution to it, together with any solution to $\mathcal{A}$, gives a solution to $\mathcal{B}$. Thus this implements the modus ponens proof rule.

That $\mathfrak{M}$ is not a Heyting algebra was proved in Sorbi [16, Theorem 5.4]. □

For either $\mathfrak{M}$ or $\mathfrak{M}_w$, Definition 2.2 amounts to saying that a propositional sentence is valid if and only if every substitution of mass problems to the propositional variables in the sentence yields a solvable problem. Let IPC denote the intuitionistic propositional calculus (see [13] for a suitable definition of axioms and rules of inference), and let Jan be the intermediate propositional logic obtained by adding to IPC the so called *weak law of the excluded middle*, i.e. the axiom scheme $\neg\alpha \vee \neg\neg\alpha$, where α is any propositional sentence. It is known (Medvedev [11], Jankov [5], Sorbi [17]) that $\mathrm{Th}(\mathfrak{M}) = \mathsf{Jan}$. Also, $\mathrm{Th}(\mathfrak{M}_w) = \mathsf{Jan}$ (announced in [17]).

By lattice theory, if L is a Brouwer algebra and $b \in L$, then the Brouwer algebra $L(\leqslant b)$ is lattice isomorphic to the quotient lattice obtained by dividing L modulo the principal filter generated by b; likewise, $L(\geqslant a)$ is isomorphic to the quotient lattice obtained by dividing L modulo the principal ideal generated by a. The difference between these two quotients, see e.g. [13], is that lattice-theoretic congruences given by ideals are also congruences of Brouwer algebras, and thus there is a surjective Brouwer homomorphism from L into $L(\geqslant a)$, giving $\mathrm{Th}(L) \subseteq \mathrm{Th}(L(\geqslant a))$ by Lemma 2.4. In order to find exact interpretations of IPC in terms of mass problems, one should then turn attention to initial segments of the Medvedev lattice, i.e. to Brouwer algebras of the form $\mathfrak{M}(\leqslant \mathbf{A})$, where $\mathbf{A}$ is a nonzero Medvedev degree.

Theorem 2.9. (Skvortsova [15]) *There exists* $\mathbf{A}$ *such that* $\mathrm{Th}(\mathfrak{M}(\leqslant \mathbf{A})) = \mathsf{IPC}$.

It is still an open problem (raised by Skvortsova [15, p.134]) whether there is a Medvedev degree $\mathbf{A}$ that is the infimum of finitely many Muchnik degrees (i.e. Medvedev degrees containing Muchnik mass problems) such that $\mathrm{Th}(\mathfrak{M}(\leqslant \mathbf{A}))$ coincides with IPC. The paper [20] is dedicated to initial segments of the Medvedev lattice and their theories as intermediate propositional logics. Note that it does not make sense to ask whether Theorem 2.9 holds for the dual of $\mathfrak{M}$, since $\mathfrak{M}$ is not a Heyting algebra by [16]. In Section 4 we show that Theorem 2.9 also holds for $\mathfrak{M}_w$, and in Section 5 we show that it holds for the dual of $\mathfrak{M}_w$.

3. Capturing IPC with Brouwer and Heyting algebras

Consider the following classic result about IPC due to McKinsey and Tarski, that provides an algebraic semantics for IPC using Brouwer algebras. (The result also follows from the results in Jaśkowski [6]).

Theorem 3.1. (Jaśkowski [6], McKinsey and Tarski [9])

$$\begin{aligned}\mathsf{IPC} &= \bigcap \big\{\mathrm{Th}(B) : B \ \textit{a finite Brouwer algebra}\big\}\\ &= \bigcap \big\{\mathrm{Th}_H(H) : H \ \textit{a finite Heyting algebra}\big\}.\end{aligned}$$

We wish to narrow down the family of Brouwer algebras and Heyting algebras needed for this result, in order to suit our needs in the next section. The result we will need later is formulated below as Corollary 3.11.

For a given lattice L, let $J(L)$ denote the partial order of nonzero join-irreducible elements of L. Recall the well known duality between finite posets and finite distributive lattices. Obviously, for every finite distributive lattice L, $J(L)$ is a poset, and conversely, for every finite poset P we obtain a finite distributive lattice $H(P)$ by considering the downwards closed subsets of P ([4, Theorem II.1.9]). These operations are inverses of each other, as $H(J(L)) \simeq L$ (as lattices), and $J(H(P)) \simeq P$ (as posets).

The following is a useful notion from the theory of categories. An equational category is a category whose objects form a variety of algebras, and whose morphisms are just the homomorphisms.

Definition 3.2. An object of an equational category L is *weakly projective* if for every onto morphism $f : L_0 \to L_1$ and every morphism $g : L \to L_1$, there exists a morphism $h : L \to L_0$ such that $g = f \circ h$. (If one changes "onto" in this definition to "epi" then one obtains the definition of *projective* object.)

Lemma 3.3. *A distributive lattice* L *is weakly projective if and only if its dual* L^{op} *is.*

Proof. It can be shown, see e.g. [1, Theorem 1.14], that in a nontrivial equational category, an object is weakly projective if and only if it is a retract of a free algebra. (Recall that A is a retract of B, if there are morphisms $f : A \to B, g : B \to A$ such that $g \circ f = 1_A$.) If L is weakly projective, and L is a retract of a free distributive lattice F, then L^{op} is a retract of F^{op} which is still free. □

When considering the category of distributive lattices, the following useful characterization of the finite weakly projective objects is available:

Theorem 3.4. [1, Corollary V.10.9] *A finite distributive lattice L is weakly projective if and only if whenever a and b are join-irreducible in L also $a \times b$ is join-irreducible.*

The following property from [23] gives an alternative characterization of finite weakly projective distributive lattices:

Definition 3.5. A finite distributive lattice L is *double diamond-like* (*dd-like*, for short) if in the poset $J(L)$ there are two incomparable elements with at least two minimal upper bounds.

Proposition 3.6. *A finite distributive lattice L is weakly projective if and only if it is not dd-like.*

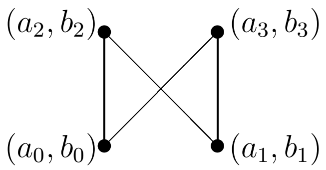
Proof. When L is weakly projective then every pair a, b of join-irreducible elements has a greatest lower bound $a \times b$ that is join-irreducible, and hence $a \times b$ is also the greatest lower bound of a and b in the poset $J(L) \cup \{0\}$. Hence L is not dd-like.

Conversely, if L is not weakly projective then there are $a, b \in J(L)$ such that $a \times b$ is join-reducible. Without loss of generality we can assume that a and b are minimal in the sense that there are no elements of $J(L)$ in between a and $a \times b$, and also no elements of $J(L)$ in between b and $a \times b$. Since any element in a finite distributive lattice can be written as a finite join of join-irreducible elements, there is a finite set $X \subseteq J(L)$ such that $a \times b = \sum X$. Since $a \times b$ itself is join-reducible, there are at least two maximal elements $x, y \in X$. Then both a and b are minimal upper bounds of x and y in $J(L)$, hence L is dd-like. $\square$

We now undertake the task of characterizing IPC by suitably restricted families of Heyting algebras and Brouwer algebras. We can in fact start from a family that was already used by Jaśkowski, by observing that it has certain additional properties. The result we will need later is formulated below as Corollary 3.9.

Lemma 3.7. *If A and B are finite distributive lattices that are not dd-like, then also their Cartesian product $A \times B$ is not dd-like.*

Proof. We need in fact that only one of A and B is not dd-like. Suppose that A is not dd-like. Note that $(a, b) \in A \times B$ is join-irreducible if and only if $a \in J(A)$ and $b \in J(B)$. Suppose that $A \times B$ is not dd-like, say $J(A \times B)$ contains the following configuration:



Here the pairs (a_2, b_2) and (a_3, b_3) are minimal upper bounds for (a_0, b_0) and (a_1, b_1) in $J(A \times B)$. Then in $J(A)$ the elements a_2 and a_3 are upper bounds for a_0 and a_1. Since by assumption A is not dd-like, not both of a_2 and a_3 are minimal upper bounds. Say a_2 is not minimal, and that $a_0, a_1 \leqslant a < a_2$

in $J(A)$. Replacing (a_2, b_2) by (a, b_2), we see that (a_2, b_2) was not a minimal upper bound of (a_0, b_0) and (a_1, b_1), contrary to assumption. □

We use the following result of Jaśkowski [6], (cited in Szatkowski [21, p41]). Given two Heyting algebras A and B, let $A + B$ be the algebra obtained by stacking B on top of A, identifying 0_B with 1_A. (This notion of sum is from Troelstra [24].) Given A and B, the Cartesian product $A \times B$ is again a Heyting algebra. Let A^n denote the n-fold product of A.

Inductively define the following sequence of Heyting algebras. Let I_1 be the two-element Boolean algebra, and let

$$I_{n+1} = I_n^n + I_1.$$

The following theorem characterizes IPC in terms of Heyting algebras:

Theorem 3.8. (Jaśkowski [6]) $\mathsf{IPC} = \bigcap_n \mathrm{Th}_H(I_n)$.

Corollary 3.9. *There is a collection $\{H_n\}_{n\in\omega}$ of finite Heyting algebras such that*

$$\mathsf{IPC} = \bigcap_n \mathrm{Th}_H(H_n),$$

and such that for every n, H_n is weakly projective.

Proof. Note that the lattices I_n defined above are all distributive lattices, and because they are finite they are automatically Heyting algebras. We claim that every I_n is not dd-like. This is clearly true for $n = 1$. Suppose that I_n is not dd-like. Then by Lemma 3.7 also I_n^n is not dd-like. It follows immediately that $I_{n+1} = I_n^n + I_1$ is also not dd-like. Hence all I_n are finite Heyting algebras that are not dd-like, and hence we can simply take $H_n = I_n$. □

Corollary 3.10. *There is a collection $\{B_n\}_{n\in\omega}$ of finite Brouwer algebras such that*

$$\mathsf{IPC} = \bigcap_n \mathrm{Th}(B_n),$$

and such that for every n, B_n is weakly projective.

Proof. Consider any propositional formula $\varphi \notin \mathsf{IPC}$. Then by Corollary 3.9 there exists a weakly projective finite distributive lattice H_n and an evaluation of p_φ^{op} for which $p_\varphi^{\mathrm{op}} \neq 1$, and thus, for this evaluation in H_n^{op}, $p_\varphi \neq 0$, showing that $\varphi \notin \mathrm{Th}(H_n^{\mathrm{op}})$. It remains to show that $B_n = H_n^{\mathrm{op}}$ is weakly projective: this follows from Lemma 3.3. □

An easy way to obtain Corollary 3.10 would be to show that every finite distributive lattice is the image of a weakly projective finite distributive lattice under a Brouwer-homomorphism. (Corollary 3.10 would then follow immediately from Theorem 3.1 and Lemma 2.4 (2).) However, this is not true: One can prove that every finite distributive lattice is the image of a weakly projective finite distributive lattice under a *lattice*-homomorphism, but in general not under a Brouwer-homomorphism.

Summarizing, we have:

Corollary 3.11. *We have*

$$\begin{aligned}\mathsf{IPC} &= \bigcap \big\{\mathrm{Th}(B) : B \textit{ a finite weakly projective Brouwer algebra}\big\}\\ &= \bigcap \big\{\mathrm{Th}_H(H) : H \textit{ a finite weakly projective Heyting algebra}\big\}.\end{aligned}$$

4. A factor of the Muchnik lattice that captures IPC

In this section we prove that there is a factor of $\mathfrak{M}_w$, obtained by dividing $\mathfrak{M}_w$ by a principal filter, that has IPC as its theory. Hence we see that the analogue of Skvortsova's result (Theorem 2.9) holds for $\mathfrak{M}_w$. We will be very liberal with notation, frequently confusing Muchnik degrees with their representatives.

The property of dd-like lattices (Definition 3.5) was used to characterize the lattices that are isomorphic to an interval of $\mathfrak{M}_w$:

Theorem 4.1. (Terwijn [23]) *For any finite distributive lattice L the following are equivalent:*

(i) *L is isomorphic to an interval in $\mathfrak{M}_w$;*
(ii) *L is not double diamond-like;*
(iii) *L does not have a double diamond-like lattice as a subinterval.*

Let $\{B_n\}_{n\in\omega}$ be the family of Brouwer algebras from Corollary 3.10. Since B_n is not dd-like by Proposition 3.6, by Theorem 4.1 there are sets $\mathcal{X}_n$ and $\mathcal{Y}_n$ such that the interval $[\mathcal{X}_n, \mathcal{Y}_n]$ in $\mathfrak{M}_w$ is isomorphic to B_n for every n. This is an isomorphism of finite distributive lattices, hence it is automatically an isomorphism of Brouwer algebras.

It is useful to remind the reader of some of the details of the construction in [23]. Let $J_n = J(B_n)$ be the set of the nonzero join-irreducible elements of B_n; since B_n is not dd-like, J_n is an initial segment of an upper semilattice. Embed J_n as an interval of the Turing degrees (this can be done, by a classical result of Lachlan and Lebeuf [8], stating that for every Turing degree $\mathbf{a}$, every countable upper semilattice with least element is isomorphic to an interval of the Turing degrees with bottom $\mathbf{a}$). For every Turing degree in the range of this embedding, choose a representative, as a function $f \in \omega^\omega$, and for convenience, let us identify J_n with the set of these chosen representatives. For every $A \subseteq J_n$, let $\hat{A}$ denote the elements of A that are $\leqslant_T$-maximal, i.e. maximal with respect to Turing reducibility.

Inspection of the proof of Theorem 3.11 in [23] shows that there is a set $\mathcal{Z}_n$ such that

$$\begin{aligned} \mathcal{X}_n &= \mathcal{Z}_n \times J_n, \\ \mathcal{Y}_n &= \mathcal{Z}_n \times \prod\bigl\{\{f\}' : f \in \hat{J}_n\bigr\}. \end{aligned} \tag{1}$$

Furthermore, we have that $\mathcal{Z}_n = \bigcup_{f\in J_n} \mathcal{Z}_n^f$, where

$$\mathcal{Z}_n^f = \bigl\{g \in \{f\}' : g|_T h \text{ for all covers } h \text{ of } f \text{ in } J_n\bigr\}. \tag{2}$$

The sets J_n come from embedding results into the Turing degrees, and we have rather great freedom in picking them. In particular, we may pick them such that they satisfy for every $n \neq m$,

$$(\forall f \in J_n)\ [\{f\} \not\geqslant_w \mathcal{Z}_m], \tag{3}$$

and

$$\left.\begin{aligned} f &\in \hat{J}_m \\ g &\in \hat{J}_n \\ h &\in J_n \end{aligned}\right\} \Longrightarrow f \oplus h >_T g. \tag{4}$$

To obtain this, it is enough to embed as an interval of the Turing degrees the upper semilattice J defined as follows: First, let

$$U = \bigcup_n \{n\} \times J_n$$

(where, again, $J_n = J(B_n)$) and in U define $(n, x) \leqslant (m, y)$ if and only if $n = m$ and, in J_n, $x \leqslant y$; finally define J by adding a least element and a greatest element to U. Clearly J is a countable upper semilattice with least element, and thus can be embedded as an interval of the Turing degrees: under this embedding each J_n is embedded as an interval of the Turing degrees, with the desired properties.

Define

$$\begin{aligned} \mathcal{Z} &= \bigcup_{n\in\omega} \mathcal{Z}_n, \\ \hat{\mathcal{X}}_n &= \mathcal{Z} \times J_n \equiv_w \mathcal{X}_n \times \mathcal{Z}, \\ \hat{\mathcal{Y}}_n &= \mathcal{Z} \times \prod\big\{\{f\}' : f \in \hat{J}_n\big\} \equiv_w \mathcal{Y}_n \times \mathcal{Z}. \end{aligned}$$

Lemma 4.2. *The interval $[\mathcal{X}_n, \mathcal{Y}_n]$ is isomorphic to the interval $[\hat{\mathcal{X}}_n, \hat{\mathcal{Y}}_n]$.*

Proof. Define a mapping from $[\mathcal{X}_n, \mathcal{Y}_n]$ to $[\mathcal{X}_n \times \mathcal{Z}, \mathcal{Y}_n \times \mathcal{Z}]$ by $\mathcal{C} \mapsto \mathcal{C} \times \mathcal{Z}$. By Lemma 2.7, the mapping is a surjective lattice-theoretic homomorphism. We check that it is also injective: Suppose that $\mathcal{C}_0$, $\mathcal{C}_1 \in [\mathcal{X}_n, \mathcal{Y}_n]$ and that $\mathcal{C}_0 \times \mathcal{Z} \equiv_w \mathcal{C}_1 \times \mathcal{Z}$. We claim that $\mathcal{C}_0 \geqslant_w \mathcal{C}_1 \times \mathcal{Z}_n$: Suppose that $g \in \mathcal{C}_0$. Then $\{g\} \geqslant_w \mathcal{X}_n = \mathcal{Z}_n \times J_n$. If $\{g\} \geqslant_w \mathcal{Z}_n$ then clearly it can be mapped to $\mathcal{C}_1 \times \mathcal{Z}_n$. If $\{g\} \not\geqslant_w \mathcal{Z}_n$ then we have $\{g\} \geqslant_w J_n$, and it follows from (2) and the fact that the Turing degrees of functions in J_n form an initial segment of the Turing degrees that $g \equiv_T k$ for some $k \in J_n$. (To see this, suppose that $\{g\} \not\geqslant_w \mathcal{Z}_n$ and $\{g\} \geqslant_w J_n$, and let $f \in J_n$ be $\leqslant_T$-maximal such that $f \leqslant_T g$. If $f \equiv_T g$ then the claim is true, so suppose that $f <_T g$. Then $g \in \{f\}'$. There is a cover $h \in J_n$ of f such that $g \not|_T h$, since otherwise $g \in \mathcal{Z}_n^f$ and hence $g \in \mathcal{Z}_n$, contrary to assumption. If $g \leqslant_T h$ then $g \equiv_T k$, for some $k \in J_n$, since the Turing degrees of the elements of J_n are an initial segment. If $h \leqslant_T g$ then $g \equiv_T h$, since we chose $f \in J_n$ maximal. Hence g has the same Turing degree of some function in J_n.) But in this case it follows from (3) and the assumption $\mathcal{C}_0 \geqslant_w \mathcal{C}_1 \times \mathcal{Z}$ that $\{g\} \geqslant_w \mathcal{C}_1 \times \mathcal{Z}_n$. Hence $\mathcal{C}_0 \geqslant_w \mathcal{C}_1 \times \mathcal{Z}_n \equiv_w \mathcal{C}_1$ (note that $\mathcal{Z}_n \geqslant_w \mathcal{C}_1$ since $\mathcal{Y}_n \geqslant_w \mathcal{C}_1$), and symmetrically we have that $\mathcal{C}_1 \geqslant_w \mathcal{C}_0$, hence $\mathcal{C}_0 \equiv_w \mathcal{C}_1$. □

Now let

$$\hat{\mathcal{Y}} = \bigcup_{n\in\omega} \hat{\mathcal{Y}}_n.$$

Lemma 4.3. *$\hat{\mathcal{Y}} + \hat{\mathcal{X}}_n \equiv_w \hat{\mathcal{Y}}_n$ for every n.*

Proof. The direction $\leqslant_w$ is immediate from $\hat{\mathcal{Y}} \leqslant_w \hat{\mathcal{Y}}_n$ and $\hat{\mathcal{X}}_n \leqslant_w \hat{\mathcal{Y}}_n$. For the other direction, suppose that $g \in \hat{\mathcal{Y}}$ and $h \in \hat{\mathcal{X}}_n$. We have to show that $g \oplus h$ computes some function in $\hat{\mathcal{Y}}_n$. Suppose that $g \in \hat{\mathcal{Y}}_m$. If $n = m$ then we are done. If either g or h is in $\langle 0\rangle^\frown\mathcal{Z}$ then we are also done because $\langle 0\rangle^\frown\mathcal{Z} \subseteq \hat{\mathcal{Y}}_n$.

In the remaining case we have $n \neq m$, $h \in \langle 1\rangle^\frown J_n$, and $g \in \{f\}'$ for some $f \in \hat{J}_m$. Let l be any element of $\hat{J}_n$. Then by (4) we have $f \oplus h >_T l$, hence $g \oplus h \geqslant_T f \oplus h \in \{l\}' \geqslant_w \hat{\mathcal{Y}}_n$. □

Theorem 4.4. *There exists a mass problem* $\hat{\mathcal{Y}}$ *such that*

$$\mathrm{Th}(\mathfrak{M}_w(\leqslant_w \hat{\mathcal{Y}})) = \mathsf{IPC}.$$

Proof. Let $\hat{\mathcal{X}}_n$, $\hat{\mathcal{Y}}_n$, and $\hat{\mathcal{Y}}$ be as above. Since by Lemma 4.3 we have $\hat{\mathcal{Y}} + \hat{\mathcal{X}}_n \equiv_w \hat{\mathcal{Y}}_n$ for every n, by Lemma 2.6 we have that

$$\mathrm{Th}\big(\mathfrak{M}_w(\leqslant_w \hat{\mathcal{Y}})\big) \subseteq \bigcap_n \mathrm{Th}\big([\hat{\mathcal{X}}_n, \hat{\mathcal{Y}}_n]\big) = \bigcap_n \mathrm{Th}(B_n) = \mathsf{IPC}.$$

The equality $\mathrm{Th}\big(\mathfrak{M}_w(\leqslant_w \hat{\mathcal{Y}})\big) = \mathsf{IPC}$ follows since $\mathsf{IPC} \subseteq \mathrm{Th}\big(\mathfrak{M}_w(\leqslant_w \hat{\mathcal{Y}})\big)$ holds for any $\hat{\mathcal{Y}}$. $\square$

5. $\mathfrak{M}_w$ as a Heyting algebra

For the dual of $\mathfrak{M}_w$ we have a similar result, but easier to prove and in fact stronger: the result, and its consequences, listed below, were already noticed in Sorbi [18], with only sketched proof.

Let $\{H_n\}_{n\in\omega}$ be the family of Heyting algebras from Corollary 3.9. The following lemma is a reformulation of a result in [23], using Proposition 3.6. The right-to left implication appeared also in [18].

Lemma 5.1. *A finite distributive lattice is isomorphic to an initial segment of the Muchnik lattice if and only if it is weakly projective, and* 0 *is meet-irreducible.*

Theorem 5.2. $\mathsf{IPC} = \mathrm{Th}_H(\mathfrak{M}_w(\geqslant 0'))$.

Proof. For every weakly projective finite distributive lattice H, define $H^+ = H + I_1$ (using the notation of Section 3.) Notice that H is isomorphic to a factor of H^+, obtained by dividing by the principal filter generated by 1_H, that is the image of the top element of H into H^+. Since filters provide congruences of Heyting algebras, we have by Lemma 2.4 (or rather, its dual version for Heyting algebras) that

$$\mathrm{Th}_H(H^+) \subseteq \mathrm{Th}_H(H).$$

It follows that

$$\mathsf{IPC} = \bigcap \big\{\mathrm{Th}_H(H) : H \text{ finite, weakly projective, with join-irreducible } 1\big\}.$$

Suppose now that H is a finite, weakly projective distributive lattice, with join-irreducible 1: let H^- be such that $H = (H^-)^+$. Embed $I_1 + H^-$ as an initial segment of $\mathfrak{M}_w$, which is possible by Lemma 5.1. Let F be the embedding, which is also a Heyting algebra embedding, since the range of F is an initial segment. Then the mapping

$$G(x) = \begin{cases} F(x), & \text{if } x \in H^-; \\ \mathbf{1}_{\mathfrak{M}_w}, & \text{if } x = 1_H \end{cases}$$

is a Heyting embedding of H into $\mathfrak{M}_w(\geqslant_w 0'))$. Thus $\mathsf{IPC} = \mathrm{Th}_H(\mathfrak{M}_w(\geqslant_w 0'))$ by Lemma 2.4. $\square$

A proof of the following result was already outlined in Sorbi [18].

Theorem 5.3. $\mathrm{Th}_H(\mathfrak{M}_w) = \mathrm{Th}(\mathfrak{M}_w) = \mathsf{Jan}$.

Proof. Let us show that $\mathrm{Th}_H(\mathfrak{M}_w) = \mathsf{Jan}$. For every Heyting algebra H let $H_+ = I_1 + H$. Let us say that a propositional formula is *positive* if it does not contain the connective $\neg$, and, for every Heyting algebra H, let $\mathrm{Th}_H^{\mathrm{pos}}(H) = \{\varphi \in \mathrm{Th}_H(H) : \varphi \text{ positive}\}$. We claim that $\mathrm{Th}_H^{\mathrm{pos}}(H_+) \subseteq \mathrm{Th}_H^{\mathrm{pos}}(H)$. Namely, one can show by induction on the complexity of a positive φ that for every $\overline{x} \in H^n$, $p_\varphi^H(\overline{x}) = p_\varphi^{H_+}(\overline{x})$. (Here, φ has, say, n variables, and p_φ^H and $p_\varphi^{H_+}$ denote the mappings $\varphi^{\mathrm{op}} : H^n \to H$ and $\varphi^{\mathrm{op}} : (H_+)^n \to H_+$, respectively, as in Definition 2.3.) Notice also that for every Heyting algebra H and any propositional formula α, we have that $\neg\alpha \vee \neg\neg\alpha \in \mathrm{Th}_H(H_+)$ (since the least element of H_+ is meet-irreducible), i.e. $\mathsf{Jan} \subseteq \mathrm{Th}_H(H_+)$. Let $H = \mathfrak{M}_w(\geqslant 0'))$, so that $H_+ = \mathfrak{M}_w$. By Theorem 5.2 we have $\mathsf{IPC} = \mathrm{Th}_H(H)$, hence $\mathsf{IPC}^{\mathrm{pos}} = \mathrm{Th}_H^{\mathrm{pos}}(H_+)$, and $\neg\alpha \vee \neg\neg\alpha \in \mathrm{Th}_H(H_+)$. Therefore one can apply a classic result due to Jankov [5], stating that Jan is the $\subseteq$-largest intermediate propositional logic I such that $\mathsf{IPC}^{\mathrm{pos}} = I^{\mathrm{pos}}$ and $\neg\alpha \vee \neg\neg\alpha \in I$. Thus we also obtain the converse inclusion $\mathrm{Th}_H(H_+) \subseteq \mathsf{Jan}$.

The proof that $\mathrm{Th}(\mathfrak{M}_w) = \mathsf{Jan}$ goes like this: let $B = \mathfrak{M}_w(\leqslant_w \hat{\mathcal{Y}})$, with $\hat{\mathcal{Y}}$ as in Theorem 4.4. Dualizing the arguments which have been used above, we obtain $\mathrm{Th}^{\mathrm{pos}}(B^+) \subseteq \mathrm{Th}^{\mathrm{pos}}(B)$, but then again by Jankov [5], $\mathrm{Th}(B^+) = \mathsf{Jan}$, and since B^+ is Brouwer embeddable into $\mathfrak{M}_w$ (use $G : B^+ \longrightarrow \mathfrak{M}_w$ which extends the embedding of B into $\mathfrak{M}_w$, by $G(1_{B^+}) = \mathbf{1}_{\mathfrak{M}_w}$) we finally get that $\mathrm{Th}(\mathfrak{M}_w) \subseteq \mathsf{Jan}$ (by Lemma 2.4), and thus $\mathrm{Th}(\mathfrak{M}_w) = \mathsf{Jan}$ since $\neg\alpha \vee \neg\neg\alpha \in \mathrm{Th}(\mathfrak{M}_w)$. $\square$

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