Almost Complete Sets

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Abstract

We show that there is a set that is almost complete but not complete under polynomial-time many-one (p-m) reductions for the class $\mathbf{E}$ of sets computable in deterministic time 2^{lin}. Here a set A in a complexity class $\boldsymbol{C}$ is almost complete for $\boldsymbol{C}$ under some reducibility r if the class of the problems in $\boldsymbol{C}$ that do not r-reduce to A has measure 0 in $\boldsymbol{C}$ in the sense of Lutz's resource-bounded measure theory. We also show that the almost complete sets for $\mathbf{E}$ under polynomial-time bounded one-one length-increasing reductions and truth-table reductions of norm 1 coincide with the almost p-m-complete sets for $\mathbf{E}$. Moreover, we obtain similar results for the class $\mathbf{EXP}$ of sets computable in deterministic time 2^{poly}.

Key words: resource-bounded measure, completeness, weak completeness, almost completeness, resource-bounded reducibilities, one-one length-increasing reductions, 1-tt -reductions
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1 Introduction

Lutz [16] introduced measure concepts for the standard deterministic time and space complexity classes that contain the class $\mathbf{E}$ of sets computable in deterministic time 2^{lin}. These measure concepts have been used for investigating

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quantitative aspects of the internal structure of the corresponding complexity classes. Most of this work focussed on the measure for **E**. The majority of the results obtained there, however, carry over to the larger complexity classes. For recent surveys of the work on resource-bounded measure, see Lutz [18] and Ambos-Spies and Mayordomo [4].

Lutz's measure theory on **E** not only has revealed relationships to older concepts such as complete and bi-immune sets, but also has led to important new concepts. The most investigated concept of this sort appears to be weak completeness, as introduced by Lutz [17]. While all sets in **E** can be reduced to a complete set (under some given polynomial time reducibility notion), for a weakly complete set, Lutz requires only that the class of the reducible sets does not have measure 0 in **E**, i.e., is a non-negligible part of **E**.

Originally, Lutz introduced weak completeness for polynomial time many-one (p-m) reducibility – the reducibility that is used in most completeness proofs in the literature – and he showed that there actually is a weakly p-m-complete set for **E** that is not p-m-complete for **E** (Lutz [17]). In fact, the class of weakly p-m-complete sets for **E** has measure 1 in **E** (Ambos-Spies, Terwijn, Zheng [7], Juedes [12]). Since it was long known that the class of p-m-complete sets for **E** has measure 0 in **E** (Mayordomo [19]), we see that weak completeness leads to a new large class of provably intractable problems.

A natural concept between completeness and weak completeness is almost completeness. Here a set A in **E** is *almost p-m-complete for* **E** if the class of problems that are p-m-reducible to A has measure 1 in **E**, i.e., if the sets in **E** that are not reducible to A can be neglected with respect to measure. Zheng and others (see e.g. [4], Section 7) raised the question of whether there are almost p-m-complete sets for **E** that are not p-m-complete for **E**. Here we answer this question affirmatively by constructing such a set.

Our result is contrasted by a result of Regan, Sivakumar and Cai [20], which implies that for the standard transitive polynomial-time reducibilities allowing more than one oracle query – such as bounded truth-table (btt), truth-table (tt), and Turing (T) reducibility – completeness and almost completeness coincide. It follows that any almost p-m-complete set for **E** is p-btt-complete for **E**, whence – in contrast to the weakly p-m-complete sets – the class of almost p-m-complete sets for **E** has measure 0 in **E**.

The above results still leave the investigation of almost completeness for the one-query reducibilities different from many-one reducibility. Here we show that the almost completeness notions coincide for the reducibilities ranging from one-to-one, length-increasing reductions to truth-table reductions of norm 1. This parallels previous results for completeness (see Berman [9] and Homer, Kurtz and Royer [11]) and weak completeness (see Ambos-Spies, Mayordomo and Zheng [5]).

The outline of the paper is as follows. In Section 2 we describe the part of Lutz's measure theory for **E** needed in the paper and we review the limiting result on almost completeness by Regan, Sivakumar, and Cai. Section 3 contains the proof of our main result, while in Section 4 the relations among the various completeness notions are discussed. In Section 5 we summarize some further results.

Our notation is fairly standard, for unexplained notation we refer to [4]. Capital letters like A, B, R, X denote sets of binary strings. Lower case letters from the end of the alphabet like x, y, z denote binary strings, whereas the other letters denote natural numbers with the exception of p and q which denote polynomials and d, f, g, h and s, t which denote general functions. Sometimes we identify strings with natural numbers by letting n be the $(n+1)$st binary string under the canonical length-lexicographical ordering.

Sets are identified with their characteristic sequence, i.e., for every natural number n, we have $A(n) = 1$ if $n \in A$ and $A(n) = 0$ if $n \notin A$. For a set A and a string x let $A{\restriction}x$ denote the restriction of A to all strings less than x, i.e., $w = A{\restriction}x$ is an initial segment of A such that the length of w is just the natural number that corresponds to x. Moreover, for $n = |x|$ we have $2^n - 1 \leq |w| < 2^{n+1} - 1$. In the context of computations that receive as input an initial segment of a set we write w or v for this prefix, whereas if the input of a computation is meant as a possible member of a set, we use x, y, or z for the input.

The polynomial time reductions considered here are general reductions of Turing type (p-T), truth-table reductions (p-tt) allowing only non-adaptive queries, bounded truth-table reductions (p-btt) in which in addition the number of queries is bounded by a constant, and the special case hereof where this constant c is fixed (btt(c)). We will represent p-btt(1)-reductions by a pair of polynomial time computable functions g and h where $g(x)$ is the string queried on input x and the unary Boolean function $h(x)$ tells how the answer of the oracle is evaluated. If the reduction is positive, i.e., $h(x)(i) = i$ for all strings x and all i in $\{0, 1\}$, we have a p-many-one-reduction (p-m) and in this case we omit h. If in addition g is one-to-one (and length-increasing) we obtain a (length-increasing) one-one reduction (p-1 and p-1-li). For r in $\{\mathrm{T}, \mathrm{tt}, \mathrm{btt}, \mathrm{btt}(1), \mathrm{m}, 1, \text{1-li}\}$ and any set A, we let the *lower* p-r-*span* of A be the class $\{B : B \leq_r^{\mathrm{p}} A\}$.

2 Measure on E and almost completeness

In this section we describe the fragment of Lutz's measure theory for the class **E** of sets computable in deterministic time 2^{lin} that we will need in the following. For a more comprehensive presentation of this theory we refer the reader to the recent surveys by Lutz [18] and by Ambos-Spies and Mayordomo [4]. Our presentation follows [4]. The $t(n)$-measure defined there slightly differs from the original definition by Lutz, but both definitions lead to the same notions of p-measure and measure on **E**.

The measure on **E** is obtained by imposing appropriate resource-bounds on a game theoretical characterization of the classical Lebesgue measure.

Definition 1 *A* betting strategy s *is a function* $s : \{0,1\}^* \to [0,1]$. *The* (normed) martingale $d_s : \{0,1\}^* \to [0,\infty)$ induced by a betting strategy s *is inductively defined by* $d_s(\lambda) = 1$ *and*

$$d_s(wi) = 2 \cdot |i - s(w)| \cdot d_s(w)$$

for $w \in \{0,1\}^*$ *and* $i \in \{0,1\}$. *A* martingale *is a martingale induced by some strategy. A martingale* d succeeds on a set A *if*

$$\limsup_{n\to\infty} d(A\upharpoonright n) = \infty,$$

and d *succeeds on a class* $\boldsymbol{C}$ *if* d *succeeds on every member* A *of* $\boldsymbol{C}$.

This definition can be motivated by the following fair betting game in which a gambler puts bets on the successive bits of a hidden sequence $A \in \{0,1\}^\infty$. The game proceeds in infinitely many rounds where at the end of round n the n-th bit of the sequence A is revealed to the gambler. The gambler starts with (normed) capital $d(\lambda) = 1$. Then, in round n, depending on the first n outcomes $w = A \upharpoonright n$, he bets a certain fraction $\alpha_w \cdot d(w)$ ($\alpha_w \in [0,1]$) of his current capital $d(w)$ on the event $A(n) = 0$ and he bets the remaining capital $(1 - \alpha_w) \cdot d(w)$ on the complementary event $A(n) = 1$. The amount put on the correct outcome is doubled, the amount put on the wrong guess is lost. Then, if the gambler uses the strategy s to determine the ratio $\alpha_w = s(w)$ for his bets, the martingale $d_s(w) = d(w)$ induced by s will describe the capital of the gambler in the course of this game. If this capital is unbounded, the gambler wins.

It can be shown that a class $\boldsymbol{C}$ has Lebesgue measure 0, $\mu(\boldsymbol{C}) = 0$, iff some martingale succeeds on $\boldsymbol{C}$. By imposing resource bounds, martingales can be used for defining resource-bounded measure concepts.

Definition 2 *Let* $t : \mathbb{N} \to \mathbb{N}$ *be a recursive function. A* $t(n)$-martingale d *is a* *martingale induced by a rational valued betting strategy s such that $s(w)$ can be computed in $\mathcal{O}(t(|w|))$ steps for all strings w.*

*A class $\boldsymbol{C}$ has $t(n)$-*measure *0, $\mu_{t(n)}(\boldsymbol{C}) = 0$, if some $t(n)$-martingale succeeds on $\boldsymbol{C}$, and $\boldsymbol{C}$ has $t(n)$-*measure *1, $\mu_{t(n)}(\boldsymbol{C}) = 1$, if the complement $\overline{\boldsymbol{C}}$ has $t(n)$-measure 0.*

Note that for $i \in \{0,1\}$ and for recursive bounds $t(n)$ and $t'(n)$ such that $t(n) \leq t'(n)$ almost everywhere,

$$\mu_{t(n)}(\boldsymbol{C}) = i \quad \Rightarrow \quad \mu_{t'(n)}(\boldsymbol{C}) = i \quad \Rightarrow \quad \mu(\boldsymbol{C}) = i.$$

In order to obtain measures for complexity classes, resource-bounded measure concepts are defined not for individual bounds but for families of bounds. In particular, working with polynomial bounds yields a measure on **E**.

Definition 3 *A* p-martingale *d is a $q(n)$-martingale for some polynomial q. A class $\boldsymbol{C}$ has* p-measure 0, *$\mu_{\mathrm{p}}(\boldsymbol{C}) = 0$, if $\mu_{q(n)}(\boldsymbol{C}) = 0$ for some polynomial $q(n)$, i.e., if some p-martingale succeeds on $\boldsymbol{C}$, and $\mu_{\mathrm{p}}(\boldsymbol{C}) = 1$ if $\mu_{\mathrm{p}}(\overline{\boldsymbol{C}}) = 0$.*

A class $\boldsymbol{C}$ has measure 0 in **E**, *$\mu(\boldsymbol{C}|\mathbf{E}) = 0$, if $\mu_{\mathrm{p}}(\boldsymbol{C} \cap \mathbf{E}) = 0$ and $\boldsymbol{C}$ has* measure 1 in **E**, *$\mu(\boldsymbol{C}|\mathbf{E}) = 1$, if $\mu(\overline{\boldsymbol{C}}|\mathbf{E}) = 0$.*

Lutz [16] has shown that this measure concept for **E** is consistent. In particular, **E** itself does not have measure 0 in **E**, namely

$$\mu_{\mathrm{p}}(\mathbf{E}) \neq 0, \text{ hence } \mu(\mathbf{E}|\mathbf{E}) \neq 0. \tag{1}$$

On the other hand, every slice of **E** has measure 0 in **E**, namely for $k \geq 1$

$$\mu_{\mathrm{p}}(\mathbf{DTIME}(2^{kn})) = 0, \text{ hence } \mu(\mathbf{DTIME}(2^{kn})|\mathbf{E}) = 0. \tag{2}$$

Based on the above measure for **E** we can now introduce the completeness notions for **E** that are central for our paper. Here $\leq_r^{\mathrm{p}}$ denotes any polynomial-time reducibility.

Definition 4 *a) (Lutz [17]) A set A is* weakly p-r-hard for **E** *if the lower* p-r-*span of A does not have measure* 0 *in* **E**, *i.e., if $\mu_{\mathrm{p}}(\{B : B \leq_r^{\mathrm{p}} A\} \cap \mathbf{E}) \neq 0$. If, in addition, A is in* **E** *then A is* weakly p-r-complete.

b) (Zheng) A set A is almost p-r-hard for **E** *if the lower* p-r-*span of A has measure* 1 *in* **E**, *i.e., if $\mu_{\mathrm{p}}(\{B : B \leq_r^{\mathrm{p}} A\} \cap \mathbf{E}) = 1$. If, in addition, A is in* **E** *then A is* almost p-r-complete.

Intuitively, a set A in **E** is weakly p-r-complete for **E** if its lower span contains a non-negligible part of **E** and it is almost p-r-complete for **E** if the part of **E** that is not contained in the lower span of A can be neglected. In particular, every p-r-complete set for $\mathbf{E}$ is almost p-r-complete for $\mathbf{E}$ and – by (1) and by additivity of μ_{p} – every almost p-r-complete set for $\mathbf{E}$ is weakly p-r-complete for $\mathbf{E}$. Moreover, since $\mathbf{P}$ has measure 0 in $\mathbf{E}$ by (2), every weakly p-r-complete set is provably intractable.

After Lutz [17] demonstrated the existence of weakly p-m-complete sets for $\mathbf{E}$ that are not p-m-complete for $\mathbf{E}$, weak completeness was extensively studied and most relations among the different weak completeness and completeness notions have been clarified (see Section 4 below).

A severe limitation on the existence of nontrivial almost complete sets is imposed by the following observation on classes that have measure 1 in $\mathbf{E}$.

Theorem 5 (Regan, Sivakumar and Cai [20]) *Let $\boldsymbol{C}$ be a class such that $\mu(\boldsymbol{C}|\mathbf{E}) = 1$ and $\boldsymbol{C}$ is either closed under symmetric difference or closed under union and intersection. Then $\boldsymbol{C}$ contains all of $\mathbf{E}$.*

Since for r in {btt,tt,T} the lower p-r-span of any set A is closed under union and intersection (as well as under symmetric difference) this shows that the concept of almost completeness is trivial for these reducibilities.

Corollary 6 *For r in* {btt,tt,T}*, every almost* p-r*-complete (almost* p-r*-hard) set for* $\mathbf{E}$ *is* p-r*-complete (*p-r*-hard) for* $\mathbf{E}$*.*

Concerning p-m-reducibility, it is immediate from Theorem 5 that in case the lower p-m-span of a set is closed under union and intersection, then the set is almost p-m-complete for $\mathbf{E}$ if and only if it is p-m-complete for $\mathbf{E}$. Applying the latter observation to a set that is p-m-complete for the class $\mathbf{NP}$, we obtain – similarly to an argument in [20] – that if $\mathbf{NP}$ has measure 1 in $\mathbf{E}$ then $\mathbf{NP} \cap \mathbf{E}$ coincides with $\mathbf{E}$ (and hence, by being closed downwards under p-m-reductions, $\mathbf{NP}$ coincides with $\mathbf{EXP}$.) For other complexity classes that are closed under union and intersection, we can argue similarly. Moreover, the above observation for example can be used to show that GI, the set of all (appropriately encoded) pairs of isomorphic graphs, is almost p-m-complete for $\mathbf{E}$ if and only if it is p-m-complete for $\mathbf{E}$. Here the closure of the lower p-m-span of GI under union and intersection follows from the fact that GI has polynomial-time computable or- and and-functions, i.e., from a list of potential members of GI we can compute in polynomial time a single pair of graphs such that this pair is in GI if and only if some pair (respectively, all pairs) in the list are in GI (for details of this construction see the textbook by Köbler et al. [15, Section 1.5]).

In general, however, the lower p-m-span of a set is neither closed under symmetric difference nor under union and intersection, hence the argument used for proving Corollary 6 does not work for almost p-m-completeness. As an immediate consequence of Corollary 6, however, almost p-m-complete sets for

E must be p-btt-complete for **E**. Since the class of p-btt-complete sets has p-measure 0 (see [6]), this also shows that almost p-m-complete sets are scarce.

Corollary 7 *Every almost* p-m-*complete (hard) set for* **E** *is* p-btt-*complete (hard) for* **E**. *In particular, the class of the almost* p-m-*complete sets for* **E** *has p-measure 0, hence measure 0 in* **E**.

In fact, as observed in [4], every almost p-m-hard set for **E** is p-btt(2)-hard for **E** (see Theorem 12 below). Despite these limitations, in the next section we will show that there are almost p-m-complete sets for **E** that are not p-m-complete for **E**. Moreover, in Section 4 we will obtain the same results for some other p-reducibilities that allow only one oracle query by showing that all these reducibilities yield the same class of almost complete sets.

Our results and proofs will use the characterization of p-measure and measure in **E** in terms of resource-bounded random sets. In the remainder of this section we shortly describe this approach (from [7]) and state some results on the measure in **E** in terms of random sets, which we will need in the following.

Definition 8 *A set R is $t(n)$-*random *if no $t(n)$-martingale succeeds on R.*

For later use, we observe the following trivial relation among random sets for increasing time bounds.

Proposition 9 *Let $t(n), t'(n)$ be recursive functions such that $t(n) \leq t'(n)$ almost everywhere. Then every $t'(n)$-random set is $t(n)$-random.*

The characterization of the p-measure and the measure in **E** in terms of random sets is as follows.

Lemma 10 (Ambos-Spies, Terwijn and Zheng [7]) *For any class $\boldsymbol{C}$,*

(i) $\mu_{\mathrm{p}}(\boldsymbol{C}) = 0$ iff there is a number k such that $\boldsymbol{C}$ does not contain any n^k-random set, and
(ii) $\mu(\boldsymbol{C}|\mathbf{E}) = 0$ iff there is a number k such that $\boldsymbol{C} \cap \mathbf{E}$ does not contain any n^k-random set.

Lemma 10 (together with Definition 3) immediately yields the following characterization of almost hardness.

Lemma 11 *A set A is almost* p-*r*-*hard for* **E** *if and only if, for some number $k \geq 1$, the lower* p-*r*-*span of A contains all n^k-random sets in* **E**, *i.e.,*

$$\forall R \in \mathbf{E}\, (R\; n^k\text{-}random \;\Rightarrow\; R \leq^{\mathrm{p}}_r A). \tag{3}$$

To illustrate how results on random sets can be turned into results on p-measure and the corresponding measure in $\mathbf{E}$, we give a proof of the strengthening of Corollary 7 mentioned above.

Theorem 12 (Ambos-Spies and Mayordomo [4]) *Every almost* p-m-*hard set for* $\mathbf{E}$ *is* p-btt(2)-*hard for* $\mathbf{E}$.

The proof of Theorem 12, which is not explicitly given in [4], requires the following two lemmas. The first lemma gives a well-known invariance property of the n^k-random sets generalizing the observation that the complement of an n^k-random set is n^k-random again.

Lemma 13 *Let* $k \geq 1$, *let the set* R *be* n^{k+1}*-random and let the set* A *be in* $\mathbf{DTIME}(2^{kn})$. *Then the symmetric difference* $R \triangle A$ *of* R *and* A *is* n^{k+1}*-random again.*

Proof. Let d be an n^{k+1}-martingale and let s be the strategy underlying d. In order to show that d does not succeed on $R \triangle A$ we convert the martingale d into an n^{k+1}-martingale d' such that d' succeeds on a set X if and only if d succeeds on $X \triangle A$. Since, by n^{k+1}-randomness of R, d' does not succeed on R, it follows that d will not succeed on $R \triangle A$.

The strategy s' underlying d' is defined by

$$s'(X \restriction x) = |A(x) - s((X \triangle A) \restriction x)|$$

Then, for any set X and any string x, $d'(X \restriction x) = d((X \triangle A) \restriction x)$ by a straightforward induction on x, hence d' has the required behavior. Moreover, given $X \restriction x$, by $A \in \mathbf{DTIME}(2^{kn})$, $A(x)$ and $(X \triangle A) \restriction x$ can be computed in $\mathcal{O}(2^{|x|} \cdot 2^{k \cdot |x|}) = \mathcal{O}(|X \restriction x|^{k+1})$ steps. Since s can be computed in $\mathcal{O}(2^{(k+1)n})$ steps, it follows that s' can be computed in $\mathcal{O}(2^{(k+1)n})$ steps too, hence d' is an n^{k+1}-martingale. □

Lemma 13 implies that every set in $\mathbf{E}$ can be presented as the symmetric difference of two n^k-random sets.

Lemma 14 *Let* A *be in* $\mathbf{E}$ *and let* $k \geq 1$. *There are* n^k*-random sets* R_1 *and* R_2 *in* $\mathbf{E}$ *such that* $A = R_1 \triangle R_2$.

Proof. Fix $k' \geq k$ such that $A \in \mathbf{DTIME}(2^{k'n})$, let R_1 be any $n^{k'+1}$-random set in $\mathbf{E}$, and let $R_2 = R_1 \triangle A$. Then, by Lemma 13, R_2 is $n^{k'+1}$-random too, hence R_1 and R_2 are n^k-random by Proposition 9. Moreover, since, for any sets X and Y, $X = Y \triangle (Y \triangle X)$, the choice of R_2 implies that $A = R_1 \triangle (R_1 \triangle A) = R_1 \triangle R_2$. □

Proof of Theorem 12. Let A be almost p-m-hard for $\mathbf{E}$. Then, by Lemma 11, there is a number $k \geq 1$ such that every n^k-random set in $\mathbf{E}$ is p-m-reducible to A. Since, by Lemma 14, every set in $\mathbf{E}$ is the symmetric difference of two n^k-random sets in $\mathbf{E}$, it follows that A is p-btt(2)-hard for $\mathbf{E}$. $\square$

Many results about p-measure exploit the fact that random sets do not contain any easy parts and that they are incompressible. Before we state the observations needed here, we first recall some definitions. A set A is $\boldsymbol{C}$-*bi-immune* for a class $\boldsymbol{C}$, if no infinite subset of A or of the complement of A is a member of $\boldsymbol{C}$. A set A is p-*incompressible* if, for any set B and for any p-m-reduction f such that $A \leq^{\mathrm{p}}_{\mathrm{m}} B$ via f, f is one-to-one almost everywhere.

Theorem 15 *(a) (Mayordomo [19]) Every n^{k+1}-random set is* $\mathbf{DTIME}(2^{kn})$*-bi-immune.*
(b) (Juedes and Lutz [13]) Every n^2-random set is p*-incompressible.*

In [19] and [13] these observations are phrased in terms of p-measure. For a proof of the theorem in the given form, see [4].

Theorem 15 (a) implies that any p-m-reduction of an n^{k+1}-random set R to a set $A \in \mathbf{DTIME}(2^{kn})$ is length-increasing almost everywhere.

Corollary 16 *Let $k \geq 1$, let A be in* $\mathbf{DTIME}(2^{kn})$*, let R be n^{k+1}-random and assume that $R \leq^{\mathrm{p}}_{\mathrm{m}} A$ via f. Then f is length-increasing almost everywhere.*

Proof. For a contradiction assume that $|f(x)| \leq |x|$ for infinitely many strings x. For $i \leq 1$ let $B_i = \{x : |f(x)| \leq |x| \ \& \ A(f(x)) = i\}$. Then $B_i \in \mathbf{DTIME}(2^{kn})$, $B_0 \subseteq \overline{R}$, $B_1 \subseteq R$, and, by assumption, B_0 or B_1 is infinite. So R is not $\mathbf{DTIME}(2^{kn})$-bi-immune contrary to Theorem 15 (a). $\square$

Another related fact needed in Section 4 is the following.

Lemma 17 *Let $k \geq 2$ be a natural number and let R be an n^k-random set. Assume that R and its complement are* p-m*-reducible to some set A via* p-m*-reductions g and h, respectively. Then the intersection of the range of g and the range of h is finite.*

Proof. For a contradiction assume that the intersection of the ranges of g and h is infinite. Since, by Lemma 13, $\overline{R}$ is n^k-random too, by symmetry, we may assume that

$$B = \{x : \exists y \leq x (g(x) = h(y))\}$$

is infinite. Note that $B \in \mathbf{DTIME}(2^{2n})$ and, for $x \in B$, the string y_x that is the least string $y \leq x$ such that $g(x) = h(y)$ can be found in $2^{2|x|}$ steps. Moreover, by choice of g and h,

$$R(x) = A(g(x)) = A(h(y_x)) = \overline{R}(y_x),$$

hence $R(x) \neq R(y_x)$ and $y_x < x$. By infinity of B this allows the definition of an n^k-martingale d that succeeds on R contrary to choice of R. The strategy s underlying d is as follows. Given an initial segment $X \upharpoonright x$, check whether $x \in B$. If not then do not bet (i.e. $s(X \upharpoonright x) = 1/2$). If so, compute y_x and bet all the current capital on the outcome $X(x) = 1 - X(y_x)$ (i.e. $s(X \upharpoonright x) = 1$ if $X(y_x) = 0$ and $s(X \upharpoonright x) = 0$ otherwise). So, when betting against R, the capital will be doubled at every string x in B and thus d succeeds on R. □

Finally, for the proof of our main theorem in the next section, we will need the following instance of the Borel-Cantelli-Lemma for p-measure (see Regan and Sivakumar [21] for a more general discussion of this lemma).

Lemma 18 *Let $\{D_1, D_2, \dots\}$ be a sequence of pairwise disjoint finite sets where D_k has cardinality k. Assume further that given x, in time $\mathrm{O}(2^{2|x|})$, firstly, one can decide whether x is in D_k for some k and, if so, secondly, one can compute the unary notation 1^k of k and a list of all strings $y < x$ in D_k. Then every n^3-random set intersects almost all of the sets D_k.*

Proof. It suffices to define an n^3-martingale d that succeeds on every set A that has an empty intersection with infinitely many of the sets D_k. We will define an appropriate betting strategy s that induces the martingale d. Here the strategy s will never bet on x being in A (i.e. $s(w) \geq 1/2$ for all w), and so s in turn is determined by specifying for all w, the stake $v(w) \geq 0$ that is bet on x not being in A.

We split the initial capital $d(\lambda) = 1$ into infinitely many parts $c_1, c_2 \dots$, where fraction $c_k = 1/2^k$ is exclusively used for bets on the strings in D_k. On input w, we let $v(w) = c_k \cdot 2^{j-1}$ in case the string $x = |w|$ to bet on is the jth element of D_k and none of the strings $y < x$ in D_k is in A (i.e. if all $j-1$ previous bets on D_k have been wins). Otherwise, we abstain from betting by letting $v(w) = 0$. Thus for all k, if A does not intersect D_k, then the capital c_k is doubled k times, i.e., the total capital originating solely from c_k equals 1. As a consequence, the gain of d on A is unbounded in the limit in case A has an empty intersection with infinitely many of the D_k.

It remains to show that $s(w)$ can be computed in time $\mathrm{O}(|w|^3)$. First we show that for every w and every prefix u of w, the stake $v(u)$ can be computed in time $\mathrm{O}(|w|^2)$. For every such u and w and for $x = |u|$, we have $2^{2|x|} \leq |w|^2$, hence by assumption within time $\mathrm{O}(|w|^2)$ we can check whether x is in some D_k and, if so, can compute 1^k and a list of the elements $y < x$ in D_k. Moreover this list, which contains less than $|w|$ elements, can be ordered in time $\mathrm{O}(|w|^2)$ and running the ordered list against w in order to check whether none of the strings y is in A requires time linear in $|w|$.

Next we will argue that in time $\mathrm{O}(|w|^3)$ we can compute the accumulated capital $d(w)$ by starting with $d(\lambda)$ and then adding up the wins and losses for all prefixes u of w. Here the values $d(u)$ are bounded by $2^{|w|}$, while the stakes $v(u)$ are bounded by 1. Moreover, the stakes $v(u)$ can always be written as multiples of $1/2^{\mathrm{O}(|w|^2)}$ because if a string $y < x$ is in some set D_k, then by the discussion in the preceding paragraph, k is in $\mathrm{O}(|w|^2)$. Thus a single addition amounts to adding two rationals that have a binary expansion of at most $|w| + \mathrm{O}(|w|^2)$ digits. So for all prefixes u of w, the computation of the stake $v(u)$, the check whether the corresponding bet was a win, and the ensueing addition can be done in time $\mathrm{O}(|w|^2)$, and consequently $d(w)$ can be computed in time $\mathrm{O}(|w|^3)$.

By definition of the terms involved, the capital $d(w0)$ can be written as $d(w)+v(w)$ and also as $2 \cdot s(w) \cdot d(w)$, hence we obtain in case $d(w) \neq 0$

$$s(w) \;=\; \frac{d(w)+v(w)}{2d(w)}. \tag{4}$$

From the preceding discussion it is immediate that the binary expansion of both numerator and denominator of the fractional representation of $s(w)$ according to (4) can be computed in time $\mathrm{O}(|w|^3)$. We leave to the reader the easy task of showing that the latter time bound is also sufficient for finding natural numbers p, q, and r where $s(w) = p/q \cdot 2^{-r}$, i.e., to find a standard representation of the rational $s(w)$ in the sense of [4]. □

3 An almost complete set that is not complete

We now turn to the main result of this paper.

Theorem 19 *There is an almost* p-m-*complete set for* **E** *that is not* p-m-*complete for* **E**.

For a proof of Theorem 19 it suffices to show the following lemma.

Lemma 20 *There are sets A and B in* **E** *such that $B \not\leq^{\mathrm{p}}_{\mathrm{m}} A$ and*

$$\textit{for all } n^3\textit{-random sets } R \textit{ in } \mathbf{EXP},\ R \leq^{\mathrm{p}}_{\mathrm{m}} A. \tag{5}$$

Then, for such sets A and B, the set A is almost p-m-complete for $\mathbf{E}$ by (5) and Lemma 11, whereas B is not p-m-reducible to A and thus A is not p-m-complete for $\mathbf{E}$. In fact, for this argument it suffices to consider $\mathbf{E}$ in place of $\mathbf{EXP}$ in (5). We will use in Section 5, however, that the extension proved here will lead simultaneously to a corresponding result for the class $\mathbf{EXP}$, i.e., the class of sets computable in time 2^{poly}.

Proof of Lemma 20. We construct sets A and B as required in stages. To be more precise we choose a strictly increasing function $h : \mathbb{N} \to \mathbb{N}$ with $h(0) = 0$ and we determine the values of A and B for all strings in the interval

$$I_k = \{x : \; h(k) \leq |x| < h(k+1)\}$$

at stage k. Here the function h is chosen to be p-constructible and, for technical reasons to be explained below, to satisfy

$$\text{(i) } k^2 < h(k) \quad \text{(ii) } k^2 \cdot p_k(h(k)) < 2^{\sqrt[k]{h(k)}} \quad \text{(iii) } p_k(h(k)) < h(k+1) \qquad (6)$$

for all $k > 0$ and $p_k(n) = n^k + k$. Note that given x, by p-constructibility of h, we can compute the index k such that $x \in I_k$, as well as $h(k)$, in $\mathrm{poly}(|x|)$ steps.

Before we define stage k of the construction formally, we first discuss the strategies to ensure the required properties of A and B and simultaneously introduce some notation required in the construction.

In order to ensure (5) we let A sufficiently resemble a p-m-complete set for $\mathbf{EXP}$. Let $\{C_e : e \geq 0\}$ be an effective enumeration of $\mathbf{EXP}$ such that $C_e(x)$ can be computed uniformly in $2^{|x|^e} + e$ steps, and let

$$E \;=\; \{1^e 0 1^{|x|^e} 0 x : x \in C_e \;\&\; e \in \mathbb{N}\}$$

be the padded disjoint union of these sets. Then E can be computed in time 2^n and, for all e, C_e is p-m-reducible to E via

$$g_e(x) \;=\; 1^e 0 1^{|x|^e} 0 x,$$

so E is p-m-complete for $\mathbf{EXP}$. Hence, if we let

$$\mathrm{CODE}_e \;=\; \mathrm{range}(g_e)$$

denote the set of strings used for coding C_e into E, then in order to satisfy (5) it suffices to meet for all numbers $e \geq 0$, the requirement

$R^1_e :$ If C_e is n^3-random then $A \cap \mathrm{CODE}_e$ is a finite variant of $E \cap \mathrm{CODE}_e$.

Namely, given an n^3-random set R in **EXP** we can choose e with $R = C_e$. Then the corresponding requirement R_e^1 will ensure that R is p-m-reducible to A via a finite variant of g_e.

In order to meet the requirements R_e^1 we will let A look like E unless the task of making B not p-m-reducible to A will force a disagreement. Since E is in $\mathbf{DTIME}(2^n)$ this procedure is compatible with ensuring that A is in $\mathbf{E}$ as long as the strings on which A and E differ can be recognized in linear exponential time. In this connection note that the sets CODE_e are pairwise disjoint and that, for given x, $\mathrm{poly}(|x|)$ steps suffice to decide whether x is a member of one of these sets and if so to compute the unique e with $x \in \mathrm{CODE}_e$.

The condition $B \not\leq_{\mathrm{m}}^{\mathrm{p}} A$ is satisfied by diagonalization. Let $\{f_k : k \geq 1\}$ be an effective enumeration of the p-m-reductions such that $f_k(x)$ can be computed uniformly in $p_k(|x|) = |x|^k + k$ steps. Then it suffices to meet the requirements

$$R_k^2 : \quad \exists x \in \{0,1\}^* \, (B(x) \neq A(f_k(x)))$$

for all numbers $k \geq 1$. We will meet requirement R_k^2 at stage k of the construction.

For this purpose we will ensure that there is a string x from a set of k^2 designated strings of length $h(k)$ such that $B(x)$ and $A(f_k(x))$ differ, while we will let B be empty and let A equal E on I_k otherwise. We will say that this action *injures* an almost completeness requirement R_e^1 if, for the chosen string x, $f_k(x)$ is in $I_k \cap \mathrm{CODE}_e$ and A and E differ on $f_k(x)$. Since A and E agree on $I_k \cap \mathrm{CODE}_e$ otherwise, the conclusion of R_e^1 will fail if and only if the requirement is injured at infinitely many stages.

To avoid injuries we will attempt to diagonalize in such a manner that injuries to the first k requirements R_e^1, $e < k$, are avoided. If the function f_k is not one-to-one on the designated strings or if $f_k(x)$ is shorter than x for some designated string x then the diagonalization will not affect A on I_k at all and thus no injuries occur. The critical case occurs if, for every designated string x, $f_k(x)$ is longer than x and is an element of some of the sets CODE_e with $e < k$. In this case, in order to diagonalize as intended we have to make $A(f_k(x))$ differ from the canonical value 0 for $B(x)$ (not vice versa, since otherwise we might fail to make B computable in exponential time) and hence some injury may occur.

By Lemma 18, however, we will be able to argue that if C_e is n^3-random and if there are infinitely many stages at which we are forced to make $A(f_k(x))$ differ from $B(x) = 0$ for some $f_k(x)$ in CODE_e, then at almost all of these stages letting A look like E on the f_k-images of the designated strings will yield the desired diagonalization. So for n^3-random C_e the requirement R_e^1 will be injured only finitely often.

We now give the formal construction. We let $B \cap I_0 = \emptyset$ and $A \cap I_0 = E \cap I_0$. Given $k > 0$, stage k of the construction is as follows. We assume that A and B have already been defined on the intervals $I_0, \dots, I_{k-1}$, and we will specify both sets on the interval I_k. For the scope of the description of stage k we call the first k^2 strings in I_k the *designated strings*. The designated strings are the potential diagonalization witnesses for requirement R_k^2, i.e., we will guarantee $B(x) \neq A(f_k(x))$ for some designated string x. Observe that every designated string has length $h(k)$ and is mapped by f_k into the union of the intervals I_0 through I_k, as follows by items (i) and (iii) in (6), respectively.

For the definition of A and B on I_k we distinguish the following four cases with respect to the images of the designated strings under the mapping f_k. Here it is to be understood that on I_k the sets A and B will always look like the set E and the empty set, respectively, unless this specification is explicitly overwritten according to one of the cases below. Moreover, as the cases are not mutually exclusive, always the first applicable case is used.

Case 1: Some designated string is not mapped to I_k.
Let x be the least such string. By the preceding discussion, $f_k(x)$ is contained in some interval I_j with $j < k$ and $A(f_k(x))$ has been defined at some previous stage. We let $B(x) = 1 - A(f_k(x))$ (thereby satisfying R_k^2).

Case 2: Two designated strings are mapped to the same string.
Let x be the least designated string such that $f_k(x) = f_k(x')$ for some designated string $x' \neq x$ and let $B(x) = 1$. (Then $B(x) = 1$ differs from $B(x') = 0$, whereas f_k maps x and x' to the same string, so R_k^2 is met.)

Case 3: Some designated string is not mapped to the set $\bigcup_{e<k} \mathrm{CODE}_e$.
Let x be the least such designated string and let $A(f_k(x)) = 1$. (Note that, by failure of Case 1, $f_k(x)$ is in I_k, and R_k^2 is met since $B(x) = 0$ by convention.)

Case 4: Otherwise.
In this case the k^2 designated strings are mapped by f_k to k^2 different strings in $\bigcup_{e<k} \mathrm{CODE}_e$, hence we can let e_k be the least $e < k$ such that f_k maps at least k designated strings to CODE_{e_k}. Let J_k be the set of the least k designated strings that are mapped to CODE_{e_k}, and let

$$F_k = \{f_k(x) : \ x \in J_k\}$$

be the f_k-image of J_k. Observe that by case assumption all strings in F_k are in I_k. In case E does not intersect F_k, we let $A(y) = 1$ where y is the maximal element in F_k and we say that $R_{e_k}^1$ is *injured at stage* k. (Then R_k^2 is met, because in case E intersects F_k there has already been a string x in J_k such that $B(x) = 0$ differs from $A(f_k(x))$ while, otherwise, we will enforce such a disagreement for some x where $f_k(x) = y$.)

This completes the construction. It remains to show that the constructed sets have the required properties. We first observe that the constructed sets are in

$\mathbf{DTIME}(2^{2n})$. We sketch the proof for the set A and leave the similar proof for the set B to the reader. Given a string y, we can compute in time poly($|y|$) the index k where y is in I_k, as well as $h(k)$. Further it takes time $\mathrm{O}(k^2 p_k(h(k)))$ to compute the list of all pairs $(x, f_k(x))$ such that x is a designated string of stage k and it takes time polynomial in the length of this list to check which of the four cases applies and to determine whether according to this case, $A(y)$ might differ from $E(y)$ at all. If not, we simply have to compute $E(y)$. Otherwise, we know that either Case 3 applies and y is in A or Case 4 applies, y is the maximal string in F_k, and y is in A iff none of the $k-1$ smaller strings in F_k is also in E. Using item (ii) in (6) it is then a routine task to show that A in fact can be computed in time 2^{2n}.

It remains to show that the requirements R^1_e, $e \geq 0$, and R^2_k, $k \geq 0$, are met. By the comments made in the individual cases of the construction, it is immediate that all the requirements R^2_k, $k \geq 0$, are met. For a proof that all the almost completeness requirements R^1_e are met, too, fix $e \geq 0$ and assume for a contradiction that R^1_e fails. Then C_e is n^3-random and $A \cap \mathrm{CODE}_e$ and $E \cap \mathrm{CODE}_e$ differ on infinitely many intervals I_k. By construction, the latter implies that there are infinitely many stages k where R^1_e is injured. Note that, for such a stage k, Case 4 applies, $e = e_k$ and consequently the set F_k is defined and

$$F_k \subseteq \mathrm{CODE}_e \cap I_k \quad \text{and} \quad F_k \cap E = \emptyset.$$

For all other stages, we define now F_k to be the set of the first k strings in $\mathrm{CODE}_e \cap I_k$ (except that for the at most finitely many k where the latter set contains less than k strings, we let F_k be an arbitrary k-element subset of CODE_e such that the sets $F_0, F_1, \ldots$ are pairwise disjoint). Then we let

$$D_k = \{g_e^{-1}(y) : y \in F_k\}.$$

Thus $C_e \cap D_k = \emptyset$ for the infinitely many stages k at which R^1_e is injured and in order to obtain the desired contradiction, it suffices to show that the sequence $D_0, D_1, \ldots$ satisfies the hypothesis of Lemma 18.

By construction, each D_k has cardinality k and the sets D_k are pairwise disjoint because they are the inverse images of the pairwise disjoint sets F_k. We show now that for given input x, in time $\mathrm{O}(2^{|x|})$ we can check whether x is in some D_k and if so, can compute 1^k and a list of the elements in D_k. Here, first, we compute $g_e(x)$ and the index k such that $g_e(x) \in I_k$. This can be done in poly($|x|$) steps. Second, we simulate stage k of the construction, check if Case 4 applies with $e = e_k$ and compute the set F_k accordingly. This can be done in time polynomial in $k^2 p_k(h(k))$, as follows by an argument similar to the one used to show that the set A can be computed in linear exponential time. Third, we compute the preimage of F_k under g_e, which can again be done in time polynomial in $k^2 p_k(h(k))$. So we are done, because the time required by the three preceding steps can be bounded for some constant c and almost all x (and the corresponding values of k) by

$$[k^2 p_k(h(k))]^c \leq 2^{c \cdot h(k)^{\frac{1}{k}}} \leq 2^{c|x|^{\frac{2e}{k}}} \leq 2^{|x|} \tag{7}$$

Here the inequalities in (7) follow, from left to right, by item (ii) of (6), because, by choice of k, $|h(k)|$ is bounded by $|g_e(x)|$ and hence is bounded by $|x|^{2e}$ for almost all x and, finally, by the asymptotic growth of the functions involved. □

4 Comparing completeness notions

The polynomial-time reducibilities allowing only one oracle query ranging from one-to-one, length-increasing reductions to truth-table reductions of norm 1 lead to the same class of complete sets for **E**. Namely, Berman [9] has shown that every p-m-complete set for **E** is in fact p-1-li-complete while Homer, Kurtz and Royer [11] have proved that every p-btt(1)-complete set for **E** is in fact p-m-complete for **E**. Corresponding results for weak completeness have been proved by Ambos-Spies, Mayordomo and Zheng [5]. By the two following theorems, the same phenomenon occurs for almost completeness.

Theorem 21 *A set is almost* p-m-*complete for* **E** *if and only if it is almost* p-1-li-*complete for* **E**.

Proof. For a proof of the nontrivial direction assume that A is almost p-m-complete for **E** and fix $k_0 \geq 1$ such that $A \in \mathbf{DTIME}(2^{k_0 n})$. By Lemma 11 and Proposition 9 choose $k > k_0$ such that all n^k-random sets in **E** are p-m-reducible to A. Then by Lemma 11 again, it suffices to show that all these sets are p-1-li-reducible to A. So let R be any n^k-random set in **E** and assume that R is p-m-reducible to A via the polynomial-time computable function f. Then, by Theorem 15 (b) and by Corollary 16, f is one-to-one and length-increasing almost everywhere, i.e.,

$$B = \{x : |f(x)| \leq |x| \text{ or } \exists y < x(f(x) = f(y))\}$$

is finite. So, in order to convert f into a p-1-li-reduction f' from R to A, it suffices to correct f on the finite set B. We do this by mapping the n-th element x_n of $B \cap R$ (if it exists) to the n-th element y_n of $A \setminus \text{range}(f)$ that is longer than x_n and, similarly, the n-th element x'_n of $B \cap \overline{R}$ (if it exists) to the n-th element y'_n of $\overline{A} \setminus \text{range}(f)$ that is longer than x'_n ($n \geq 1$). Since B is finite, the function f' defined in this way is polynomial-time computable and it is a p-1-li-reduction from R to A.

It remains to show, however, that f' is well-defined, i.e., that the required strings y_n and y'_n actually exist. For this sake it suffices to show that $A \setminus \text{range}(f)$ and $\overline{A} \setminus \text{range}(f)$ are infinite. This is done as follows. By Lemma 13, the complement $\overline{R}$ of R is n^k-random too. Hence, by choice of k, $\overline{R}$ is p-m-reducible to A, say via g. Moreover, since $\overline{R} \notin \mathbf{P}$, the intersections $\text{range}(g) \cap A$ and $\text{range}(g) \cap \overline{A}$ are infinite, whence it suffices to show that the ranges of f and g have only finitely many elements in common. But this is true by Lemma 17. $\square$

Theorem 22 *A set is almost* p-btt(1)*-complete for* $\mathbf{E}$ *if and only if it is almost* p-m*-complete for* $\mathbf{E}$.

Proof. For a proof of the nontrivial direction assume that A is almost p-btt(1)-complete for $\mathbf{E}$. Then, by Lemma 11, firstly, we can assume that for some $k \geq 2$ the lower p-btt(1)-span of A contains all n^k-random sets in $\mathbf{E}$ and, secondly, it suffices to show that every n^k-random set in $\mathbf{E}$ is p-m-reducible to A. So let R be an n^k-random set in $\mathbf{E}$ and, for a contradiction, assume that R is not p-m-reducible to A. We will obtain the desired contradiction by constructing an n^k-random set R' in $\mathbf{E}$ that is not p-btt(1)-reducible to A.

Let $\{(g_e, h_e) : e \geq 0\}$ be an effective enumeration of the p-btt(1)-reductions with nonconstant evaluators, i.e., with $h_e(x)(0) \neq h_e(x)(1)$ for all strings x, and such that $g_e(x)$ and $h_e(x)$ can be uniformly computed in $2^{|x|} + e$ steps. Then, for every $e \geq 1$ we define a variant R_e of R by letting

$$R_e(x) \;=\; \begin{cases} R(x) & \text{if } h_e(x)(0) < h_e(x)(1) \\ 1 - R(x) & \text{if } h_e(x)(0) > h_e(x)(1) \end{cases}.$$

Note that the set R_e is not p-btt(1)-reducible to A via (g_e, h_e) since otherwise R would be p-m-reducible to A via g_e contrary to the assumption. In fact, by closure under finite variants of the lower p-m-span of A, the reduction (g_e, h_e) fails to reduce R_e to A for infinitely many arguments. Hence we can construct a partition of $\{0,1\}^*$ into easily recognizable intervals I_e, $e \geq 0$, such that (g_e, h_e) fails to reduce R_e to A for some string in I_e. Moreover, by a standard delayed diagonalization argument (see e.g. Chapter 7 of [8]), we can choose the partition in such a way that, for any x, the index e of the interval I_e containing x can be computed in $|x|^2$ steps and such that $|x| \geq e$. Now define R' by letting R' agree with R_e on the interval I_e. Then R' will not be p-btt(1)-reducible to A via any of the reductions (g_e, h_e), $e \geq 0$. Since any p-btt(1)-reduction to A can be easily converted into a reduction with nonconstant evaluators, this shows that R' is not p-btt(1)-reducible to A. Moreover, by choice of the intervals I_e, R' is in $\mathbf{E}$ and the set

$$D \;=\; \{x : \exists e \geq 0 \;\; (x \in I_e \;\&\; h_e(x)(0) > h_e(x)(1))\},$$

consisting of the strings for which the evaluator h_e corresponding to the interval I_e containing x is negative, can be computed in time $\mathcal{O}(2^n)$. Since $R' = R \triangle D$, by Lemma 13 this implies that R' is n^k-random. $\square$

Previous results in the literature together with the results of this paper clarify most of the relations among the different completeness notions for $\mathbf{E}$. If we let $\mathcal{C}(\mathbf{E}, r)$ denote the class of p-r-complete sets for $\mathbf{E}$, and if $\mathcal{AC}(\mathbf{E}, r)$ and $\mathcal{WC}(\mathbf{E}, r)$ denote the corresponding classes of almost and weakly complete sets, respectively, the known relations among the classes are summarized in Figure 1.

$$\begin{array}{ccccc}
\mathcal{C}(\mathbf{E}, \text{1-li}) & \subset & \mathcal{AC}(\mathbf{E}, \text{1-li}) & \subset & \mathcal{WC}(\mathbf{E}, \text{1-li}) \\
\| & & \| & & \| \\
\mathcal{C}(\mathbf{E}, \text{m}) & \subset & \mathcal{AC}(\mathbf{E}, \text{m}) & \subset & \mathcal{WC}(\mathbf{E}, \text{m}) \\
\| & & \| & & \| \\
\mathcal{C}(\mathbf{E}, \text{btt}(1)) & \subset & \mathcal{AC}(\mathbf{E}, \text{btt}(1)) & \subset & \mathcal{WC}(\mathbf{E}, \text{btt}(1)) \\
\cap & & \cap & & \cap \\
\mathcal{C}(\mathbf{E}, \text{btt}) & = & \mathcal{AC}(\mathbf{E}, \text{btt}) & \subset & \mathcal{WC}(\mathbf{E}, \text{btt}) \\
\cap & & \cap & & \cap \\
\mathcal{C}(\mathbf{E}, \text{tt}) & = & \mathcal{AC}(\mathbf{E}, \text{tt}) & \subseteq & \mathcal{WC}(\mathbf{E}, \text{tt}) \\
\cap & & \cap & & \cap \\
\mathcal{C}(\mathbf{E}, \text{T}) & = & \mathcal{AC}(\mathbf{E}, \text{T}) & \subseteq & \mathcal{WC}(\mathbf{E}, \text{T})
\end{array}$$

Fig. 1. The figure shows the known relations among the completeness notions discussed in this paper. Here '$\subset$' means that a class is a proper subclass, while '$\subseteq$' indicates that it is not known if the inclusion is strict.

In Figure 1, the inclusions from top to bottom and from left to right are immediate by definition. The two equalities in the first column have been demonstrated by Berman [9] and by Homer et al. [11] (see the beginning of this section), while the strictness of the remaining three inclusions in this column has been established by Watanabe [22], who separated the standard completeness notions for reducibilities that allow more than one query. The two equalities in the second column are proved in Theorems 21 and 22 above. It follows with Theorem 19 that the first three inclusions from column 1 to column 2 are proper, while the coincidence of completeness and almost completeness for the other three reducibilities follows from Corollary 6 above due to Regan et al. [20]. This corollary also yields that the last two inclusions in column 2 are proper. That the class $\mathcal{AC}(\mathbf{E}, \mathrm{btt}(1))$ is a proper subclass of the class $\mathcal{AC}(\mathbf{E}, \mathrm{btt})$ follows from Theorem 12, since Watanabe [22] has shown that there is a p-btt-complete set for $\mathbf{E}$ that is not p-btt(2)-complete. The relations stated in the third column have been established by Ambos-Spies et al. in [5] where weak completeness notions are compared.

The strictness of the first four inclusions between the second and the third column follows by considering the measure in $\mathbf{E}$ of the classes involved. By Corollary 7, the class $\mathcal{AC}(\mathbf{E}, \mathrm{btt})$ and all classes contained in it have measure 0 in $\mathbf{E}$. On the other hand, $\mathcal{WC}(\mathbf{E}, \mathrm{m})$ has nonzero measure in $\mathbf{E}$ ([12]), in fact measure 1 in $\mathbf{E}$ ([7]), i.e., all the classes of weakly complete sets shown in the third column have measure 1 in $\mathbf{E}$. Note that the measure in $\mathbf{E}$ of the remaining four classes (the complete and almost complete sets for p-tt- and p-T-reducibility) is hitherto unknown.

Finally, the question whether the last two inclusions between the second and the third column are proper is still open. It has been shown, however, that these questions cannot be resolved by relativizable techniques: namely, Allender and Strauss [1] have shown that, relative to some oracle, all n^2-random sets are p-tt-complete whereas Ambos-Spies, Lempp, and Mainhardt [3] and, independently, Buhrman et al. [10] have given oracles relative to which no n^2-random set is p-T-complete for $\mathbf{E}$. This also shows that the measure in $\mathbf{E}$ of the classes of complete and almost complete sets for p-tt- and p-T-reducibility is oracle dependent.

5 Further results

In this paper we looked at the concept of almost completeness only for the class $\mathbf{E}$ of sets computable in linear exponential time. Similar results, however, can be obtained for other complexity classes. In particular all of our results can be also shown for Lutz's measure on the class $\mathbf{EXP}$ of sets computable in time 2^{poly}. The analog of our main theorem (Theorem 19) in this setting follows directly from Lemma 20 by the characterization of the measure in $\mathbf{EXP}$ in terms of $2^{(\log n)^k}$-random sets corresponding to Lemma 10, while analogs of the other results require only minor changes in the proofs. The relations among the different completeness notions in Figure 1 will remain the same if we replace $\mathbf{E}$ by $\mathbf{EXP}$.

While it is well-known that p-m-hardness for $\mathbf{E}$ and $\mathbf{EXP}$ coincide, Juedes and Lutz [14] have shown that every weakly p-m-hard set for $\mathbf{E}$ is also weakly p-m-hard for $\mathbf{EXP}$ but that there are weakly p-m-complete sets for $\mathbf{EXP}$ in $\mathbf{E}$ that are not weakly p-m-complete for $\mathbf{E}$. By refining the technique used in the proof of our main theorem, Ambos-Spies has shown that the concepts of almost p-m-hardness for **E** and **EXP** are independent and that witnesses for the independence can be found in **E**. Moreover, there is an almost p-m-complete set for **EXP** that is not even weakly p-m-hard for **E**.

Ambos-Spies [2] has also investigated almost hardness for **E** and **EXP** under bounded query reducibilities of constant norm, namely under the adaptive p-Turing reducibility p-bT(c) of constant norm c and the nonadaptive p-truth-table reducibility p-btt(c) of norm c. He has shown that the corresponding notions of almost hardness are nontrivial and he proved hierarchy theorems clarifying the relations among these new concepts.

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