CALIBRATING RANDOMNESS

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§1. Introduction. We report on some recent work centered on attempts to understand when one set is more random than another. We look at various methods of calibration by initial segment complexity, such as those introduced by Solovay [129], Downey, Hirschfeldt, and Nies [38], Downey, Hirschfeldt, and LaForte [35], and Downey [30]; as well as other methods such as lowness notions of Kučera and Terwijn [71], Terwijn and Zambella [137], Nies [104, 105], and Downey, Griffiths, and Reid [33]; higher level randomness notions going back to the work of Kurtz [73], Kautz [61], and Solovay [129]; and other calibrations of randomness based on definitions along the lines of Schnorr [121].

These notions have complex interrelationships, and connections to classical notions from computability theory such as relative computability and enumerability. Computability figures in obvious ways in definitions of effective randomness, but there are also applications of notions related to randomness in computability theory. For instance, an exciting by-product of the program we describe is a more-or-less natural *requirement-free* solution to Post's Problem, much along the lines of the Dekker deficiency set.

This paper is self-contained (though we assume basic notions from computability theory, at the level of the first few chapters of Soare [125]), with some representative proofs being sketched. We begin with a quick recapitulation of the classical approaches to algorithmic randomness, through measure theory, unpredictability, and incompressibility. After that we turn to Solovay's approach to calibrating the complexity of left computably enumerable (left-c.e.) reals, and the Kučera-Slaman Theorem which implies that there is essentially *only one* random left-c.e. real, Chaitin's Ω. After looking at deficiencies of the notion of Solovay reducibility we turn to other initial segment measures of relative randomness.

We next introduce the notion of K-triviality. A set is K-trivial if its initial segment prefix-free Kolmogorov complexity is the same as that of the sequence of all ones (up to an additive constant). Noncomputable K-trivial sets exist but are all Δ^0_2. The c.e. sets among them are solutions to Post's Problem. We explore Nies' results on the Turing degrees of such sets, which form a natural Σ^0_3 ideal in the c.e. Turing degrees, and the connections between K-triviality, lowness notions, and other forms of computational weakness related to randomness. We then discuss the complexity of c.e. sets and a theorem of Kummer which characterizes c.e. degrees containing complex c.e. sets.

After this we turn to other notions of randomness such as Schnorr randomness and computable randomness, where the notion of what constitutes a random set actually changes. Again, there have been many recent results in this area, such as a machine characterization of Schnorr randomness, and beautiful lowness characterizations related to the hyperimmune-free degrees. We also discuss lowness notions for various flavors of randomness.

Next we look at arithmetical versions of randomness (n-randomness). We include a bit of background material here, as it seems not widely known, and the lovely recent result that a set is 2-random (a relativized prefix-free complexity notion) iff its initial segment complexity is maximal infinitely often when measured by plain Kolmogorov complexity. Many "typical" random phenomena only really occur for 2- or even 3-random sets. For instance, if we concentrate on the left-c.e. reals, then we get the impression that 1-random sets look like Ω and are computationally rich. However, 2-random sets are computationally very weak, and look much more like *low* sets. The next section includes several results of Miller and Yu, which explore another measure of relative randomness and further expand on this story, showing for instance that there is a relationship between levels of randomness and initial segment complexity. We follow this with a look at relativizing randomness by considering Ω as an operator, which is complicated by the fact that it is a c.e. operator but is not c.e. in *and above*.

In the last section we take a brief look at effective Hausdorff dimension and related notions of partial randomness. This is a huge area of research, particularly in computer science, and we will only mention a few recent results.

The topic of this paper is a mix of computability theory, algorithmic information theory, and measure theory. Our computability-theoretic notation generally follows Odifreddi [110, 111] and Soare [125]. We deal with several degree structures, but when we mention degrees without further specification, we mean Turing degrees. We will denote the e-th partial computable function by Φ_e, the e-th partial computable function with oracle X by Φ_e^X, and the e-th computably enumerable set by W_e. When we write $\log n$, we mean the base 2 logarithm of n, rounded up to the nearest integer. We will use C to denote plain Kolmogorov complexity, and K to denote prefix-free Kolmogorov complexity.

This paper is not a general introduction to recent work on algorithmic randomness, but rather an attempt to give the reader insight into what we feel are some of the high points in the program to understand relative randomness, the Kolmogorov complexity of sets, and the relationships of these topics to classical computability theory. This is a fast-growing area of research, and we have necessarily omitted many important results and even entire fruitful lines of investigation. There is a wealth of open questions in this area; we mention a few below, but refer the reader to Miller and Nies [96] for a more comprehensive list.

Although we do not focus on the history of the field, we have tried to give motivating historical definitions and intuitions, as well as related results by, for instance, Kurtz [73] and Kautz [61]. Among several historically important papers not explicitly mentioned below, we may cite Shannon [123], Solomonoff [127], Chaitin [23], and van Lambalgen [75]. Details of the results here and a version of the legendary unpublished notes of Solovay [129] will appear in a forthcoming book by Downey and Hirschfeldt [34], Solovay's material appearing with his permission. A forthcoming book by Nies [107] contains details on some of the results here, in particular where the application of randomness notions in computability theory is concerned (for instance lowness properties and priority-free solutions to Post's problem). It also contains a chapter on formalizing the intuitive notion of randomness via effective descriptive set theory, studied in [50].

§2. Sets, measure, and martingales.

2.1. Sets and measure. The Cantor space of all infinite binary sequences is denoted by 2^ω. This space is endowed with the tree topology, which has as basic clopen sets

$$[\sigma] := \{X \in 2^\omega : \sigma \prec X\},$$

where $\sigma \in 2^{<\omega}$. The *uniform* or *Lebesgue measure* on 2^ω is induced by giving each basic open set $[\sigma]$ measure $\mu([\sigma]) := 2^{-|\sigma|}$.

We identify an element X of 2^ω with the set $\{n : X(n) = 1\}$. The space 2^ω is measure-theoretically identical with the real interval $[0,1]$, although the two are not homeomorphic as topological spaces, so we can also think of elements of 2^ω as elements of $[0,1]$. Thus we refer to elements of 2^ω as *sets* or *reals*. We will use the former term except when we wish to emphasize the identification of 2^ω with $[0,1]$, in particular when dealing with left computably enumerable reals. A real is *left computably enumerable* (*left-c.e.*) if it is the limit of a computable increasing sequence of rationals, or equivalently, if its left cut is c.e. Such reals are often simply called *c.e. reals*, but we wish to avoid any confusion between c.e. *reals* and c.e. *sets*. A real is *strongly c.e.* if it is of the form $0.A$ for a c.e. set A. It is not hard to see that not every left-c.e. real is strongly c.e.

The collection of strings (finite initial segments of sets) is denoted by $2^{<\omega}$. For strings σ and τ, let $\sigma \preccurlyeq \tau$ denote that σ is an initial segment of τ. Similarly, for a set X and a string σ, let $\sigma \prec X$ denote that σ is an initial segment of X. We denote the length of a string σ by $|\sigma|$. For a set X, we denote the string consisting of the first n bits of X by $X \upharpoonright n$.

One particularly important class of measurable subsets of 2^ω consists of the unions of prefix-free collections of basic clopen sets. A set $P \subset 2^{<\omega}$ is called *prefix-free* if for all σ, τ, if $\sigma \prec \tau$ and $\sigma \in P$, then $\tau \notin P$. Note that for such a set P, if $\mathcal{P} = \bigcup_{\sigma \in P}[\sigma]$, then

$$\mu(\mathcal{P}) = \sum_{\sigma \in P} 2^{-|\sigma|}.$$

2.2. Martingales. A different treatment of measure, important for our story, is the one of Ville using martingales. A *martingale* is a function $d : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ that satisfies for every $\sigma \in 2^{<\omega}$ the averaging condition

$$2d(\sigma) = d(\sigma 0) + d(\sigma 1).$$

Similarly, d is a *supermartingale* if it satisfies

$$2d(\sigma) \geqslant d(\sigma 0) + d(\sigma 1).$$

A (super)martingale d *succeeds on* a set A if $\limsup_{n\to\infty} d(A \upharpoonright n) = \infty$. We say that d succeeds on, or *covers*, a class $\mathcal{A} \subseteq 2^\omega$ if d succeeds on every $A \in \mathcal{A}$. The *success set* $S[d]$ of d is the class of all sets on which d succeeds. The reader should think of a martingale as a betting strategy. The function d assigns a portion of our capital to be bet on the string σ. The success set of d is thus the collection of sets on which this betting strategy allows us to increase our capital arbitrarily much.

The following classical result shows how the concept of a martingale relates to measurability.

THEOREM 2.1 (Ville [138]). *For any class $\mathcal{A} \subseteq 2^\omega$ the following statements are equivalent:*

(i) *$\mathcal{A}$ has Lebesgue measure zero,*
(ii) *There exists a martingale that succeeds on $\mathcal{A}$.*

Martingales will prove important when we look at refinements of classical Martin-Löf randomness. Martingales are the key to looking at measure and Hausdorff dimension in small classes such as polynomial time, a fact first realized by Lutz [83]. (See also Lutz [86].)

§3. Three approaches to randomness. Historically[1], there have been three main approaches to the definition of an algorithmically random sequence. They are via what we call

(i) the measure-theoretic paradigm,
(ii) the unpredictability paradigm, and
(iii) the incompressibility paradigm.

3.1. The measure-theoretic paradigm. Among the oldest definitions of randomness are those saying that a random set should have certain stochastic properties. For instance, a random set should have about as many 0's as 1's. Von Mises, in his remarkable paper [100], defined a notion of randomness based on such stochastic properties, and noted that for any *countable* collection of such properties a nonempty notion of randomness results. But he did not have a canonical choice of such a countable collection at hand. Later Church made the connection with the theory of computability by suggesting that one should take all *computable* stochastic properties. Martin-Löf then noted that these are a special kind of measure zero subsets of 2^ω, and that a more general and smooth definition could be obtained by considering all effectively measure zero sets. We discuss this approach of Martin-Löf below. For more discussion and references on the original stochasticity approach see Ambos-Spies and Kučera [1].

The measure-theoretic paradigm is that the random sets should be those with no *effectively rare* properties. If a property constitutes an effective null set, then a random set should not have this property.

A collection of sets that is effectively enumerated is a Σ^0_1-*class*. We can represent a Σ^0_1-class U as $\bigcup_{\sigma\in W}[\sigma]$ for some prefix-free c.e. set of strings W. We say that W is a *presentation* of U. Whenever we mention a Σ^0_1-class U, we assume we have a fixed presentation W of U, and identify U with W. So, for instance, for $\sigma \in 2^{<\omega}$, we write $\sigma \in U$ to mean $\sigma \in W$. We will not explicitly mention presentations unless necessary for clarity.

Now a *test* is a sequence $\{U_i\}_{i\in\omega}$ of such Σ^0_1-classes that are shrinking in size. A set passes a test $\{U_i\}_{i\in\omega}$ if it is not in the intersection $\bigcap_i U_i$. The main idea is that a random set should pass all effective tests. This leads to the following definition.

DEFINITION 3.1 (Martin-Löf [88]). A collection of sets $\mathcal{A} \subseteq 2^\omega$ is *Martin-Löf null* (or Σ^0_1-*null*) if there is a uniformly c.e. sequence $\{U_i\}_{i\in\omega}$ of Σ^0_1-classes (called a *Martin-Löf test*) such that $\mu(U_i) \leqslant 2^{-i}$ and $\mathcal{A} \subseteq \bigcap_i U_i$.

[1]In these notes, we will try to avoid discussion of the history of the evolution of the notion of algorithmic randomness. There are thorough discussions in the monograph of Li and Vitányi [81], the paper of Zvonkin and Levin [146], and van Lambalgen's thesis [74].

A set $A \in 2^\omega$ is *Martin-Löf random*, or Σ^0_1-*random*, or 1-*random*, if $\{A\}$ is not Σ^0_1-null.

Solovay gave the following definition of randomness equivalent to that of Martin-Löf. It is not hard to prove that the two notions are the same.

DEFINITION 3.2 (Solovay [129]). A *Solovay test* is a uniformly c.e. sequence $\{U_i\}_{i\in\omega}$ of Σ^0_1-classes with $\sum_i \mu(U_i) < \infty$. A set A is *Solovay random* if for all Solovay tests $\{U_i\}_{i\in\omega}$, we have $A \in U_i$ for only finitely many i. Note that the definition does not change if we replace the Σ^0_1-classes U_i by basic clopen sets $[\sigma_i]$.

One very interesting fact, due to Martin-Löf, is that there is a *universal* Martin-Löf test $\{U_n\}_{n\in\omega}$, meaning that a set A is Martin-Löf random iff $A \notin \bigcap_n U_n$. To define such a test, fix a computable enumeration $\{V_i^m\}_{i,m\in\omega}$ of all Martin-Löf tests (with the m-th test being $\{V_i^m\}_{i\in\omega}$), and let $U_n = \bigcup_k V^k_{n+k+1}$. Similarly, there is a universal Solovay test.

3.2. The unpredictability paradigm. If you ask someone why they think a certain event is random they will most often give the answer that the event is "unpredictable". In particular, a set $A = a_0a_1\dots$ should be random if we cannot predict any of its bits given other bits. One way to implement this idea is to use martingales.

DEFINITION 3.3 (Schnorr [120]). We say that a (super)martingale d is *effective* (also called Σ^0_1 or *computably enumerable*) if the reals $d(\sigma)$ for $\sigma \in 2^{<\omega}$ are uniformly left-c.e.

The reader might have expected that an effective martingale would be one where d is a computable function, rather than one with computable *approximations*. We will return to this very important point in Section 10.

Martingales of varying complexities are a convenient way of introducing various effective measures. This approach was first taken by Schnorr [120, 121], and later applied with much success in complexity theory by Lutz [83].

Schnorr proved the following effective version of Ville's Theorem 2.1. Its proof is a direct effectivization of that of Theorem 2.1.

THEOREM 3.4 (Schnorr [121], Satz 5.3). *A class $\mathcal{A} \subseteq 2^\omega$ is Martin-Löf null iff there is a c.e. supermartingale d such that $\mathcal{A} \subseteq S[d]$.*

This theorem remains true if we replace "c.e. supermartingale" by "c.e. martingale".

Since c.e. supermartingales are Σ^0_1 objects, it comes as no surprise that we can effectively list all of them in an enumeration $d_0, d_1, \dots$. The sum $d = \sum_i 2^{-i} d_i$ is then again a c.e. supermartingale, and its success set $S[d]$ is the maximal Martin-Löf null set, so d is a *universal* c.e. supermartingale. Additionally, d is an *optimal* c.e. supermartingale, in the sense that for any other nontrivial c.e. supermartingale d', there is a c such that $cd(\sigma) \geqslant d'(\sigma)$ for all σ. The existence of an optimal c.e. supermartingale is also implicit in Levin's construction of a universal c.e. semimeasure (see Zvonkin and Levin [146]). The fact that there is no optimal c.e. martingale is one of the reasons to consider supermartingales. (This fact was implicit in [146]; see [34] for a proof.)

3.3. The incompressibility paradigm. A third approach to defining the notion of a random set is the one essentially due to Kolmogorov [65]. Here we regard a *string* as random iff it has no short description, that is, there is no short program to generate the string, meaning that the only way to generate it is essentially to hardwire it into the machine. (As opposed to, e.g., 101010 repeated 1000 times, which can be generated by a short program.) We then use this idea to define randomness of sets by considering the lengths of shortest descriptions of its initial segments. We mention only a few basic results about Kolmogorov complexity; for more on the subject, see Li and Vitányi [81] or Calude [14].

3.3.1. *Plain Kolmogorov complexity.* Fix a universal Turing machine U. Given a string $\sigma \in 2^{<\omega}$, define the *plain Kolmogorov complexity* of σ by

$$C(\sigma) := \min\{|\tau| : U(\tau) = \sigma\}.$$

Two basic facts concerning C are that (i) the choice of U does not matter up to an additive constant and (ii) $C(\sigma) \leqslant |\sigma| + O(1)$ for all σ.

For $n \in \mathbb{N}$, let $C(n) = C(0^n)$. The specific encoding of natural numbers into strings used in this definition does not matter, since if $f : 2^{<\omega} \to 2^{<\omega}$ is computable, then $C(f(\sigma)) \leqslant C(\sigma) + O(1)$.

We can define σ to be *k-random* if $C(\sigma) \geqslant |\sigma| - k$. (This definition will only be used in this section, and should not be confused with the notion of n-randomness we will introduce in Section 12.) An easy counting argument shows that random strings exist: For each n, there are $\sum_{i=0}^{n-k-1} 2^{-i} = 2^{n-k} - 1$ programs of length $< n - k$, so there are at least $2^n - 2^{n-k} + 1$ many k-random strings of length n. For every k, the set of k-random strings is an *immune* Π^0_1 set, i.e., it does not contain any infinite c.e. subsets. As a function, C is not computable. If $m(x) = \min\{C(y) : y \geqslant x\}$, then m is unbounded (because we eventually run out of short programs), but grows slower than any unbounded nondecreasing partial computable function.

We would like to extend the definition of randomness for finite strings to a definition for infinite strings. Naively, we could define a set A to be random iff there is a k such that every $\sigma \prec A$ is k-random. However, Martin-Löf showed that such sets do not exist! This can be seen using the following argument of Katseff [60]. Let $\sigma_0, \sigma_1, \dots$ be an effective listing of all strings, with $|\sigma_n| = \log n$. If $A \upharpoonright m = \sigma_n$, then from the *length* of $A \upharpoonright n$ we can recover $A \upharpoonright m$. Thus, to generate $A \upharpoonright n$, we need only generate the string τ such that $A \upharpoonright n = \sigma_n \tau$ and compute n from $|\tau| = n - \log n$, which gives us σ_n. This shows that for every A,

$$\exists^\infty n\,[C(A \upharpoonright n) \leqslant n - \log n + O(1)].$$

The basic intuition for what goes wrong in trying to use plain Kolmogorov complexity to define randomness for sets is that the Kolmogorov complexity of τ should be the length of the shortest string σ such that τ can be obtained from the *bits* of σ. The *length* of σ seems to give an additional $\log n$ many bits of information. This idea is explicitly used above to demonstrate that, using plain Kolmogorov complexity, we will always get complexity oscillations in the initial segments of a set. Levin [78, 79], Schnorr [122], and Chaitin [20, 22] introduced methods to get around this problem, as we now discuss.

It should be noted that there *is* now a plain complexity characterization of 1-randomness, given recently by Miller and Yu [97].

THEOREM 3.5 (Miller and Yu [97]). *A set A is 1-random iff for every computable function g such that $\sum_n 2^{-g(n)} < \infty$,*

$$C(A \upharpoonright n) \geqslant n - g(n) - O(1).$$

3.3.2. *Prefix-free Kolmogorov complexity.* Call a Turing machine M a *prefix-free* (or *self-delimiting*) *machine* if $\mathrm{dom}(M) \subseteq 2^{<\omega}$ is prefix-free. A prefix-free machine U is *universal* if for each prefix-free machine M there is a string ρ_M such that

$$\forall \sigma\,[U(\rho_M \sigma) = M(\sigma)].$$

We call $|\rho_M|$ the *coding constant* of M in U. Note that if c is the coding constant of M in U then

$$\forall \sigma\, \exists \tau\,[|\tau| \leqslant |\sigma| + c \wedge U(\tau) = M(\sigma)].^2$$

It is easy to construct a universal prefix-free machine U, by letting

$$U(1^m 0 \sigma) = T_m(\sigma)$$

for an effective enumeration $T_0, T_1, \dots$ of all prefix-free machines.

We can now define the *prefix-free complexity*[3] of a string σ by

$$K(\sigma) := \min\{|\tau| : U(\tau) = \sigma\}.$$

Again, the choice of U matters only up to an additive constant. For $n \in \mathbb{N}$, let $K(n) = K(0^n)$.

[2] A universal prefix-free machine is sometimes defined to be one for which such a constant c exists for each prefix-free machine M. The distinction between this weaker notion and the one we adopt here is most often irrelevant. It becomes more important when we relativize the notion of universal prefix-free machine, as we do in Section 14.

[3] There are alternative approaches to modifying the definition of Kolmogorov complexity, such as monotone complexity and process complexity, which are motivated by similar considerations as prefix-free complexity. See Li and Vitányi [81] for details.

Unlike plain complexity, prefix-free complexity is *subadditive*: $K(\sigma\tau) \leqslant K(\sigma) + K(\tau) + O(1)$.

The prefix-free encoding $\widehat{\sigma} = 1^{|\sigma|}0\sigma$ gives $K(\sigma) \leqslant 2|\sigma| + O(1)$, and the prefix-free encoding $\widehat{|\sigma|}\sigma$ gives $K(\sigma) \leqslant |\sigma| + 2\log|\sigma| + O(1)$. (Here we identify $|\sigma|$ with its binary representation.) This process can be continued to obtain tighter upper bounds on $K(\sigma)$, but for any c there is a σ such that $K(\sigma) > |\sigma| + \log|\sigma| + c$.

The following is a fundamental result about K.

THEOREM 3.6 (Counting Theorem, Chaitin [20]).
(i) $\max\{K(\sigma) : |\sigma| = n\} = n + K(n) \pm O(1)$.
(ii) $\left|\{\sigma : |\sigma| = n \wedge K(\sigma) \leqslant n + K(n) - r\}\right| \leqslant 2^{n-r+O(1)}$, *where the constant $O(1)$ does not depend on n and r.*

We are now in a position to define randomness for sets in terms of initial segment complexity.

DEFINITION 3.7 (Levin [78], Chaitin [20]). A set A is *Levin-Chaitin random* (or Kolmogorov-Levin-Chaitin random) if there is a constant c such that $K(A \upharpoonright n) \geqslant n - c$ for every n.

Again we arrive at the same concept of randomness as above.

THEOREM 3.8 (Schnorr, see Chaitin [20]). *A set $A \in 2^\omega$ is Martin-Löf random iff it is Levin-Chaitin random.*

The proof of Theorem 3.8 given below uses the effective version of Kraft's Inequality [66], which is a fundamental tool in this area. This version is usually known as the Kraft-Chaitin Theorem, as it appears in Chaitin [20] (where it is attributed to Pippinger), but according to Gács, it appeared earlier in Levin's dissertation [77] and is implicit in Schnorr's paper [122]. We retain the terminology "Kraft-Chaitin" for this theorem and certain associated concepts defined below for the sake of terminological consistency with many of the papers we discuss. If U is a prefix-free machine then the open set presented by the domain of U is measurable. The Kraft-Chaitin Theorem is a kind of converse to this fact, and implies that each left-c.e. real is the measure of the domain of some prefix-free machine.

THEOREM 3.9 (Levin [77], Schnorr [122], Chaitin [20]). *Let $\langle d_i, \tau_i\rangle_{i\in\omega}$ be a computable sequence of pairs (which we call* requests*), with $d_i \in \mathbb{N}$ and $\tau_i \in 2^{<\omega}$, such that $\sum_i 2^{-d_i} \leqslant 1$. Then there is a prefix-free machine M and strings σ_i of length d_i such that $M(\sigma_i) = \tau_i$. Thus $K(\tau_i) \leqslant K_M(\tau_i) + O(1) \leqslant d_i + O(1)$.*

PROOF. It is enough to define effectively a prefix-free sequence of strings $\sigma_0, \sigma_1, \ldots$ with $|\sigma_n| = d_n$. The following organizational device is due to

Joe Miller. For each n, let $x^n = x^n_1 \dots x^n_m$ be a binary string such that $0.x^n_1 \dots x^n_m = 1 - \sum_{j \leqslant n} 2^{-d_j}$. We will define the σ_n so that the following holds for each n: for each m with $x^n_m = 1$ there is a string μ^n_m of length m so that $S_n = \{\sigma_i : i \leqslant n\} \cup \{\mu^n_m : x^n_m = 1\}$ is prefix-free.

We begin by letting σ_0 be 0^{d_0}. Notice that $x^0_m = 1$ iff $0 < m \leqslant d_0$, so if we define $\mu^0_m = 0^{m-1}1$, then $\{\sigma_0\} \cup \{\mu^0_m : x^0_m = 1\}$ is prefix-free.

Now assume we have defined $\sigma_0, \dots, \sigma_n$ and μ^n_m for $x^n_m = 1$ so that $S_n = \{\sigma_i : i \leqslant n\} \cup \{\mu^n_m : x^n_m = 1\}$ is prefix-free.

If $x^n_{d_{n+1}} = 1$ then x^{n+1} is the same as x^n except that $x^{n+1}_{d_{n+1}} = 0$. So we can let $\sigma_{n+1} = \mu^n_{d_{n+1}}$ and $\mu^{n+1}_m = \mu^n_m$ for all $m \neq d_{n+1}$, and then $S_{n+1} = \{\sigma_i : i \leqslant n+1\} \cup \{\mu^{n+1}_m : x^{n+1}_m = 1\}$ is equal to S_n, and hence is prefix-free.

Otherwise, find the largest $j < d_{n+1}$ such that $x^n_j = 1$. Such a j must exist since otherwise $1 - \sum_{j \leqslant n} 2^{-d_j} < 2^{-d_{n+1}}$, which would mean that $\sum_{j \leqslant n+1} 2^{-d_j} > 1$. In this case x^{n+1} is the same as x^n except for positions $j, \dots, d_{n+1}$, where we have $x^{n+1}_j = 0$ and $x^{n+1}_m = 1$ for $j < m \leqslant d_{n+1}$. Let $\sigma_{n+1} = \mu^n_j 0^{d_{n+1}-j}$. For $m < j$ or $m > d_{n+1}$, let $\mu^{n+1}_m = \mu^n_m$, and for $j < m \leqslant d_{n+1}$, let $\mu^{n+1}_m = \mu^n_j 0^{m-j-1}1$. Then $S_{n+1} = \{\sigma_i : i \leqslant n+1\} \cup \{\mu^{n+1}_m : x^{n+1}_m = 1\}$ is the same as S_n except that μ^n_j is replaced by a pairwise incomparable set of superstrings of μ^n_j. This clearly ensures that S_{n+1} is prefix-free.

This completes the definition of the σ_i. Each finite subset of $\{\sigma_0, \sigma_1, \dots\}$ is contained in some S_n, and is hence prefix-free. Thus the whole set is prefix-free. Since the σ_i are chosen effectively, we can define a prefix-free machine M by letting $M(\sigma_i) = \tau_i$ for each i. $\dashv$

We call an effectively enumerated set of requests $\langle d_i, \tau_i \rangle_{i \in \omega}$ such that $\sum_i 2^{-d_i} \leqslant 1$ a *Kraft-Chaitin set*. The *weight* of this set is $\sum_i 2^{-d_i}$. As an illustration of the use of the Kraft-Chaitin Theorem, we give a proof of Schnorr's Theorem 3.8.

PROOF OF THEOREM 3.8. (Only if) Let U be the universal prefix-free machine relative to which K is defined. Let $R_k = \bigcup\{[\sigma] : K(\sigma) \leqslant |\sigma| - k\}$. Notice that $\{R_k\}_{k \in \omega}$ is a uniformly c.e. sequence of Σ^0_1-classes. We now show that it is a Martin-Löf test.

Let P_k be the set of σ such that $K(\sigma) \leqslant |\sigma| - k$ but $K(\tau) > |\tau| - k$ for all $\tau \prec \sigma$. Then P_k is prefix-free and $R_k = \bigcup_{\sigma \in P_k} [\sigma]$. Furthermore, for each $\sigma \in P_k$, there is a string σ^* such that $U(\sigma^*) = \sigma$ and $|\sigma^*| \leqslant |\sigma| - k$. Since U is prefix-free, $\sum_{\sigma \in P_k} 2^{-|\sigma^*|} \leqslant \sum_{U(\tau)\downarrow} 2^{-|\tau|} \leqslant 1$. So

$$\mu(R_k) = \sum_{\sigma \in P_k} 2^{-|\sigma|} \leqslant \sum_{\sigma \in P_k} 2^{-(|\sigma^*|+k)} = 2^{-k} \sum_{\sigma \in P_k} 2^{-|\sigma^*|} \leqslant 2^{-k}.$$

Thus $\{R_k\}_{k\in\omega}$ is a Martin-Löf test. Now if A is Martin-Löf random, then $A \notin \bigcap_k R_k$, so there is a k such that $K(A \upharpoonright n) > n-k$ for all n, and thus A is Levin-Chaitin random.

(If) Let $\{U_k\}_{k\in\omega}$ be the universal Martin-Löf test. As discussed above, we identify each U_k with a particular fixed presentation of U_k. Let

$$L = \{\langle |\sigma| - k, \sigma\rangle : \exists k \geqslant 1\, [\sigma \in U_{2k}]\}.$$

Then L is a Kraft-Chaitin set, since it is clearly c.e. and

$$\sum_{k\geqslant 1}\sum_{\sigma\in U_{2k}} 2^{-|\sigma|+k} = \sum_{k\geqslant 1} 2^k \mu(U_{2k}) \leqslant \sum_{k\geqslant 1} 2^k 2^{-2k} = \sum_{k\geqslant 1} 2^{-k} = 1.$$

So by the Kraft-Chaitin Theorem there is a c such that $K(\sigma) \leqslant |\sigma| - k + c$ for all $k \geqslant 1$ and $\sigma \in U_{2k}$. Now if A is Levin-Chaitin random then there is a k such that $K(A \upharpoonright n) \geqslant n - k + c$ for all n, which implies that $A \notin U_{2k}$, and hence that A is Martin-Löf random. ⊣

Remark. We will often use the Kraft-Chaitin Theorem in conjunction with Kleene's Recursion Theorem (see [125, Theorem II.3.1]). If we uniformly enumerate Kraft-Chaitin sets K_n for $n \in \mathbb{N}$ and use the Kraft-Chaitin theorem to obtain corresponding prefix-free machines M_n, then, since the proof of the Kraft-Chaitin Theorem is effective, the Recursion Theorem implies that there is an n such that M_n has coding constant n. Thus, when we build a prefix-free machine M via the Kraft-Chaitin Theorem, we can assume we know the coding constant c of M in advance and use it in enumerating our requests (as long as the enumeration yields a Kraft-Chaitin set for any value of c).

Recently, Miller and Yu [97] have proved significant generalizations of Schnorr's Theorem 3.8, which can be interpreted as saying that *not only is a set* 1*-random iff its initial segment complexity is always above* n*, but the initial segment complexity of* 1*-random sets is "well above"* n *most of the time.*

Theorem 3.10 (Miller and Yu [97]). *Let* A *be* 1*-random.*

(i) $\sum_n 2^{n-K(A\upharpoonright n)} < \infty$.

(ii) *For any function* f *such that* $\sum_n 2^{-f(n)} = \infty$,

$$\exists^\infty n\, [K(A \upharpoonright n) > n + f(n)].$$

Proof. Part (ii) follows easily from part (i). We give a proof of part (i) due to Nies. Let

$$d(\sigma) = \sum_{\tau\preccurlyeq\sigma} 2^{|\tau|-K(\tau)} + \sum_{\sigma\prec\tau} 2^{|\sigma|-K(\tau)}.$$

It is easy to check that d is a c.e. martingale, and that $\sum_n 2^{n-K(A\upharpoonright n)} \leqslant \limsup_n d(A \upharpoonright n)$. But this limsup is finite, since A is 1-random. ⊣

Part (i) of Theorem 3.10 is known as the Ample Excess Theorem, and improves a result of Chaitin [22], who showed that if A is 1-random then $\lim_n K(A \upharpoonright n) - n = \infty$. Part (ii) had been proved for *computable* functions f by Solovay [129]. It is easy to check that if A is not 1-random, then (i) and (ii) fail, so these conditions are actually *equivalent* to 1-randomness.

In this section we have seen that for certain natural notions of randomness, various approaches to the definition of randomness lead to the same class, the 1-random sets. Later we will look at certain criticisms of this notion and variations generated by such criticisms. We now turn to our first approach to calibrating randomness.

§4. Solovay reducibility, and a characterization of 1-random left-c.e. reals. Notice that a consequence of the Kraft-Chaitin Theorem is that *a real is left-c.e. iff it is the measure of the domain of a prefix-free machine*. Thus, in this setting, left-c.e. reals occupy the same place as c.e. sets do in classical computability theory.

We have not yet seen an example of a 1-random set. Since the universal Martin-Löf test defines an effectively null set, the collection of Martin-Löf random reals has measure 1. One might well ask to what extent they resemble one another, and how such resemblance might be measured. Since for each k the set $P_k = \{A : \forall n\, [K(A \upharpoonright n) \geqslant n - k]\}$ is a Π^0_1-class containing only 1-random sets, there are 1-random sets of low Turing degree (by the Low Basis Theorem [58]), and 1-random left-c.e. reals (since the leftmost path of a Π^0_1-class is a left-c.e. real). On the other hand, we have the following result.

THEOREM 4.1 (Kučera [67]). *If a* 1*-random set A has c.e. degree, then* $A \equiv_T \emptyset'$.

The most famous explicitly defined 1-random set is Chaitin's Ω [20]:

$$\Omega := \sum_{U(\sigma)\downarrow} 2^{-|\sigma|} = \mu(\mathrm{dom}(U)),$$

where U is a universal prefix-free machine. This is the *halting probability* of U. Notice that Ω is a left-c.e. real.

Here is a short proof that Ω is 1-random. It follows from Theorem 4.1, and is not hard to check directly, that $\Omega \equiv_{\mathrm{T}} \emptyset'$, so in particular Ω is not rational. Thus for each n there is an s with $\Omega_s \upharpoonright n = \Omega \upharpoonright n$, where $\Omega_s := \sum_{U_s(\sigma)\downarrow} 2^{-|\sigma|}$. We build a prefix-free machine M. By the Recursion Theorem, we can assume we know its coding constant c in U. Whenever at a stage s we have $U_s(\tau) = \Omega_s \upharpoonright n$ for some τ such that $|\tau| < n - c$ (which means that $K_U(\Omega_s \upharpoonright n) < n - c$), we choose a string μ not in the range of U_s and declare $M(\tau) = \mu$. Since M is coded in U with coding constant c, there must be a ν such that $|\nu| \leqslant |\tau| + c < n$ and $U(\nu) = M(\tau) = \mu$.

Since $\mu \notin \mathrm{rng}(U_s)$, it follows that $\nu \notin \mathrm{dom}(U_s)$, so $\Omega - \Omega_s \geqslant 2^{-|\nu|} > 2^{-n}$, and hence $\Omega \upharpoonright n \neq \Omega_s \upharpoonright n$. This procedure ensures that if $|\tau| < n - c$ then $U(\tau) \neq \Omega \upharpoonright n$, whence $K_U(\Omega \upharpoonright n) \geqslant n - c$ for all n.

Of course, the definition of Ω depends on the choice of universal prefix-free machine, so we should really say that Ω is *a* halting probability, rather than *the* halting probability, as it is commonly referred to. However, the analog of Ω in classical computability is the halting problem $\emptyset' := \{i : \Phi_i(i)\downarrow\}$, and we usually talk about *the* halting problem, although that situation is analogous, in that the definition of $\emptyset'$ depends on the choice of enumeration of the partial computable functions. What allows us to disregard this enumeration-dependence is Myhill's Theorem (see [125, Theorem II.4.6]), which says that all halting problems are essentially the same, since they are all equivalent modulo a very strong reducibility.

Solovay [129] recognized this situation, and sought to introduce appropriate reducibilities to establish an analog to Myhill's Theorem. As we now discuss, Solovay's program has been recently realized by the joint work of several authors. Our starting point is the notion of Solovay reducibility, or domination, introduced by Solovay in his manuscript [129].

DEFINITION 4.2 (Solovay [129]). We say that a real α is *Solovay reducible* to β (or that β *dominates* α), and write $\alpha \leqslant_{\mathrm{S}} \beta$, if there are a constant c and a partial computable function f so that for all $q \in \mathbb{Q}$ with $q < \beta$,

$$\alpha - f(q) < c(\beta - q).$$

One way to look at this definition is that a sequence of rationals converging to β can be used to generate one converging to α at the same rate or faster. Indeed, if we have increasing computable sequences of rationals $\{r_n\}_{n\in\omega}$ and $\{q_n\}_{n\in\omega}$ converging to α and β, respectively, then $f(q_n)\downarrow$ for all n, and for each n we can effectively find a k such that $f(q_n) < r_k < \alpha$. This observation yields the following characterization of Solovay reducibility.

LEMMA 4.3 (Calude, Coles, Hertling, and Khoussainov [16]). *For left-c.e. reals α and β, let $\{r_n\}_{n\in\omega}$ and $\{q_n\}_{n\in\omega}$ be increasing computable sequences of rationals converging to α and β, respectively. Then $\alpha \leqslant_S \beta$ iff there exist a total computable function g and a constant c such that for all n,*

$$\alpha - r_{g(n)} < c(\beta - q_n).$$

Solovay [129] observed that this "analytic" version of m-reducibility could be used to extend many results about Ω to a class of left-c.e. reals with a machine-independent definition, namely the Ω-*like*, or *Solovay complete*, left-c.e. reals, which are those left-c.e. reals α such that $\beta \leqslant_{\mathrm{S}} \alpha$

for all left-c.e. reals β. (It is not hard to show that Ω is Ω-like.) In particular, if a left-c.e. real is Ω-like, then it is 1-random. This result follows from the following property of Solovay reducibility.

LEMMA 4.4 (Solovay [129]). *If $\alpha \leqslant_S \beta$ then $K(\alpha \upharpoonright n) \leqslant K(\beta \upharpoonright n) + O(1)$.*

PROOF SKETCH. The proof relies on the following fact observed by Solovay: *For each d there is a k such that, for all $n \geqslant 1$ and all σ, τ of length n with $|0.\sigma - 0.\tau| < d2^{-n}$, we have $|K(\tau) - K(\sigma)| \leqslant k$.* Solovay's observation (which is also true for C) follows easily from the fact that there are only $O(d)$ many such τ for a fixed σ.

Now let f and c be as in Definition 4.2. Let $\beta_n = 0.(\beta \upharpoonright n)$. Since β_n is rational and $\beta - \beta_n < 2^{-(n+1)}$, we have $\alpha - f(\beta_n) < c2^{-(n+1)}$. Thus, by the observation, $K(\alpha \upharpoonright n) \leqslant K(f(\beta_n) \upharpoonright n) + O(1)$. Since f is computable, this implies that $K(\alpha \upharpoonright n) \leqslant K(\beta \upharpoonright n) + O(1)$. ⊣

Using the Kraft-Chaitin Theorem, Calude, Hertling, Khoussainov, and Wang [17] proved that if α is Ω-like, then α is the halting probability of a universal prefix-free machine. We give a short proof based on a characterization of Solovay reducibility by Downey, Hirschfeldt, and Nies [38].

THEOREM 4.5 (Calude, Hertling, Khoussainov, and Wang [17]). *Let α be a left-c.e. real such that $\Omega \leqslant_S \alpha$. Then α is a halting probability. That is, there is a universal prefix-free machine $\widehat{U}$ such that $\mu(\operatorname{dom}(\widehat{U})) = \alpha$.*

PROOF. Let U be a universal prefix-free machine with $\Omega = \mu(\operatorname{dom}(U))$. In [38], it is shown that $\Omega \leqslant_{\mathrm{S}} \alpha$ implies that there are sequences of rationals $0 = \Omega_0 < \Omega_1 < \cdots$ and $0 = \alpha_0 < \alpha_1 < \cdots$ converging to Ω and α, respectively, and a constant c such that $\Omega_{s+1} - \Omega_s < 2^c(\alpha_{s+1} - \alpha_s)$ for all s. Assume we have chosen c large enough so that $\alpha + 2^{-c} < 1$ and $2^{-c} < \alpha$.

Now $\beta = \alpha + 2^{-c}(1 - \Omega)$ is a left-c.e. real, so by the Kraft-Chaitin Theorem there is a prefix-free machine M such that $\mu(\operatorname{dom}(M)) = \beta$, and we can assume that there is a string ρ such that $|\rho| = c$ and $M(\rho)\downarrow$. Define a prefix-free machine $\widehat{U}$ by letting $\widehat{U}(\sigma) = M(\sigma)$ if $\sigma \not\succcurlyeq \rho$ and $\widehat{U}(\rho\tau) = U(\tau)$. Then $\widehat{U}$ is universal, since it codes U, and $\mu(\operatorname{dom}(\widehat{U})) = \mu(\operatorname{dom}(M)) - 2^{-c}(1 - \Omega) = \alpha$. ⊣

Kučera and Slaman finished the story by proving the following.

THEOREM 4.6 (Kučera and Slaman [70]). *Every 1-random left-c.e. real is Ω-like.*

PROOF SKETCH. Suppose that α is a 1-random left-c.e. real and β is a left-c.e. real. We need to show that $\beta \leqslant_{\mathrm{S}} \alpha$. We enumerate a Martin-Löf test $\{R_n\}_{n\in\omega}$ in stages. Let $\{\alpha_s\}_{s\in\omega}$ and $\{\beta_s\}_{s\in\omega}$ be increasing sequences of rationals converging to α and β, respectively. At stage s, if $\alpha_s \in R_{n,s}$, do nothing, and otherwise put $(\alpha_s, \alpha_s + 2^{-n}(\beta_{s+1} - \beta_{t_s}))$ into R_n, where t_s is the last stage at which something was put into R_n. Now $\mu(R_n) < 2^{-n}\beta < 2^{-n}$, and thus $\{R_n\}_{n\in\omega}$ is a Martin-Löf test. As α is 1-random, there is an n such that $\alpha \notin R_n$, which implies that $\beta \leqslant_{\mathrm{S}} \alpha$ with constant 2^n. ⊣

Theorem 4.6 gives great insight into the structure of the 1-random left-c.e. reals. All that is needed for α to be 1-random is that $K(\alpha \upharpoonright n) \geqslant n - O(1)$. But of course $K(\alpha \upharpoonright n)$ can be near $n + K(n)$. In fact, we know that the complexity of the initial segments of Ω must oscillate near this bound, and, indeed, by the work of Solovay mentioned after the proof of Theorem 3.10, all 1-random sets exhibit such oscillations. The Kučera-Slaman Theorem says that all 1-random left-c.e. reals exhibit the *same* pattern of complexity oscillation, and that in a strong sense, there is essentially only one 1-random left-c.e. real.

Results such as the above motivate us to understand the structure of left-c.e. reals under $\leqslant_{\mathrm{S}}$. Naturally, this reducibility gives rise to equivalence classes, called *Solovay degrees*. We denote the Solovay degree of α by $\deg_{\mathrm{S}}(\alpha)$. It was observed by Solovay and others, such as Calude, Hertling, Khoussainov, and Wang [17], that the Solovay degrees of left-c.e. reals form an upper semilattice, with the join operation induced by addition (or equivalently, multiplication); that is, $\deg_{\mathrm{S}}(\alpha) \vee \deg_{\mathrm{S}}(\beta) = \deg_{\mathrm{S}}(\alpha + \beta) = \deg_{\mathrm{S}}(\alpha \cdot \beta)$. Downey, Hirschfeldt, and Nies [38] showed that this upper semilattice is distributive, and established some of its structural properties.

THEOREM 4.7 (Downey, Hirschfeldt, and Nies [38]).

(i) *The Solovay degrees of left-c.e. reals are dense. That is, if $\alpha <_S \beta$ are left-c.e. reals, then there is a left-c.e. real γ such that $\alpha <_S \gamma <_S \beta$.*

(ii) *If $\beta <_S \alpha <_S \Omega$ are left-c.e. reals, then there exist left-c.e. reals $\gamma_1 \,|_S\, \gamma_2$ such that $\beta <_S \gamma_1, \gamma_2$ and $\alpha = \gamma_1 + \gamma_2$. In other words, every incomplete Solovay degree of left-c.e. reals splits over each lesser degree.*

(iii) *If α and β are left-c.e. reals such that $\Omega = \alpha + \beta$ then either $\alpha \equiv_S \Omega$ or $\beta \equiv_S \Omega$. In other words, the complete Solovay degree of left-c.e. reals does not split in the Solovay degrees of left-c.e. reals.*[4]

Items (ii) and (iii) above demonstrate that 1-random left-c.e. reals are qualitatively different from all other left-c.e. reals in the sense that they cannot be split into two left-c.e. reals of lesser Solovay degree. It is important to realize that this is only true of left-c.e. reals. To see this, note that if $\Omega = .a_0a_1\ldots$ and we let $\alpha = .a_00a_20a_40\ldots$ and $\beta = .0a_10a_30\ldots$, then clearly neither α nor β can be 1-random, yet $\alpha + \beta = \Omega$. But α and

[4]According to Kučera (personal communication), item (iii) in Theorem 4.7 had been proved earlier by Demuth.

β are not left-c.e. The fact that addition induces the join operation on left-c.e. reals leads to another characterization of Solovay reducibility on left-c.e. reals.

THEOREM 4.8 (Downey, Hirschfeldt, and Nies [38]). *Let α and β be left-c.e. reals. Then $\alpha \leqslant_S \beta$ iff there exist a constant c and a left-c.e. real γ such that*

$$c\beta = \alpha + \gamma.$$

Before we leave the Solovay degrees of left-c.e. reals, we note that their structure is quite complicated.

THEOREM 4.9 (Downey, Hirschfeldt, and LaForte [36]). *The first-order theory of the Solovay degrees of left-c.e. reals is undecidable.*

The proof of Theorem 4.9 uses Nies' method of interpreting effectively dense Boolean algebras (see [103]), together with a technical construction of a certain class of (strongly) c.e. reals.

Calude and Nies [18] proved that the 1-random left-c.e. reals are all wtt-complete. This also follows from the result in Downey, Hirschfeldt, and LaForte [35] that if β is a left-c.e. real and α is a strongly c.e. real, then $\alpha \leqslant_{\mathrm{S}} \beta$ implies $\alpha \leqslant_{\mathrm{wtt}} \beta$ (and even $\alpha \leqslant_{\mathrm{sw}} \beta$, which will be defined in Section 5.1). If we combine this result with the following theorem of Demuth, and the fact that if a 1-random set has c.e. Turing degree then it is Turing complete, we see that while a 1-random left-c.e. real is wtt-complete, it is tt-incomparable with all noncomputable, incomplete c.e. sets. In particular, no 1-random left-c.e. real can be tt-complete.

THEOREM 4.10 (Demuth [29]). *If A is 1-random and $B \leqslant_{tt} A$ is noncomputable, then there is a 1-random set $C \equiv_T B$.*

Proofs of Demuth's Theorem can be found in Kautz [61, Theorem IV.3.16] and Downey and Hirschfeldt [34].

§5. Other reducibilities that calibrate randomness. In [35], a transitive preordering $\leqslant$ on sets was said to be a *measure of relative randomness* if it satisfies the *Solovay property*:

$$\text{If } A \leqslant B, \text{ then } K(A \upharpoonright n) \leqslant K(B \upharpoonright n) + O(1).$$

This view of what constitutes a measure of relative randomness is too restrictive, as it is tailored to reducibilities motivated by the incompressibility approach to randomness, so we will instead use the term *K-measure of relative randomness*. (In Section 5.3 we will see a reducibility motivated by the unpredictability approach to randomness.)

Solovay reducibility is a K-measure of relative randomness, but it is certainly not the only one. Moreover, it behaves reasonably only on the left-c.e. reals; it is quite easy to construct a real α and a *computable* real

β with $\beta \not\leqslant_{\mathrm{S}} \alpha$. Even on the left-c.e. reals, S-reducibility is too fine and uniform (as we will see), and badly fails to capture relative complexity exactly.

5.1. $\leqslant_{\mathbf{sw}}$ and $\leqslant_{\mathbf{rK}}$. In [35], Downey, Hirschfeldt, and LaForte introduced another K-measure of relative randomness called *sw-reducibility* (strong weak truth table reducibility).

DEFINITION 5.1. We say that A is *sw-reducible* to B, and write $A \leqslant_{\mathrm{sw}} B$, if there is a functional Γ such that $\Gamma^B = A$ and the use of $\Gamma^B(n)$ is bounded by $n + O(1)$.

Again it is not difficult to prove that $\leqslant_{\mathrm{sw}}$ is a K-measure of relative randomness. This reducibility is quite close to one considered by Csima [26] and Soare [126] in connection with work of Nabutovsky and Weinberger [102] in differential geometry, the difference being that the use is bounded by n in their case. Lewis and Barmpalias [80] have recently given an interesting characterization of sw-reducibility in terms of Lipschitz continuity.

Downey, Hirschfeldt, and LaForte [35] showed that $\leqslant_{\mathrm{sw}}$ agrees with $\leqslant_{\mathrm{S}}$ on the *strongly* c.e. reals, but the two notions are incomparable on the left-c.e. reals, in the sense that there exist left-c.e. reals α, β, γ, δ with $\alpha \leqslant_{\mathrm{S}} \beta$ but $\alpha \not\leqslant_{\mathrm{sw}} \beta$, and $\gamma \leqslant_{\mathrm{sw}} \delta$ but $\gamma \not\leqslant_{\mathrm{S}} \delta$. Furthermore, if α is a noncomputable left-c.e. real, then there is a noncomputable strongly c.e. real $\beta \leqslant_{\mathrm{sw}} \alpha$, but this is *not* true in general for $\leqslant_{\mathrm{S}}$, as shown by the following theorem, which is proved by a gap/co-gap argument.

THEOREM 5.2 (Downey, Hirschfeldt, and LaForte [35]). *There exists a noncomputable left-c.e. real α such that all strongly c.e. reals S-below α are computable.*

By and large, however, sw-reducibility is very badly behaved, as witnessed by the next two theorems.

THEOREM 5.3 (Downey, Hirschfeldt, and LaForte [35]). *The sw-degrees of left-c.e. reals do not form an upper semilattice.*

THEOREM 5.4 (Yu and Ding [142]). *There is no sw-complete left-c.e. real. Thus the analog of the Kučera-Slaman Theorem 4.6 cannot hold for sw-reducibility.*

Theorem 5.4 says that while the initial segment complexity of all 1-random left-c.e. reals is the same, there is no natural uniform way to get the *bits* of one version of Ω from those of another.

On the other hand, there is something we can say about sw-hardness with respect to c.e. *sets*.

THEOREM 5.5 (Downey and Hirschfeldt [34]). *Let α be a 1-random left-c.e. real. Then $B \leqslant_{sw} \alpha$ for any c.e. set B. Thus, not only is Ω wtt-complete, but it is sw-hard for c.e. sets.*

PROOF. Since α is 1-random, there is a d such that $K(\alpha \upharpoonright n) > n-d$ for all n. We build a prefix-free machine M using the Kraft-Chaitin Theorem. By the Recursion Theorem, we can assume we know the coding constant c of M in the universal prefix-free machine U. Whenever we see $n > c+d$ enter B at stage s, we enumerate a request $\langle n-c-d, \alpha_s \upharpoonright n\rangle$. The total cost of these requests is bounded by $\sum_{n>c+d} 2^{-(n-c-d)} = 1$, so the hypotheses of the Kraft-Chaitin Theorem are satisfied. Thus for all $n > c+d$, if n enters B at stage s, then $K(\alpha_s \upharpoonright n) \leqslant K_M(\alpha_s \upharpoonright n)+c \leqslant n-c-d+c = n-d$, which implies that $\alpha \upharpoonright n \neq \alpha_s \upharpoonright n$. So to compute $B(n)$ for $n > c+d$, it is enough to run the approximation of α until a stage s such that $\alpha \upharpoonright n = \alpha_s \upharpoonright n$, and then $n \in B$ iff $n \in B_s$. Since the use of this computation is n, it is an sw-reduction. $\dashv$

The above proof can easily be modified to show that if α is a left-c.e. real such that $\forall n\, [K(\alpha \upharpoonright n) > \varepsilon n]$ for some $\varepsilon > 0$, then α is wtt-complete.

There have been several recent results on sw-reducibility. For instance, Barmpalias [5] showed that there are no sw-maximal c.e. sets, and Barmpalias and Lewis [6] showed that there are left-c.e. reals that are not sw-below any 1-random left-c.e. real (cf. the comment following Theorem 12.1). It is not known whether there is a maximal sw-degree, although Lewis and Barmpalias [80] showed that no 1-random set can have maximal sw-degree. On the other hand, they also showed that there are sw-degrees $\deg_{\mathrm{sw}}(A)$ that are *quasi-maximal*, in the sense that if $A \leqslant_{\mathrm{sw}} B$ then $B \equiv_{\mathrm{T}} A$. Indeed, they showed that every 1-random set has quasi-maximal sw-degree.

Both S-reducibility and sw-reducibility are uniform in a way that relative initial-segment complexity is not. Motivated by this idea, Downey, Hirschfeldt, and LaForte [35] introduced the following notion.

DEFINITION 5.6. Let A and B be sets. We say that B is *relative K-reducible* (rK-reducible) to A, and write $B \leqslant_{\mathrm{rK}} A$, if there exist a partial computable binary function f and a constant k such that for each n there is a $j \leqslant k$ for which $f(A \upharpoonright n, j)\downarrow = B \upharpoonright n$.

THEOREM 5.7 (Downey, Hirschfeldt, and LaForte [35]).

(i) *$\leqslant_{rK}$ is a K-measure of relative randomness.*
(ii) *If $A \leqslant_{sw} B$, then $A \leqslant_{rK} B$.*
(ii) *If α and β are left-c.e. reals and $\alpha \leqslant_S \beta$, then $\alpha \leqslant_{rK} \beta$.*
(iii) *A left-c.e. real α is rK-complete iff it is 1-random.*
(iv) *If $A \leqslant_{rK} B$ then $A \leqslant_T B$.*

The most interesting characterization of rK-reducibility (and the reason for its name) is given by the following result, which shows that there is a very natural sense in which rK-reducibility is an exact measure of relative randomness. The prefix-free complexity $K(\tau \mid \sigma)$ of τ *relative to* σ is the length of the shortest string μ such that $U^\sigma(\mu) = \tau$, where U is a prefix-free machine that is universal with respect to any oracle. We can define $C(\tau \mid \sigma)$ analogously.

THEOREM 5.8 (Downey, Hirschfeldt, and LaForte [35]). *Let A and B be sets. Then $A \leqslant_{rK} B$ iff $K(A \upharpoonright n \mid B \upharpoonright n) \leqslant O(1)$ (or, equivalently, $C(A \upharpoonright n \mid B \upharpoonright n) \leqslant O(1)$).*

The rK-degrees have many of the same nice structural properties as the S-degrees.

THEOREM 5.9 (Downey, Hirschfeldt, and LaForte [35]).

(i) *The rK-degrees of left-c.e. reals form an upper semilattice with least degree that of the computable sets and highest degree that of Ω.*
(ii) *For left-c.e. reals α and β, we have $\deg_{rK}(\alpha) \vee \deg_{rK}(\beta) = \deg_{rK}(\alpha + \beta)$.*
(iii) *The rK-degrees of left-c.e. reals are dense.*
(iv) *For any rK-degrees $\mathbf{a} < \mathbf{b} < \deg_{rK}(\Omega)$ of left-c.e. reals, there are rK-degrees $\mathbf{c_0}$ and $\mathbf{c_1}$ of left-c.e. reals such that $\mathbf{a} < \mathbf{c_0}, \mathbf{c_1} < \mathbf{b}$ and $\mathbf{c_0} \vee \mathbf{c_1} = \mathbf{b}$.*

Of course, Theorem 4.7 (iii) implies that $\deg_{\mathrm{rK}}(\Omega)$ cannot be split in the rK-degrees of left-c.e. reals. We do not know whether the rK-degrees are distributive. The theories of the sw- and rK-degrees have not yet been proved to be undecidable, though this must surely be the case.

5.2. The basic measures $\leqslant_K$ and $\leqslant_C$. Of course, we are particularly interested in the measure of relative complexity *defined* by the Solovay property: We say that A is *K-reducible* to B, and write $A \leqslant_K B$, if

$$K(A \upharpoonright n) \leqslant K(B \upharpoonright n) + O(1).$$

The preordering $\leqslant_C$ is defined analogously. Note that $\leqslant_K$ is not really a *reducibility*, but simply a transitive preordering measuring relative complexity. This is best seen by the following result.

THEOREM 5.10 (Yu, Ding, and Downey [144]). $|\{A : A \leqslant_K \Omega\}| = 2^{\aleph_0}$.

In Theorem 5.10, we can replace Ω by any 1-random set. In Section 13 we will see that Joe Miller has proved that the K-degree of any 1-random set is countable. (This is consistent, since there is no natural join operator on the K-degrees, or indeed any join operator at all; see Corollary 13.4.)

In Section 6 we will see that $\leqslant_K$ does not imply $\leqslant_{\mathrm{T}}$, even on the c.e. sets. Interestingly, $\leqslant_C$ does imply $\leqslant_{\mathrm{T}}$ on the left-c.e. reals.

THEOREM 5.11 (Stephan (personal communication); see [34]). *If α and β are left-c.e. reals such that $\alpha \leqslant_C \beta$, then $\alpha \leqslant_T \beta$.*

Theorem 5.11 generalizes an old result of Chaitin [20] (which generalizes an even older result of Loveland [82]).

THEOREM 5.12 (Chaitin [20]). *Suppose that either* $C(A \upharpoonright n) \leqslant C(n) + d$ *for an infinite computable set of* n *or* $C(A \upharpoonright n) \leqslant \log n + d$ *for an infinite computable set of* n*. Then* A *is computable. Furthermore, for a given constant* d*, there are only* $O(2^d)$ *many such* A*.*

Before we turn to the very interesting relationship of $\leqslant_K$ to $\leqslant_{\mathrm{T}}$, we look at the structure of the left-c.e. reals under $\leqslant_K$. There are obvious similarities between Theorems 4.7 and 5.9. Downey and Hirschfeldt [34] have proved the following generalization, which applies to $\leqslant_K$ in particular.

THEOREM 5.13 (Downey and Hirschfeldt [34]). *Let* $\leqslant_Q$ *be any* K*-measure of relative randomness with a* Σ^0_3 *definition on the left-c.e. reals, such that the least* Q*-degree of left-c.e. reals contains the computable reals, the top* Q*-degree of left-c.e. reals is that of* Ω*, and* $+$ *induces a join on the* Q*-degrees of left-c.e. reals. Then the following hold.*

(i) *The* Q*-degrees of left-c.e. reals are dense.*
(ii) *For any* Q*-degrees* $\mathbf{a} < \mathbf{b} < \deg_Q(\Omega)$ *of left-c.e. reals, there are* Q*-degrees* $\mathbf{c_0}$ *and* $\mathbf{c_1}$ *of left-c.e. reals such that* $\mathbf{a} < \mathbf{c_0}, \mathbf{c_1} < \mathbf{b}$ *and* $\mathbf{c_0} \vee \mathbf{c_1} = \mathbf{b}$.

At a talk by the first author in Heidelberg, in May 2003, Alexander Shen pointed out that a natural measure of relative randomness could be obtained by replacing the constant in the definition of K-reducibility by $O(\log n)$. That is, he suggested considering the ordering defined by letting $A \leqslant B$ if $K(A \upharpoonright n) \leqslant K(B \upharpoonright n) + O(\log n)$. The reason for this suggestion is that various approaches to defining relative randomness are equivalent up to a log factor, and hence this definition would be independent of the choice of approach. We will not discuss this line of research here, but point to a paper by Chernov, Muchnik, Romashchenko, Shen, and Vereshchagin [24]. Notice that this ordering is still Σ^0_3.

The measures $\leqslant_C$ and $\leqslant_K$ seemed at first difficult to deal with directly, and even now there is much about them that is not known. In view of Theorem 5.10, it was not even clear whether there are uncountably many K-degrees. This question was recently solved by showing that while the cardinality of the collection of sets K-below a given set can be large, its measure is always small.

THEOREM 5.14 (Yu, Ding, and Downey [144]). *For any set* B*, we have* $\mu(\{A : A \leqslant_K B\}) = 0$*. Hence there are uncountably many* K*-degrees of* 1*-random sets.*

Using Theorem 5.14, Yu and Ding [143] established the following result.

THEOREM 5.15 (Yu and Ding [143]). *There are* $2^{\aleph_0}$ *many* K*-degrees of* 1*-random sets.*

It was later noticed that this result follows directly from Theorem 5.14 by Silver's Theorem [124] that any coanalytic equivalence relation on 2^ω

with uncountably many equivalence classes has continuum many equivalence classes. Of course, it also follows from Miller's result (Theorem 13.9) that the K-degree of any 1-random set is countable.

The following are examples of basic questions about the K-degrees that remain open. (A *minimal pair* is a pair of degrees $\mathbf{a}, \mathbf{b}$ such that if $\mathbf{c}$ is below both $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{c} = \mathbf{0}$.)

QUESTION 5.16. Are there minimal pairs of K-degrees of left-c.e. reals? Do the K-degrees of left-c.e. reals form a lattice?

For reducibilities such as $\leqslant_{\mathrm{S}}$ and $\leqslant_{\mathrm{rK}}$, the existence of minimal pairs follows from the existence of minimal pairs in the Turing degrees. A minimal pair of K-degrees (not containing left-c.e. reals) was recently constructed by Csima and Montalbán [27], using Theorem 6.4 below.

In Downey and Hirschfeldt [34] it is shown that neither the S-degrees nor the rK-degrees of left-c.e. reals form a lattice, by a straightforward adaptation of Jockusch's proof [55] of the corresponding fact for the wtt-degrees of left-c.e. reals.

There are a number of exciting recent results on $\leqslant_K$ and $\leqslant_C$ due to Liang Yu and Joe Miller. We will discuss some of these in Section 13.

5.3. Other ways to compare randomness. It is possible to define measures of relative randomness based on other approaches to randomness. In unpublished work, Downey, Griffiths, and Hirschfeldt studied *supermartingale reducibility*, where $A \leqslant_{\mathrm{su}} B$ if $d(B \upharpoonright n) = O(d(A \upharpoonright n))$, where d is an optimal c.e. supermartingale, as defined in Section 3.2. Clearly, there is a greatest su-degree, consisting of the 1-random sets (which implies that $\leqslant_{\mathrm{su}}$ is not a K-measure of relative randomness). Downey, Griffiths, and Hirschfeldt showed that the computable sets form the least su-degree, and that addition induces a join on the su-degrees of left-c.e. reals. Thus Theorem 5.13 applies to $\leqslant_{\mathrm{su}}$. It is not known whether there is an exact characterization of $\leqslant_{\mathrm{su}}$ in terms of initial segment complexity.

It would be interesting to define a measure of relative randomness based on the measure-theoretic approach.

§6. K-triviality, Post's Problem, and generalizing the Kučera-Slaman Theorem.

6.1. K-trivial sets. We return to the fascinating interrelationship between $\leqslant_K$ and $\leqslant_{\mathrm{T}}$. A natural question is whether $\leqslant_K$ implies $\leqslant_{\mathrm{T}}$. We have seen that if $A \leqslant_C \emptyset$ then A must be computable. Using a relativization of the method of the proof of this fact, Chaitin showed the following.

THEOREM 6.1 (Chaitin [21]). *If $K(A \upharpoonright x) \leqslant K(x) + O(1)$, then $A \leqslant_T \emptyset'$.*

Surprisingly, we cannot replace $\emptyset'$ by $\emptyset$ in the above result. That is, although A may look identical to the computable sets in terms of initial segment prefix-free complexity, we cannot conclude that A is computable, even for c.e. sets A.

We say that a set A is *K-trivial* if $A \leqslant_K \emptyset$. Solovay [129] was the first to construct a noncomputable K-trivial set; this construction was adapted to the case of c.e. sets by Zambella [145] (see also Calude and Coles [15]). In [39], Downey, Hirschfeldt, Nies, and Stephan gave a new construction of a noncomputable K-trivial c.e. set, which we present below. (A similar construction had been produced independently by Kummer in unpublished work.) As we will later see, this construction gives a priority-free, and even a requirement-free, solution to Post's Problem.

THEOREM 6.2 (Zambella [145], after Solovay [129]). *There is a noncomputable c.e. set A such that $K(A \upharpoonright n) \leqslant K(n) + O(1)$.*

REMARK. While the proof below is easy, it is slightly hard to see why it works. So, by way of motivation, suppose that we were to asked to "prove" that $\emptyset$ has the same initial segment complexity complexity as ω. A complicated way to do this would be to build our own prefix-free machine M whose only job is to compute initial segments of $\emptyset$. The idea would be that if the universal prefix-free machine U converges to 1^n on input σ then $M(\sigma)\downarrow = 0^n$. Notice that, in fact, using the Kraft-Chaitin Theorem it would be enough to build M *implicitly*, enumerating the length request $\langle|\sigma|, 0^n\rangle$. We are guaranteed that $\sum_{\tau\in\mathrm{dom}(M)} 2^{-|\tau|} \leqslant \sum_{\sigma\in\mathrm{dom}(U)} 2^{-|\sigma|} \leqslant 1$, and hence the Kraft-Chaitin Theorem applies. Note also that we could, for convenience and as we do in the main construction, use a string of length $|\sigma| + 1$, in which case we would ensure that $\sum_{\tau\in\mathrm{dom}(M)} 2^{-|\tau|} < 1/2$.

PROOF OF THEOREM 6.2. We will build a noncomputable c.e. set A in place of $\emptyset$ in the remark and, as above, we will slavishly follow the universal prefix-free machine U on n in the sense that whenever U enumerates, at stage s, a shorter σ with $U(\sigma) = n$, we will enumerate a request $\langle|\sigma| + 1, A_s \upharpoonright n\rangle$ for our machine M. To make A noncomputable, we will also sometimes make $A_s \upharpoonright n \neq A_{s+1} \upharpoonright n$. Then for each j with $n \leqslant j \leqslant s$, for the currently shortest string σ_j computing j, we will also need to enumerate a request $\langle|\sigma_j|, A_{s+1} \upharpoonright j\rangle$ for M. The construction works by making this extra measure added to the domain of M small.

We are ready to define A:

$$A := \{n : \exists e\, \exists s\, [W_{e,s} \cap A_s = \emptyset \wedge n > 2e \wedge n \in W_{e,s} \wedge \sum_{n\leqslant j\leqslant s} 2^{-K_s(j)} < 2^{-(e+2)}]\},$$

where $W_{e,s}$ is the stage s approximation to the e-th c.e. set W_e and $K_s(j)$ is the stage s approximation to the K-complexity of j.

Clearly A is c.e. Since $\sum_{j \geqslant m} 2^{-K(j)}$ goes to zero as m increases, if W_e is infinite then $A \cap W_e \neq \emptyset$. Since A is also coinfinite, this implies that A is noncomputable. Finally, the extra measure put into the domain of M, beyond one half of that which enters the domain of U, is bounded by $\sum_e 2^{-(e+2)}$ (corresponding to at most one initial segment change for each e), whence

$$\sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|} \leqslant \sum_{\sigma \in \mathrm{dom}(U)} 2^{-(|\sigma|+1)} + \sum_e 2^{-(e+2)} \leqslant \frac{1}{2} + \frac{1}{2} = 1.$$

So the Kraft-Chaitin Theorem applies, and M is a well-defined prefix-free machine. Thus $K(A \upharpoonright n) \leqslant K(n) + O(1)$. $\dashv$

The above proof can be modified to prove the result, due to Muchnik (see [12]), that there exists a noncomputable c.e. set A that is *low for* K, in the sense that K-complexity relativized to A is the same as K-complexity, up to an additive constant, i.e., $K(\sigma) \leqslant K^A(\sigma) + O(1)$. Such an A is both K-trivial and low for 1-randomness (which will be formally defined in Section 7, but means that relativizing the definition of 1-randomness to A does not change the class of 1-random sets). We will later see that these concepts are all equivalent, as shown by the work of Nies (see Sections 7 and 8).

Clearly the above proof also admits many variations. For instance, we can make A promptly simple, or below any nonzero c.e. degree. We cannot control the jump or make A Turing complete, since the K-trivial sets are nonhigh, as shown by Downey, Hirschfeldt, Nies, and Stephan [39] (and in fact low, as shown by Nies [105]); see Sections 6.2, 7, and 8.

As we will see in Section 6.2, the construction above automatically yields a Turing incomplete c.e. set. It is thus an *injury-free* solution to Post's Problem. It is not, however, *priority-free*, in that it depends on an ordering of the simplicity requirements, with stronger requirements allowed to use up more of the domain of the machine M. We can do methodologically better by giving a priority-free solution to Post's Problem, in the sense that no explicit diagonalization (such as that of W_e above) occurs in the construction of the incomplete c.e. set, and therefore the construction of this set (as opposed to the verification that it is noncomputable) does not depend on an ordering of requirements. We now sketch this method, which is due to Downey, Hirschfeldt, Nies, and Stephan [39], and is rather more like that of Solovay's original proof of the existence of a noncomputable K-trivial set.

Let us reconsider the key idea in the proof of Theorem 6.2. At certain stages we wish to change an initial segment of A for the sake of diagonalization. Our method is to make sure that the total measure added to the domain of our machine M (which proves the K-triviality of A) due to such changes is bounded by 1. Suppose, on the other hand, that we were fortunate enough to have the universal machine itself "cover" the measure needed for these changes. That is, suppose we were at a stage s where we desired to put n into $A_{s+1} - A_s$, and at that very stage $K_s(j)$ changed for all $j \in \{n, \ldots, s\}$. That means that *in any case* we would need to enumerate new requests describing $A_{s+1} \upharpoonright j$ for all $j \in \{n, \ldots, s\}$, whether or not these initial segments change. Thus at that very stage, we could also change $A_s \upharpoonright j$ for all $j \in \{n, \ldots, s\}$ at no extra cost.

Notice that we would not need to copy the universal prefix-free machine U at every stage. We could enumerate a collection of stages $t_0, t_1, \ldots$ and only update M at stages t_i. Thus, for the lucky situation outlined above to occur, we would only need the approximation to $K(j)$ to change for all $j \in \{n, \ldots, t_s\}$ at some stage u with $t_s \leqslant u \leqslant t_{s+1}$. This observation would seem to allow a greater possibility for this lucky situation to occur, since many stages can occur between t_s and t_{s+1}.

The key point in this discussion is the following. Let $t_0, t_1, \ldots$ be a computable collection of stages. *Suppose that we construct a set* $A = \bigcup_s A_{t_s}$ *so that for* $n \leqslant t_s$, *if* $A_{t_{s+1}} \upharpoonright n \neq A_{t_s} \upharpoonright n$ *then* $K_{t_s}(j) > K_{t_{s+1}}(j)$ *for all* j *with* $n \leqslant j \leqslant t_s$. *Then* A *is* K*-trivial.* We are now ready to define A in a priority-free way.

A PRIORITY-FREE SOLUTION TO POST'S PROBLEM. Let $t_0, t_1, \ldots$ be a collection of stages such that t_i as a function of i dominates all primitive recursive functions. (Actually, we do not need $i \mapsto t_i$ to be quite this fast growing; see below for more details.) At each stage u, let $\{a_{i,u} : i \in \omega\}$ list $\overline{A}_u$. Define

$$A_{t_{s+1}} = A_{t_s} \cup \{a_{n,t_s}, \ldots, t_s\},$$

where n is the least number $\leqslant t_s$ such that $K_{t_{s+1}}(j) < K_{t_s}(j)$ for all $j \in \{n, \ldots, t_s\}$. (Naturally, if no such n exists, $A_{t_{s+1}} = A_{t_s}$.) Requiring the complexity change for all $j \in \{n, \ldots, t_s\}$, rather than just $j \in \{a_{n,t_s}, \ldots, t_s\}$, ensures that A is coinfinite, since for each n there are only finitely many s such that $K_{t_{s+1}}(n) < K_{t_s}(n)$.

Note that there is no priority used in the definition of A. It is like the Dekker deficiency set or the so-called "dump set" (see Soare [125], Theorem V.2.5).

It remains to prove that A is noncomputable. By the Recursion Theorem, we can build a prefix-free Turing machine M and know the coding constant c of M in the universal prefix-free machine U. That is, if we declare $M(\sigma) = j$ then we will have $U(\tau) = j$ for some τ such that $|\tau| \leqslant |\sigma| + c$. Note further that if we put σ into the domain of M at stage t_s, then τ will be in the domain of U by stage $t_{s+1} - 1$. (This is why we required $i \mapsto t_i$ to dominate the primitive recursive functions. In fact, we only need this function to dominate the overhead of the Recursion Theorem; that is, we only need the property that if σ enters the domain of M at stage t_s, then there is a τ such that $|\tau| \leqslant |\sigma| + c$ and

$U_{t_{s+1}-1}(\tau)\downarrow = M(\sigma)$. The use of a fast growing sequence of stages was the key insight in Solovay's original construction.)

Now the proof looks like that of Theorem 6.2. We devote $2^{-(e+1)}$ of the domain of our machine M to ensuring that A satisfies the e-th simplicity requirement. When we see a_{n,t_s} occur in W_{e,t_s}, where $\sum_{n\leqslant j\leqslant t_s} 2^{-K_{t_s}(j)} < 2^{-(e+c+2)}$, we provide shorter M_{t_s} descriptions of all j with $n \leqslant j \leqslant t_s$ so that $K_{t_{s+1}}(j) < K_{t_s}(j)$ for all such j. The cost of this change is bounded by $2^{-(e+1)}$, and a_{n,t_s} will enter $A_{t_{s+1}}$, as required. ⊣

While the above proof does make use of an ordering of the simplicity requirements, it does so only in the verification of the fact that A is not computable, and not in the construction of A, which remains priority-free.

One remarkable fact about the K-trivial sets is that there are few of them for any given witnessing constant (cf. Theorem 5.12).

THEOREM 6.3 (Zambella [145]). *For each d there are $O(2^d)$ many sets A such that $K(A \upharpoonright n) \leqslant K(n) + d$ for all n.*

Proofs of this theorem can be found in [39] and [34].

Recently, Csima and Montalbán [27] have shown that there is a gap between the initial segment complexity of K-trivial sets and that of non-K-trivial sets, mirroring the gap in initial segment complexity between 1-random sets and non-1-random sets illustrated by results such as Theorem 3.10.

THEOREM 6.4 (Csima and Montalbán [27]). *There is a nondecreasing unbounded function f such that if A is not K-trivial then $K(A \upharpoonright n) > K(n) + f(n) - O(1)$.*

As mentioned above, Csima and Montalbán [27] used this result to show that there is a minimal pair of K-degrees.

6.2. K-trivial sets solve Post's Problem. While we will see improvements on the following results when we consider Nies' work in Sections 7 and 8, we remark that K-trivial sets are necessarily Turing incomplete, and indeed not of high degree, and hence form a somewhat natural solution to Post's Problem.

THEOREM 6.5 (Downey, Hirschfeldt, Nies, and Stephan [39]). *If a set A is K-trivial then A is Turing incomplete. Indeed, A is not even high.*

We discuss the proof of this theorem in Section 8. The original proof can be found in [39].

Downey, Hirschfeldt, Nies, and Stephan [39] proved that the class of K-trivial sets is closed under wtt-reduction and $\oplus$. The former will be improved to Turing reduction in Section 8. As Nies has remarked, this means that the K-trivial c.e. sets are the only known natural nontrivial Σ^0_3 ideal in the (c.e.) Turing degrees.

Note also that, since $\leqslant_C$ implies $\leqslant_{\mathrm{T}}$ for c.e. sets, the existence of noncomputable K-trivial c.e. sets means that $\leqslant_K$ does not imply $\leqslant_C$.

QUESTION 6.6. Does $\leqslant_C$ imply $\leqslant_K$ (for the left-c.e. reals)? We conjecture that the answer is no.

6.3. Generalizing Kučera-Slaman. Theorem 4.6 implies that there is essentially only one 1-random left-c.e. real, and it is Ω. So the strongest possible extension for the left-c.e. reals of the fact that there are noncomputable K-trivial sets would be that if α is a nonrandom left-c.e. real then there is a left-c.e. real $\beta \equiv_K \alpha$ such that $\beta \nleqslant_{\mathrm{T}} \alpha$. The following theorem clarifies the situation.

THEOREM 6.7 (Downey and Yang [44]). *Suppose that a left-c.e. real α has the property that for every left-c.e. real $\beta \leqslant_K \alpha$, we have $\beta \leqslant_T \alpha$. Then α is Turing complete.*

Notice that there are such nonrandom left-c.e. reals α. Indeed we can take $\alpha = \emptyset'$, since $\beta \leqslant_{\mathrm{wtt}} \emptyset'$ for every left-c.e. real β.

§7. Lowness for 1-randomness. In this section, we discuss relativized randomness and lowness properties for the class of 1-random sets.

Relativized randomness was studied by several authors, including van Lambalgen [74], Kurtz [73], and Kautz [61]. The definition of 1-randomness relative to A is obtained by substituting "c.e." by "A-c.e." in Definition 3.1. That is, X is 1-random relative to A if there is no uniformly A-c.e. sequence $\{U_i\}_{i\in\omega}$ of Σ_1^A-classes with $\mu(U_i) \leqslant 2^{-i}$ such that $X \in \bigcap_i U_i$. We can similarly relativize notions such as c.e. martingale and prefix-free Kolmogorov complexity, and obtain the relativized versions of Theorems 3.4 and 3.8.

In computability theory a set A is called *low* if $A' \leqslant_{\mathrm{T}} \emptyset'$, where A' is the halting problem relativized to A; that is, if the complexity of the halting problem does not increase when relativized to A (and hence the class of Δ_2^0 sets does not change when relativized to A). In complexity theory, if a class $\mathcal{C}$ has a definition that relativizes, a set A is called *low for* $\mathcal{C}$ (or $\mathcal{C}$*-low*) if $\mathcal{C} = \mathcal{C}^A$. So the low sets from computability theory are those that are low for the class of Δ_2^0 sets. Similarly, a set A such that every 1-random set is 1-random relative to A is called *low for the* 1*-random sets*, or *low for* 1*-randomness*.

Clearly, every computable set is low for 1-randomness. M. van Lambalgen and D. Zambella asked whether there exist noncomputable sets that are low for 1-randomness. (The question was first stated in Zambella [145].) This question was raised in the context of a comparison between randomness properties in classical dynamic systems (specifically, Bernoulli sequences) and computability-theoretic randomness. A result of Kamae [59] showed that the infinite binary sequences that have no information about Bernoulli sequences (normal sequences) are precisely those with zero entropy. This fact raised the issue of whether a similar characterization exists for sets that have no information about 1-random sets, and motivated the question of the existence of noncomputable sets that are low for 1-randomness, which was answered by Kučera and Terwijn [71].

THEOREM 7.1 (Kučera and Terwijn [71]). *There exists a noncomputable c.e. set that is low for* 1*-randomness.*

PROOF. It is not difficult to build a c.e. operator I such that I^A is a universal Solovay test relative to A for every oracle A. (Here we think of Solovay tests as collections of basic clopen sets.) Given a set A, we can attempt to cover I^A with an unrelativized Solovay test J by adding $[\sigma]$ to J at stage s whenever $[\sigma]$ is in I^{A_s} at stage s. We then have $J \supseteq I^A$, but we also need to build A to ensure that $\mu(\bigcup J) = \sum_{[\sigma]\in J} \mu([\sigma]) < \infty$, and hence J is a Solovay test. That is, we need to build A so that the total measure of the "mistakes" we make in approximating I^A is not too big.

The crucial idea comes from [71]: Let $M_s(n)$ be the collection of σ which are in I^{A_s} at stage s with use greater than n. If $n \notin A_s$ and we enumerate n into A, then the elements of $M_s(n)$ may be in $J - I^A$, where J is as above, so we need to ensure that $\mu(\bigcup M_s(n))$ for $n \in A_{s+1} - A_s$ is small, while still making A noncomputable. Thus at each stage s, for the least $e < s$ such that $A_s \cap W_{e,s} = \emptyset$, if there is an $n \in W_{e,s}$ such that $n > 2e$ and $\mu(\bigcup M_s(n)) < 2^{-e}$, then we put the least such n into A.

It is easy to see that such an enumeration can happen at most once for each e, and hence the total measure of our mistakes, namely the sets in $J - I^A$, is bounded by $\sum_e 2^{-e}$, which implies that J is a Solovay test.

If X is not 1-random relative to A then X is in infinitely many elements of I^A, and hence X is in infinitely many elements of J, which implies that X is not 1-random. Thus A is low for 1-randomness, so the only thing left to prove is that A is noncomputable. Since A is clearly coinfinite, it is enough to show that if W_e is infinite then $A \cap W_e \neq \emptyset$.

Given e such that W_e is infinite, let s be a stage such that for all $i < e$ we have $A \cap W_i \neq \emptyset \Rightarrow A_s \cap W_{i,s} \neq \emptyset$. Suppose for a contradiction that $A \cap W_e = \emptyset$. It is easy to build sequences $2e < n_0 < n_1 < n_2 < \cdots$ and $s \leqslant s_0 < s_1 < s_2 < \cdots$ such that the $M_{s_i}(n_i)$ are pairwise disjoint and $n_i \in W_{e,s_i}$ for each i. Since $n_i \notin A$ for all i, it must be the case that $\mu(\bigcup M_{s_i}(n_i)) \geqslant 2^{-e}$ for all i. But $M_{s_i}(n_i) \subset J$ for all i, so $\mu(\bigcup J) = \infty$, contradicting the fact that J is a Solovay test. ⊣

Kučera and Terwijn left open questions about the possible complexity of sets that are low for 1-randomness. As we will see, this complexity is restricted in various ways. For instance, if A is low for 1-randomness then it is GL_1, meaning that $A' \leqslant_{\mathrm{T}} A \oplus \emptyset'$ (Kučera [69]; see also Corollary 7.8), and has certain traceability properties.

Definition 7.2 (Zambella, see [135]). A set A is *c.e.-traceable* if there is a computable function p such that, for each function $f \leqslant_{\mathrm{T}} A$, there is a computable function h (called a *trace* for f) satisfying, for all n,

(i) $|W_{h(n)}| \leqslant p(n)$ and
(ii) $f(n) \in W_{h(n)}$.

It is not hard to check that the above definition does not change if we replace "there is a computable function p" by "for every unbounded nondecreasing computable function p" (cf. Proposition 11.2). Since one can uniformly enumerate all c.e. traces for a fixed bound p, there is a universal trace with bound p, that is, one that traces each function $f \leqslant_{\mathrm{T}} A$ on almost all inputs.

Theorem 7.3 (Terwijn and Zambella [137]). *If A is low for 1-randomness then A is c.e.-traceable.*

Nies [104] has recently improved this result by replacing c.e.-traceability with a stronger property called jump-traceability; see [104] for details.

Kučera, Terwijn, and Zambella asked whether all sets that are low for 1-randomness are Δ^0_2. It was also noted by Downey, Hirschfeldt, Nies, and Stephan [39] that the construction in the proof of Theorem 7.1 bears a close resemblance to that of a K-trivial set. This is no coincidence, as the two classes are in fact the same!

Theorem 7.4 (Nies [105]). *A set A is low for 1-randomness iff A is K-trivial.*

Both directions of this theorem are rather hard to prove, particularly the "if" direction, which will be discussed in Section 8. The proof of this direction is based on Nies' proof that the class of K-trivial sets is closed downward under Turing reducibility, and is necessarily nonuniform, in a sense that will be made precise below. As we will see in Section 8, it can be extended to show that every K-trivial set is low for K (as defined in Section 6.1).

The proof of the "only if" direction of Theorem 7.4 went through various stages; see Nies [106] for details. One approach is to convert the hypothesis that A is low for 1-randomness into a combinatorial condition, which can then be used to establish the K-triviality of A. For instance, an early version of the proof used the condition, due to Frank Stephan, that for some Σ^0_1 open set $R \subset 2^\omega$ of measure less than 1,

$$\exists b \in \omega \, \forall \sigma \in 2^{<\omega} \, [K^A(\sigma) \leqslant |\sigma| - b \ \Rightarrow \ [\sigma] \subseteq R],$$

where K^A denotes prefix-free complexity relative to A. This condition says in essence that A cannot be used to reduce the prefix-free complexity of very many strings. A more recent version of the proof uses martingales, and will be sketched in the proof of Theorem 11.12 below. We will see another approach to establishing this direction of Theorem 7.4 in Theorem 8.10 below.

For c.e. sets A, however, there is an easier proof. In fact, Nies, Stephan, and Terwijn [109] have shown that in this case it suffices to require that A be low for Ω, which is defined as follows.

DEFINITION 7.5. A set B is *low for* Ω if Ω is 1-random relative to B.

THEOREM 7.6 (Nies, Stephan, and Terwijn[109]). *If a c.e. set A is low for Ω then it is K-trivial.*

The hypothesis that A be c.e. cannot be entirely dropped. As we will see in Theorem 12.8 below, every 2-random set is low for Ω, so the class of sets that are low for Ω has measure 1! On the other hand, Theorem 8.10 will show that Theorem 7.6 can be extended to all Δ^0_2 sets A.

PROOF SKETCH OF THEOREM 7.6. We enumerate a Martin-Löf test $\{R^A_d\}_{d\in\omega}$ relative to A, and use the fact that there is a d such that $\Omega \notin R^A_d$ to define a Kraft-Chaitin set L_d that shows that A is K-trivial (that is, for each n there is a request $\langle r, A \restriction n\rangle \in L_d$ for some $r \leqslant K(n) + d + 1$).

We define L_d as a union $S \cup C_d$, where S supplies a new request whenever the approximation to $K(n)$ decreases, and C_d does the same when $A \restriction n$ changes (after some delay).

Let $S = \{\langle K_s(n) + 2, A_s \restriction n\rangle : n, s \in \omega \wedge K_s(n) < K_{s-1}(n)\}$. It is easy to check that S is a Kraft-Chaitin set of weight $\leqslant \Omega/2 < 1/2$.

When k enters A at stage s, we want to enumerate requests $\langle K_s(n) + d + 1, A_s \restriction n\rangle$ into C_d for each n with $k < n \leqslant s$. We also want to ensure that C_d is a Kraft-Chaitin set of weight at most $1/2$, which implies that $L_d = S \cup C_d$ is a Kraft-Chaitin set. To do so, we "force" the approximation to Ω to increase by $2^{-K_s(n)-d}$ before we put $\langle K_s(n)+d+1, A_s \restriction n\rangle$ into C_d. This allows the weight of requests enumerated into C_d to be "accounted against" increases of Ω. (That is, it ensures that increases in the weight of C_d are bounded by increases in the approximation to Ω, or more precisely, one half of such increases.)

The increase in Ω is achieved by putting an interval $[\Omega_s, \Omega_s + 2^{-K_s(n)-d}]$ into R^A_d with an A-use n. Either A changes below n (in which case we do not need the new request anymore, and have not enumerated it into R^A_d), or Ω has to move out of the interval, since we are assuming that $\Omega \notin R^A_d$. $\dashv$

Notice that this proof shares some elements with that of Kučera and Slaman [70] showing that each 1-random left-c.e. real is Solovay complete (Theorem 4.6).

From Theorem 7.4 we can conclude that all K-trivial sets are low, as we now see.

Theorem 7.7 (Nies and Stephan (unpublished)). *If a Δ^0_2 set A is B-random, then B is GL_1.*

Proof. Let $f(n)$ be the least s such that $\forall t \geqslant s\,[A_t \upharpoonright n = A_s \upharpoonright n]$. Note that $f \leqslant_{\mathrm{T}} \emptyset'$. Let $\widehat{\mathcal{R}}_e$ be the open set $[A_{s_e} \upharpoonright e+1]$ where s_e is the stage at which $\Phi^B_e(e)$ converges (or $\widehat{\mathcal{R}}_e = \emptyset$ if $\Phi^B_e(e)\uparrow$). Let $\mathcal{R}_i = \bigcup_{e \geqslant i} \widehat{\mathcal{R}}_e$. Clearly, $\{\mathcal{R}_i\}_{i\in\omega}$ is a Martin-Löf test relative to B. Since $A \notin \bigcap_i \mathcal{R}_i$, only finitely many $\widehat{\mathcal{R}}_e$'s contain A. Thus $f(e) \geqslant s_e$ for almost all e such that $\Phi^B_e(e)\downarrow$. So using f and B, we can compute whether $\Phi^B_e(e)\downarrow$, which implies that $B' \leqslant_{\mathrm{T}} B \oplus \emptyset'$. ⊣

Applying this result with $A = \Omega$, we obtain the following corollary, first proved by Nies, Stephan, and Terwijn [109].

Corollary 7.8 (Nies, Stephan, and Terwijn [109]). *Every set that is low for Ω is GL_1.*

Since all K-trivial sets are Δ^0_2, by Theorem 7.4 we have the following result.

Corollary 7.9 (Nies [105]). *Every K-trivial set is low.*

§8. Characterizing the K-trivial sets. In this section we discuss the harder direction of Theorem 7.4, along with several related results about the class $\mathcal{K}$ of K-trivial sets. These results indicate that $\mathcal{K}$ is a robust class, in the sense that it captures several different intuitive notions of computational weakness related to randomness, and is of great computability-theoretic interest.

Recall from Section 6.1 that A is low for K if $K(\sigma) \leqslant K^A(\sigma) + O(1)$; that is, using A as an oracle does not help us to reduce the prefix-free complexity of any string (up to a constant). Let $\mathcal{M}$ denote the class of sets that are low for K. This class was studied by Muchnik; as mentioned above, he showed that it contains a noncomputable set (see [12]).

If $A \leqslant_{\mathrm{T}} B$ then $K^B(\sigma) \leqslant K^A(\sigma) + O(1)$, so $\mathcal{M}$ is closed downward under $\leqslant_{\mathrm{T}}$. Since both 1-randomness and K-triviality are defined in terms of prefix-free complexity, if a set is low for K then it is both low for 1-randomness and K-trivial.

In Nies [105], Theorem 7.4 is proved in two separate pieces, by showing that both the class of sets that are low for 1-randomness and $\mathcal{K}$ actually coincide with $\mathcal{M}$. We have already discussed the fact that lowness for 1-randomness implies K-triviality (and will return to it at the end of this section, when we discuss bases for 1-randomness). The converse follows immediately from the following theorem.

Theorem 8.1 (Nies and Hirschfeldt, see Nies [105]). *Every K-trivial set is low for K.*

The proof of this result is a reasonably straightforward modification of that of the following theorem.

THEOREM 8.2 (Nies [105]). *The class $\mathcal{K}$ is closed downward under Turing reducibility.*

As we will discuss below, the proof of Theorem 8.2 also yields the result that the construction in the proof of Theorem 6.2 is in fact a characterization of $\mathcal{K}$, in that it is in essence the only way to obtain a K-trivial set!

PROOF SKETCH OF THEOREMS 8.1 AND 8.2. We first discuss the easier proof that every K-trivial set is Turing incomplete, but present it as a "projection" of the proofs of Theorems 8.1 and 8.2 in Nies [105]. We then briefly discuss the modifications needed to convert this argument into proofs of Theorems 8.1 and 8.2.

Assume for a contradiction that $\forall n\,[K(A \upharpoonright n) \leqslant K(n) + b]$ and A is Turing complete. By Theorem 6.1, we can choose a Δ^0_2 approximation $\{A_s\}_{s\in\omega}$ to A. We will enumerate requests into a Kraft-Chaitin set L (in fact enumerating at most one request $\langle r, n\rangle$ for each $n \in \omega$, with $r \in \omega$). The *weight* of $X \subseteq \omega$ is

$$\mathrm{wt}(X) := \sum\{2^{-r} : \langle r, n\rangle \in L \ \wedge\ n \in X\}.$$

This weight coincides with the measure of the corresponding descriptions given by the prefix-free machine M_d defined by L (using the Kraft-Chaitin Theorem). Roughly, we will enumerate appropriate requests $\langle r, n\rangle$ into L in order to ensure that $K(n) \leqslant r + d$. Since A is K-trivial via b, at some stage s after this enumeration the opponent has to give a short description of $A_s \upharpoonright n$, by ensuring that $U_s(\sigma) = A_s \upharpoonright n$ for some σ with $|\sigma| \leqslant r+d+b$, where U is the universal prefix-free machine that is being used to define K. We use the Turing completeness of A to make the approximation to $A \upharpoonright n$ change after stage s, so that the opponent has to come up with a short description of the new approximation to $A \upharpoonright n$. If we can change A often enough, then the measure corresponding to the opponent's descriptions exceeds 1, which is impossible.

More precisely, by the Recursion Theorem we can assume we have an index d such that M_d is a prefix-free machine corresponding to L in the sense of the Kraft-Chaitin Theorem. If we let $c = b + d$, then as an answer to our enumeration of the request $\langle r, n\rangle$ into L, the opponent has to provide a U-description of $A \upharpoonright n$ of length $\leqslant r + c$. Let $k = 2^{c+1}$. If we manage to put requests of total weight $1/2$ into L and also force the approximation to $A \upharpoonright n$ to change at least k times for each n mentioned in our requests, then the total measure of the opponent's descriptions will exceed 1. (Here we only count a change in the approximation to

$A \upharpoonright n$ if the opponent has provided a short description of the previous approximation to this initial segment.)

The proof is much easier if we assume that $\emptyset' \leqslant_{\mathrm{wtt}} A$. In this case, we build an auxiliary c.e. set B, and by the Recursion Theorem we can assume we are given a total wtt-reduction Γ such that $B = \Gamma^A$, with use bounded by a computable increasing function g. Let $n = g(k)$. We put the single request $\langle 0, n\rangle$ into L. (The weight of this single request is 1.) Each time $\Gamma^{A_s}(j)$ converges to $B_s(j)$ for all $j < k$ and the opponent provides a U-description of $A_s \upharpoonright n$ of length $\leqslant c$, we force the approximation to $A \upharpoonright n$ to change by putting into B the largest number $< k$ that is not yet in B. Once we reach k such changes, the total measure of U-descriptions is at least $k2^{-c} > 1$, which is a contradiction.

For the Turing case, we still build B and have a total reduction $\Gamma^A = B$ given by the Recursion Theorem, but now there is no computable bound on the use γ^A of Γ^A. The problem now is that, when we have an m such that $\gamma^{A_s}(m) = n$ and we put a request $\langle r, n\rangle$ into L, the opponent might, before providing a short description of $A_s \upharpoonright n$, move $\gamma^A(m)$ beyond n, thereby depriving us of the possibility of causing further changes in the approximation to $A \upharpoonright n$ by enumerating numbers $< m$ into B. Broadly speaking, the solution is to carry out many attempts, based on different computations $\Gamma^A(m)$. Each time the use of such a computation changes, some of what we placed in L for this attempt becomes "garbage", but as the reduction Γ is total, this only happens finitely often for each m. We have to ensure that the total weight of the garbage produced by all our attempts is limited, since otherwise L will not be a Kraft-Chaitin set.

For each s, we can effectively determine a stage $f(s) > s$ such that $\forall n < s\,[K_{f(s)}(A_{f(s)} \upharpoonright n) \leqslant K_{f(s)}(n) + b]$. Let $s_0 = 0$ and $s_{i+1} = f(s_i)$. The construction is restricted to *stages* in $\{s_i : i \in \omega\}$. The following is a way to keep track of the number of times the opponent has had to give new descriptions of approximations to $A \upharpoonright n$. We say that the number n is in a j-set if this has happened j times. More precisely, for $1 \leqslant j \leqslant k$, we say that a finite set $E \subseteq \omega$ is a *j-set at stage t* if, for all $n \in E$, at some *stage* $u < t$ a request $\langle r, n\rangle$ went into L and at stage t there are at least j distinct strings σ of length n such that $K_t(\sigma) \leqslant r + c$. A c.e. set E with an enumeration $E = \bigcup_t E_t$ is a *j-set* if E_t is a j-set at each *stage* t. In our construction, the strings σ will have the form $A_s \upharpoonright n$ for certain *stages* s with $u \leqslant s \leqslant t$. Since the opponent has to match every description of n we provide via L with descriptions of $A \upharpoonright n$ that are at most c longer, we have the following straightforward but important fact. (Recall that $k = 2^{c+1}$.)

If the c.e. set E is a k-set, then $\mathrm{wt}(E) \leqslant 1/2$.

As in the wtt case, our construction will build a k-set C_k of weight $> 1/2$ to reach a contradiction.

The procedure P_j $(2 \leqslant j \leqslant k)$ enumerates a j-set C_j. The construction begins by calling P_k, which calls P_{k-1} several times, and so on down to P_2, which enumerates L (and C_2).

Each procedure P_j has rational parameters $q, \beta \in [0,1]$. The *goal* q is the weight it wants C_j to reach, and the *garbage quota* β is how much it is allowed to waste.

We now describe the procedure $P_j(q,\beta)$, where $1 < j \leqslant k$, and the parameters $q = 2^{-x}$ and $\beta = 2^{-y}$ are such that $x \leqslant y$.

1. Choose m large.
2. Wait until $\Gamma^A(m)\downarrow$.
3. Let $v \geqslant 1$ be the number of times P_j has gone through step 2.

$j = 2$: Pick a large number n. Put $\langle r_n, n\rangle$ into L, where $2^{-r_n} = 2^{-v}\beta$. Wait for a *stage* t such that $K_t(n) \leqslant r_n + d$, and put n into C_1. (If M_d is a prefix-free machine corresponding to L, then t exists.)

$j > 2$: Call $P_{j-1}(2^{-v}\beta, \beta')$, where $\beta' = \min(\beta, 2^{j-k-w-1})$ and w is the number of P_{j-1} procedures started so far. (The most important point to understand here is that the goals of the P_{j-1} procedures called by a P_j procedure are related to the garbage quota of the P_j procedure, which ensures that even if all such procedures are canceled, the wasted measure will still be below this garbage quota. Another important point is that the garbage quotas are chosen so that their sum over all procedures started during the construction is less than $1/2$.)

In any case, if $\mathrm{wt}(C_{j-1}) < q$ then repeat step 3, and otherwise return.

4. Put m into B. This forces A to change below $\gamma(m) < \min(C_{j-1})$, and hence makes C_{j-1} a j-set (if we assume inductively that C_{j-1} is a $(j-1)$-set). So put C_{j-1} into C_j, and declare $C_{j-1} = \emptyset$.

If $\gamma^A(m)$ changes during the execution of the loop at step 3, then cancel the run of all subprocedures, and go to step 2. Despite the cancelations, C_{j-1} is now a j-set because of this very change. (This is an important point, as it ensures that the measure associated with numbers already in C_{j-1} is not wasted.) So put C_{j-1} into C_j, and declare $C_{j-1} = \emptyset$.

This completes the description of the procedures. The construction consists of calling $P_k(1, 1/4)$. One can check that, because of the way the garbage quotas are chosen, L is a Kraft-Chaitin set. The set C_k is a k-set, and therefore should have weight at most $1/2$. But, since Γ^A is total, each procedure returns unless canceled, so the initial procedure P_k, which is never canceled, eventually ensures that C_k has weight $> 1/2$, which is a contradiction.

We can visualize this construction by thinking of a machine similar to Lerman's pinball machine (see [125, Chapter VIII.5]). However, since we enumerate rational quantities instead of single objects, we replace the balls in Lerman's machine by amounts of a precious liquid, say 1955 Biondi-Santi Brunello wine. Our machine consists of decanters $C_k, C_{k-1}, \dots, C_0$. At any stage C_j is a j-set. We put C_{j-1} above C_j so that C_{j-1} can be emptied into C_j. The height of a decanter is changeable. The procedure $P_j(q, \beta)$ wants to add weight q to C_j, by filling C_{j-1} up to q and then emptying it into C_j. The emptying corresponds to adding one more A-change.

The emptying device is a hook (the $\gamma^A(m)$-marker), which besides being used on purpose may go off finitely often by itself. When C_{j-1} is emptied into C_j then $C_{j-2}, \dots, C_0$ are spilled on the floor, since the new hooks emptying $C_{j-1}, \dots, C_0$ may be much longer (the $\gamma^A(m)$-marker may move to a much bigger position), and so we cannot use them any more to empty those decanters in their old positions.

We first pour wine into the highest decanter C_0, representing the left domain of L, in portions corresponding to the weight of requests entering L. We want to ensure that at least half the wine we put into C_0 reaches C_k. Recall that the parameter β is the amount of garbage $P_j(q, \beta)$ allows. If v is the number of times the emptying device has gone off by itself, then P_j lets P_{j-1} fill C_{j-1} in portions of size $2^{-v}\beta$. Then when C_{j-1} is emptied into C_j, at most $2^{-v}\beta$ much liquid can be lost because of being in higher decanters $C_{j-2}, \dots, C_0$. The procedure $P_2(q, \beta)$ is special but limits the garbage in the same way: it puts requests $\langle r_n, n\rangle$ into L where $2^{-r_n} = 2^{-v}\beta$. Once it sees the corresponding $A \upharpoonright n$ description, it empties C_0 into C_1 (but C_0 may be spilled on the floor before that because of a lower decanter being emptied).

We briefly sketch how to show that the class of K-trivial sets is closed downward under Turing reducibility. Let A be K-trivial and let a Turing reduction $B = \Gamma^A$ be given. We cannot change B at will any more, since we do not directly control it. However, if B does not change enough, we can build a Kraft-Chaitin set W showing that B is K-trivial. We now have a tree of runs of procedures. The root node is the single run of P_k, which as before tries to reach a k-set of weight 1. The leaves behave like the P_2 procedure above. A node $P_{j,\tau}$ $(2 < j \leqslant k)$ calls procedures $P_{j-1,\sigma}$, at stages where $U_s(\sigma) = m$ and $B_s = \Gamma^{A_s}$ converges on all inputs $< m$. For in this case we want to enumerate a request $\langle |\sigma| + d, B_s \upharpoonright m\rangle$ into the Kraft-Chaitin set W built at node $P_{j,\tau}$ (where d is an appropriate constant depending on the node $P_{j,\tau}$).

As before, once $P_{j-1,\sigma}$ returns, $P_{j,\tau}$ needs an $A \upharpoonright \gamma(m)$ change. If such a change happens sufficiently often then $P_{j,\tau}$ reaches its goal. Otherwise, the cost of changes of B, in the sense of the proof of Theorem 6.2, is small, so W is a Kraft-Chaitin set, showing that B is K-trivial. There must be a run on the tree where this cost is small (called the "golden run" in [105]), as otherwise the root node would reach its goal. However, the proof is nonuniform since one cannot identify the golden run effectively, as we will see below.

The proof of Theorem 8.1 proceeds in a similar way, except that $P_{j,\tau}$ calls procedures $P_{j-1,\sigma}$ based on computations $U^A(\sigma) = y$ (since we now want to enumerate requests $\langle |\sigma| + d, y \rangle$), and the marker $\gamma(m)$ is replaced by the use of this computation. The details can be found in Nies [105]. $\dashv$

We next discuss two theorems from Nies [105] that can be obtained by extending the methods in the proof of Theorem 8.2. The first shows that the construction in the proof of Theorem 6.2 actually provides a characterization of the K-trivial c.e. sets. That is, each K-trivial c.e. set A can be thought of as being built by such a construction, for an appropriate effective enumeration. The proof of this result uses the proof of downward closure of $\mathcal{K}$, for the special case of the identity functional. We state it here for c.e. sets, but a version for Δ^0_2 sets is also given in [105].

THEOREM 8.3 (Nies [105]). *For any c.e. set A, the following are equivalent.*

(i) *A is K-trivial.*
(ii) *There is a c.e. approximation $\{A_s\}_{s\in\omega}$ to A such that*
$$\sum\{\widetilde{c}(x,s) : x \text{ is minimal in } A_s(x) - A_{s-1}(x)\} < 1,$$
where $\widetilde{c}(x,s) = \sum_{x<y\leqslant s} 2^{-K_s(y)}$.

Note that $\widetilde{c}(x,s)$ is the cost of putting x into A at stage s.

As an application of this characterization in the Δ^0_2 case, one obtains the fact that K-triviality is, in essence, a notion about c.e. sets.

THEOREM 8.4 (Nies [105]). *For each K-trivial set A, there is a K-trivial c.e. set D such that $A \leqslant_{tt} D$.*

A further application of the methods in the proof of Theorem 8.2 is that there is a uniform listing of the c.e. sets in $\mathcal{K}$ that includes the constants via which K-triviality holds. (This result can be extended to all of $\mathcal{K}$; see [39].)

THEOREM 8.5 (Downey, Hirschfeldt, Nies, and Stephan [39]). *There is an effective listing $\{\langle\{B_{e,s}\}_{s\in\omega}, d_e\rangle\}_{e\in\omega}$ of c.e. approximations such that every K-trivial set occurs as a $B_e = \lim_s B_{e,s}$, and each B_e is K-trivial via the constant d_e.*

Nies [104] proved that Theorem 8.5 fails if one replaces the notion "K-trivial via d" by the notion "low for K via d". In other words, one can not list the c.e. sets B in $\mathcal{K}$ while also providing constants d such that $\forall y\,[K(y) \leqslant K^B(y) + d]$. The reason is that from such a constant one can effectively obtain an index for the lowness of B [105, Proposition 2.8]. On the other hand, for any sequence $\{B_e\}_{e\in\omega}$ of uniformly low c.e. sets, an extension of the construction in Theorem 6.2 provides an $A \in \mathcal{K} = \mathcal{M}$ that is not Turing below any B_e, and hence the sequence $\{B_e\}_{e\in\omega}$ does not exhaust $\mathcal{K}$. Thus the nonuniformity in the proof of Theorem 8.1 is necessary. Details can be found in [104, Theorem 5.9].

COROLLARY 8.6 (Nies [104]). *There is no effective way to obtain from a pair (A, d), where A is a c.e. set that is K-trivial via d, a constant $\widetilde{d}$ such that A is low for K via $\widetilde{d}$.*

We summarize the degree-theoretic properties of $\mathcal{K}$. Recall that A is ω-c.e. iff $A \leqslant_{\mathrm{wtt}} \emptyset'$. It follows from Theorem 8.4 that every K-trivial set is ω-c.e. In fact, as shown in Downey, Hirschfeldt, Miller, and Nies [37], every K-trivial set is a *d.c.e. real*, that is, the difference of two left-c.e. reals.

THEOREM 8.7 (Nies [105]). *The K-trivial sets form a nonprincipal Σ^0_3 ideal in the low ω-c.e. Turing degrees, which is generated by its c.e. members.*

PROOF. That $\mathcal{K}$ is an ideal follows from Theorem 8.2 and the closure of $\mathcal{K}$ under join mentioned in Section 6.2. By Corollary 7.9 and Theorem 8.4, this ideal is contained in the low ω-c.e. Turing degrees, and is generated by its c.e. members. The ideal is nonprincipal because, by the above-mentioned extension of the construction in the proof of Theorem 6.2, one can build a K-trivial set not Turing below a given low c.e. set. ⊣

COROLLARY 8.8 (Nies [105]). *There is a low$_2$ c.e. set E such that $A \leqslant_T E$ for every K-trivial set A.*

PROOF. By Theorem 8.4, it suffices to give such a bound E for the K-trivial c.e. sets. By work of Nies to be published in [34] and [107], any proper Σ^0_3 ideal in the c.e. degrees has a low$_2$ c.e. upper bound. ⊣

Another notion of computational weakness related to 1-randomness is that of bases for 1-randomness.

DEFINITION 8.9. A set A is a *basis for 1-randomness* if there is an $X \geqslant_{\mathrm{T}} A$ such that X is 1-random relative to A.

Kučera [67] and Gács [47] showed that every set can be computed by some 1-random set (see Theorem 12.1 below), so if A is low for 1-randomness then A is a basis for 1-randomness. In the other direction, Kučera [69] showed that every basis for 1-randomness is GL_1. More recently, Hirschfeldt, Nies, and Stephan [51] gave an exact characterization of the bases for 1-randomness.

THEOREM 8.10 (Hirschfeldt, Nies, and Stephan [51]). *A set is K-trivial iff it is a basis for 1-randomness.*

PROOF SKETCH. If a set is K-trivial then it is low for 1-randomness, and hence is a basis for 1-randomness.

The converse is proved by what has been called the "hungry sets" construction. Suppose that A is a basis for 1-randomness, and let Z and Φ be such that $\Phi^Z = A$ and Z is 1-random relative to A. We enumerate a Kraft-Chaitin set L_d for each $d \in \omega$. We want to ensure that there is a d such that L_d contains a request $\langle K(|\tau|) + d + 2, \tau\rangle$ for each $\tau \prec A$. The idea is to build sets $C_d^\tau \subseteq 2^\omega$ for $d \in \omega$ and $\tau \in 2^{<\omega}$ with the following properties. (These are the "hungry sets".)

(i) The C_d^τ are uniformly c.e.
(ii) For each fixed d, the C_d^τ are pairwise disjoint.
(iii) If we let $U_d = \bigcup_{\tau \prec A} C_d^\tau$, then the following hold.
  (a) $\{U_d\}_{d\in\omega}$ is a Martin-Löf test relative to A.
  (b) If $Z \notin U_d$ then $\mu(C_d^\tau) = 2^{-K(|\tau|)-d}$ for all $\tau \prec A$.

We then define L_d by enumerating a request $\langle K_s(|\tau|) + d + 2, \tau\rangle$ at stage s whenever we have not previously enumerated such a request and $\mu(C_d^\tau[s]) \geqslant 2^{-K_s(|\tau|)-d-1}$. Since the C_d^τ are pairwise disjoint, this is a Kraft-Chaitin set. Since Z is 1-random relative to A, we have $Z \notin U_d$ for some d and hence $\mu(C_d^\tau) = 2^{-K(|\tau|)-d}$ for all $\tau \prec A$, which implies that $\langle K(|\tau|) + d + 2, \tau\rangle \in L_d$ for all $\tau \prec A$, as desired.

To build the C_d^τ, as long as $\mu(C_d^\tau) < 2^{-K_s(|\tau|)-d}$, we look for strings σ such that $\tau \preccurlyeq \Phi^\sigma$ and $\mu(C_d^\tau) + 2^{-|\sigma|} \leqslant 2^{-K_s(|\tau|)-d}$, and put $[\sigma]$ into C_d^τ. To keep our sets pairwise disjoint, we then ensure that no $[\sigma']$ such that σ' is compatible with σ is later put into any C_d^ν. If $Z \notin U_d$, then no $[\sigma]$ with $\sigma \prec Z$ is ever put into any C_d^τ, which means that the measure of each C_d^τ with $\tau \prec A = \Phi^Z$ must eventually reach $2^{-K(|\tau|)-d-1}$. ⊣

One corollary of this result is the easier direction of Theorem 7.4, namely that every set that is low for 1-randomness is K-trivial. Another is an extension of Theorem 7.6: if A is Δ_2^0 and low for Ω, then it is a basis for 1-randomness, and hence K-trivial.

One way to look at Theorem 8.10 is in connection with the following classical theorem.

THEOREM 8.11. (de Leeuw, Moore, Shannon and Shapiro [76], Sacks [119]). *If A is not computable then $\mu(\{X : X \geqslant_T A\}) = 0$.*

There is a sense in which this result cannot be effectivized, since $\{X : X \geqslant_{\mathrm{T}} A\}$ is never Martin-Löf null, as by Theorem 12.1 it always contains a 1-random set. However, if A is not a basis for 1-randomness, then $\{X : X \geqslant_{\mathrm{T}} A\}$ is contained in the universal Martin-Löf test relative to A, and hence is Martin-Löf null *relative to A*. Thus we have the following consequence of Theorem 8.11, which can be taken to say that the K-trivial sets are exactly those relative to which Theorem 8.11 cannot be effectivized.

Corollary 8.12 (Hirschfeldt, Nies, and Stephan [51]). *A set A is not K-trivial iff $\{X : X \geqslant_T A\}$ is Martin-Löf null relative to A.*

Theorem 8.10 has found a surprising application to computability theory. A *Scott set* is a Turing ideal $\mathcal{S}$ such that for each infinite binary tree $T \in \mathcal{S}$, there is an infinite path of T in $\mathcal{S}$. Scott sets occur naturally in various contexts, such as the study of models of arithmetic and reverse mathematics. H. Friedman and A. McAllister independently asked the following question: if $\mathcal{S}$ is a Scott set and $X \in \mathcal{S}$ is not computable, does there necessarily exist a $Y \in \mathcal{S}$ such that $X \mid_{\mathrm{T}} Y$? Kučera showed that the answer is positive if X is not a basis for 1-randomness, by the following argument. If $X \in \mathcal{S}$, then $\mathcal{S}$ contains an infinite binary tree all of whose infinite paths are 1-random relative to X, so there is a $Y \in \mathcal{S}$ that is 1-random relative to X. Of course $Y \nleqslant_{\mathrm{T}} X$, but if X is not a basis for 1-randomness, then also $X \nleqslant_{\mathrm{T}} Y$, so $X \mid_{\mathrm{T}} Y$. Slaman (personal communication) has recently used Theorems 8.1 and 8.10 to handle the case in which X is a basis for 1-randomness, thus giving a full positive answer to the question.

Most of the topics in this Section are surveyed in [108] and the corresponding proceedings paper. The interaction of K-triviality and 1-randomness via Turing reducibility is in the focus of current research; see [96, Section 4].

§9. Kummer complex c.e. sets, array noncomputability, and c.e.-traceability. As we saw in Theorem 7.3, Terwijn and Zambella [137] showed that the sets that are low for 1-randomness are c.e.-traceable. In Theorem 11.6, we will see that an analogous concept to c.e.-traceability is relevant to the characterization of another class of sets satisfying a lowness notion. In the case of c.e. sets, there is a fascinating connection between Kolmogorov complexity and the notion of array computability introduced by Downey, Jockusch, and Stob [40], which, as shown by Ishmukhametov [53], coincides with c.e.-traceability on the c.e. degrees.

Again we are concerned with initial segment complexity, this time of c.e. sets. We work with plain complexity. The following result is well-known.

Theorem 9.1 (Barzdins' Lemma [7]). *Let A be a c.e. set. Then $C(A \upharpoonright n \mid n) \leqslant \log n + O(1)$ and $C(A \upharpoonright n) \leqslant 2\log n + O(1)$.*

Proof. To compute $A \upharpoonright n$ given n, it suffices to know the *number* of elements $\leqslant n$ in A, since we can run the enumeration of A until this many elements appear. So $A \upharpoonright n$ can be described given n with at most $\log n + O(1)$ many bits. Similarly, to compute $A \upharpoonright n$, it suffices to know n and the number of elements $\leqslant n$ in A, which can be encoded in a string of length $2 \log n$. ⊣

A longstanding open question was whether the $2\log n$ is optimal in the second part of Theorem 9.1. The best we could hope for is to have $C(A \upharpoonright n) \geqslant 2\log n - O(1)$ for infinitely many n, since the following is known (and implies that there is no c.e. set A such that $C(A \upharpoonright n) \geqslant 2\log n - O(1)$ for every n).

THEOREM 9.2 (Solovay (unpublished)). *There is no c.e. set A such that $C(A \upharpoonright n \mid n) \geqslant \log n - O(1)$ for every n.*

PROOF. Define $g(n)$ so that $A \upharpoonright g(n)$ has exactly 2^n elements. Note that $\log g(n) \geqslant n$. We can compute $A \upharpoonright g(n)$ given $g(n)$ and n, by running the enumeration of A until 2^n elements enter A below $g(n)$. Thus we can describe $A \upharpoonright g(n)$ given $g(n)$ using only $\log n + O(1)$ bits of information, and hence $C(A \upharpoonright g(n) \mid g(n)) \not\geqslant \log g(n) - O(1)$. ⊣

Solovay explicitly asked whether it is possible for $C(A \upharpoonright n) \geqslant 2\log n - O(1)$ to happen infinitely often for a c.e. set A.

DEFINITION 9.3. We say that a c.e. set A is *(Kummer) complex* if for each d there are infinitely many n such that $C(A \upharpoonright n) \geqslant 2\log n - d$.

Kummer proved that such complex sets do exist. The classification of the degrees containing Kummer complex sets is an interesting interplay between computability theory and algorithmic complexity. We need the following definition.

DEFINITION 9.4 (Downey, Jockusch, and Stob [40]).

(i) Let $D_0, D_1, \ldots$ be a standard enumeration of the finite sets. A *strong array* is a set of the form $\{D_{f(x)} : x \in \mathbb{N}\}$ for a computable function f.
(ii) A strong array $\{D_{f(x)} : x \in \mathbb{N}\}$ is called a *very strong array* if $|D_{f(x)}| > |D_{f(y)}|$ for all $x > y$.
(iii) For a very strong array $\mathcal{F} = \{D_{f(x)} : x \in \mathbb{N}\}$, we say that a c.e. set A is *$\mathcal{F}$-array noncomputable* if for each c.e. set W there exists a k such that
$$W \cap D_{f(k)} = A \cap D_{f(k)}.$$
(iv) A c.e. set is *array noncomputable* (*a.n.c.*) if it is $\mathcal{F}$-array noncomputable for some very strong array $\mathcal{F}$. A degree is *array noncomputable* if it contains an array noncomputable set; otherwise it is *array computable.*

This definition was designed to capture a certain kind of multiple permitting construction. The intuition is that for A to be $\mathcal{F}$-a.n.c., A needs $|D_{f(k)}|$ many permissions in order to agree with W on $D_{f(k)}$.

In Downey, Jockusch, and Stob [41], a new definition of array noncomputability was introduced, based on domination properties of functions. We first recall that $f \leqslant_{\mathrm{wtt}} A$ (for a function f and a set A) means that there are an index e and a computable function b such that $f = \Phi_e^A$ and, furthermore, for each n, the use of the computation $\Phi_e^A(n)$ does not exceed $b(n)$. It is easily seen that $f \leqslant_{\mathrm{wtt}} \emptyset'$ iff there are computable functions $h(.,.)$ and $p(.)$ such that, for all n, we have $f(n) = \lim_s h(n,s)$ and $|\{s : h(n,s) \neq h(n,s+1)\}| \leqslant p(n)$.

Definition 9.5. A degree $\mathbf{a}$ is *array noncomputable* if for each $f \leqslant_{\mathrm{wtt}} \emptyset'$ there is a function g computable in $\mathbf{a}$ such that $g(n) \geqslant f(n)$ for infinitely many n. Otherwise, $\mathbf{a}$ is *array computable.*

Theorem 9.6 (Downey, Jockusch, and Stob [40, 41]). *Let $\mathbf{a}$ be a c.e. degree and let $\{F_n\}_{n\in\mathbb{N}}$ be a very strong array. Then the following are equivalent:*

(i) *The degree $\mathbf{a}$ is a.n.c. in the sense of Definition 9.4.*
(ii) *There is a c.e. set A of degree $\mathbf{a}$ such that $\forall e\, \exists n\, [W_e \cap F_n = A \cap F_n]$.*
(iii) *The degree $\mathbf{a}$ is a.n.c. in the sense of Definition 9.5.*

Hence for c.e. degrees, the two definitions of array noncomputability coincide, and the first definition is independent of the choice of very strong array.

It is well known that an arbitrary degree $\mathbf{a}$ is in $\overline{\mathrm{GL}_2}$ (i.e., $(\mathbf{a} \cup \mathbf{0}')' < \mathbf{a}''$) iff for each function f computable in $\mathbf{a} \cup \mathbf{0}'$ there is a function g computable in $\mathbf{a}$ such that $g(n) \geqslant f(n)$ for infinitely many n. From this fact it immediately follows that if $\mathbf{a} \in \overline{\mathrm{GL}_2}$, then $\mathbf{a}$ is a.n.c., and if $\mathbf{a}$ is Δ_2^0 and array computable, then $\mathbf{a}$ is low$_2$.

There are a number of other characterizations of the array noncomputable c.e. degrees (see [40, 41]). For example, the a.n.c. c.e. degrees are precisely those that bound c.e. sets A_1, A_2, B_1, B_2 such that $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$ and every separating set for A_1, A_2 is Turing incomparable with every separating set for B_1, B_2. In fact, they are the degrees that bound disjoint c.e. sets A, B that have no separating set of degree $\mathbf{0}'$. The a.n.c. c.e. degrees also form an invariant class for the perfect thin Π_1^0-classes, which form an orbit in the lattice of Π_1^0-classes, in the same way that the maximal sets realize all high c.e. degrees and are an invariant orbit for the high c.e. degrees (see Cholak, Coles, Downey, and Herrmann [25]).

Of relevance here is the following result.

Theorem 9.7 (Ishmukhametov [53]). *A c.e. degree is array computable iff it is c.e.-traceable.*

Using this characterization, Ishmukhametov proved the following remarkable theorem. A degree $\mathbf{m}$ is a *strong minimal cover* of a degree $\mathbf{a} < \mathbf{m}$ if for all degrees $\mathbf{d} < \mathbf{m}$, we have $\mathbf{d} \leqslant \mathbf{a}$.

Theorem 9.8 (Ishmukhametov [53]). *A c.e. degree is array computable iff it has a strong minimal cover.*

We can now state Kummer's classification of the c.e. degrees containing complex c.e. sets. Again there is a deep connection with traceability.

THEOREM 9.9 (Kummer's Gap Theorem [72]).

(i) *A c.e. degree contains a complex set iff it is array noncomputable.*

(ii) *In addition, if A is c.e. and of array computable degree, then for every unbounded, nondecreasing, total computable function f,*

$$C(A \upharpoonright n) \leqslant \log n + f(n) + O(1).$$

(iii) *Hence the c.e. degrees exhibit the following gap phenomenon: for each c.e. degree $\mathbf{a}$, either*

  (a) there is a c.e. set $A \in \mathbf{a}$ such that $C(A \upharpoonright n) \geqslant 2 \log n - O(1)$ for infinitely many n, or

  (b) there are no c.e. set $A \in \mathbf{a}$ and $\varepsilon > 0$ such that $C(A \upharpoonright n) \geqslant (1 + \varepsilon) \log n - O(1)$ for infinitely many n.

Thus we have the remarkable fact that a c.e. degree contains a c.e. set whose initial segment complexity is as large as possible iff it has a strong minimal cover!

In Theorem 10.30 we will see that the degrees containing Kummer complex c.e. sets are the same as those containing sets that are random relative to a variation of Kurtz randomness. It is natural to ask whether there is a classification of, say, all jump classes in terms of initial segment complexity.

§10. Other notions of algorithmic randomness. We now return to our consideration of the basic definition of randomness. We have seen that the three approaches (through measure theory, unpredictability, and incompressibility) all yield the same notion of randomness. But consider the characterization of 1-randomness in terms of martingales, namely that no *computably enumerable* martingale succeeds on the given set. In [121], Schnorr gave this characterization, then analyzed it. He argued that it demonstrates a clear *failure* of the intuition behind the notion of Martin-Löf randomness. He argued that randomness should be concerned with defeating *computable* strategies rather than computably enumerable ones, since the latter are fundamentally asymmetric, in the same way that a c.e. set is semi-decidable rather than decidable. We can make a similar argument about Martin-Löf tests being effectively null (in the sense that we know how fast they converge to zero), but not effectively given, in the sense that the test sets V_n themselves are not computable, but rather c.e. (The discussion may have been obscured by the fact that for a Martin-Löf test $\{V_n\}_{n \in \omega}$, the sets V_n can always be chosen to be computable (as sets of finite initial segments). However, their *measures* are not necessarily computable.)

Armed with this fundamental insight, and following Schnorr [121], we will look at two notions of randomness that refine the notion of Martin-Löf randomness. Both notions are natural, one being inspired by the measure-theoretic approach and the other by the martingale approach. The first notion we introduce is most naturally defined using tests.

DEFINITION 10.1 (Schnorr [121]).

(i) We say that a Martin-Löf test $\{V_n\}_{n\in\omega}$ is a *Schnorr test* if $\mu(V_n) = 2^{-n}$ for all n. In this case we call any subset of $\bigcap_n V_n$ a *Schnorr null set.*
(ii) A set A is *Schnorr random* if $A \notin \bigcap_n V_n$ for all Schnorr tests $\{V_n\}_{n\in\omega}$.

In his original version of Definition 10.1, Schnorr only required that the numbers $\mu(V_n)$ be uniformly computable, which is easily seen to yield the same notions of null set and random set as the definition given here. When dealing with Schnorr randomness, we will use whichever version of the definition is most convenient.

Our next notion of randomness is based on computable martingales.

DEFINITION 10.2 (Schnorr [121]).

(i) A martingale $f : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ is *computable* if its values $f(\sigma)$ are uniformly computable reals.
(ii) A set A is *computably random* if no computable martingale succeeds on A.

Schnorr proved that we can take the range of the martingales in the definition of computable randomness to be the non-negative rationals $\mathbb{Q}^+ \cup \{0\}$.

LEMMA 10.3 (Schnorr [121]). *For each computable martingale F there is a computable martingale $f : 2^{<\omega} \to \mathbb{Q}^+ \cup \{0\}$ such that $S[F] = S[f]$.*

The notion of computable randomness has enjoyed considerable popularity in complexity theory since it naturally admits complexity-theoretic versions, such as polynomial time randomness, which can be used to explore both randomness and Hausdorff dimension in small complexity classes. For more discussion and references, see Lutz [83, 84] and Ambos-Spies and Kučera [1].

It is clear that Martin-Löf random implies computably random, and Theorem 10.5 below shows that computably random implies Schnorr random. Neither of these implications can be reversed. That there are computably random sets that are not Martin-Löf random was proved by Schnorr [121], and that there are Schnorr random sets that are not computably random was proved by Wang [139]. (See also Section 15.3.) The precise separation of these concepts in terms of Turing degrees was determined by Nies, Stephan, and Terwijn [109]; see Theorem 10.13.

We remark that a natural algorithmic randomness notion $\mathcal{C}$ ought to be invariant under a computable rearrangement of the bits. Thus one should have $Z \in \mathcal{C} \to \rho(Z) \in \mathcal{C}$ for any computable permutation ρ. For Martin-Löf and Schnorr randomness this is clear, since the underlying test concepts are invariant under such permutations, and likewise for Kurtz randomness (defined in Section 10.3). Merkle, Miller, Nies, Reimann, and Stephan [93] showed that closure under computable permutations also holds for computable randomness.

10.1. Schnorr randomness. Although the critique in Schnorr's book [121] was subtle and well-put, the notion of Schnorr randomness initially did not attract the attention that it deserved. Perhaps part of the reason was that Martin-Löf randomness was *good enough* for many results. Another reason was that the notion is much less tractable than Martin-Löf randomness. For instance, Schnorr proved that *there is no universal Schnorr test.* Many basic questions remained open for a long time. For instance, one of the cornerstones of the theory of 1-randomness is that the three characterizations, via machine incompressibility, tests, and martingales, all coincide, and hence the notion of 1-randomness is mathematically robust. It is thus interesting to ask whether similar equivalent characterizations exist for the notions of randomness proposed by Schnorr.

Schnorr gave the following martingale characterization of Schnorr randomness. The crucial notion is that of an order.

Definition 10.4 (Schnorr [121]). An *order* is an unbounded nondecreasing function $h : \mathbb{N} \to \mathbb{N}$. (Note that an "Ordnungsfunktion" in Schnorr's terminology is always computable, whereas we prefer to leave the complexity of orders unspecified in general.) For a martingale d and an order h we define

$$S_h[d] := \{X : \limsup_{n\to\infty} \frac{d(X \upharpoonright n)}{h(n)} \geqslant 1\}.$$

Schnorr pointed out that the rate of success of a c.e. martingale d can be so slow that it cannot be computably detected. Thus, rather than working with null sets contained in sets of the form $S[d]$ with $d \in \Sigma^0_1$, he worked with null sets contained in sets of the form $S_h[d]$ where both d and h are computable. He showed that these null sets are the same as the Schnorr null sets from Definition 10.1. The following result gives Schnorr's characterization of Schnorr randomness in terms of computable martingales and the "speed of success". (Note that it implies that every computably random set is Schnorr random.)

Theorem 10.5 (Schnorr [121], Sätze 9.4, 9.5). *$\mathcal{A} \subseteq 2^\omega$ is Schnorr null iff there are computable functions d and h such that $\mathcal{A} \subseteq S_h[d]$.*

It had been a longstanding open problem to provide a machine characterization for Schnorr randomness. Terwijn [135] had made some progress in this area. Downey and Griffiths gave the following machine characterization of Schnorr randomness.

DEFINITION 10.6 (Downey and Griffiths [31]). We say a prefix-free machine M is *computable* if

$$\mu(\mathrm{dom}(M)) = \sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|}$$

is a computable real.

THEOREM 10.7 (Downey and Griffiths [31]). *A set A is Schnorr random iff $K_M(A \upharpoonright n) \geqslant n - O(1)$ for all computable prefix-free machines M.*

The proof of Theorem 10.7 filtered through a Solovay test characterization of Schnorr randomness. (An equivalent definition in terms of martingales is given in Wang [139].)

DEFINITION 10.8 (Downey and Griffiths [31]). A *total Solovay test* is a computable collection of c.e. open sets $\{V_i\}_{i\in\omega}$ such that $\sum_i \mu(V_i)$ is finite and computable. A set A passes this total Solovay test if $A \in V_i$ for at most finitely many i.

THEOREM 10.9 (Downey and Griffiths [31]). *A set is Schnorr random iff it passes all total Solovay tests.*

The Kučera-Slaman Theorem 4.6 shows that all 1-random left-c.e. reals are wtt-complete, since they are Solovay-complete. (As we have seen in Theorem 4.1, Kučera [67] was the first to prove that they are all Turing complete.) There is also a characterization of the Turing degrees of Schnorr random left-c.e. reals: Downey and Griffiths [31] proved that every Schnorr random left-c.e. real has high Turing degree, and they also proved that there is a Turing incomplete Schnorr random left-c.e. real. Later, Downey, Griffiths, and LaForte [32] proved that every high c.e. Turing degree contains a Schnorr random left-c.e. real. This also follows from Theorem 10.13 below, which is due to Nies, Stephan, and Terwijn [109].

The machine characterization of Martin-Löf randomness allows us to calibrate randomness via $\leqslant_K$, and we can similarly calibrate the complexity of sets in terms of their Schnorr complexity.

DEFINITION 10.10 (Downey and Griffiths [31]). We write $A \leqslant_{Sch} B$ if for each computable prefix-free machine M there is a computable prefix-free machine $\widehat{M}$ such that

$$K_{\widehat{M}}(A \upharpoonright n) \leqslant K_M(B \upharpoonright n) + O(1),$$

where the constant depends on M.

Clearly, if $\alpha \leqslant_{\mathrm{Sch}} \beta$ for all left-c.e. reals α, then β is Schnorr random. Virtually nothing is known about $\leqslant_{\mathrm{Sch}}$. Downey and Griffiths constructed a "Schnorr trivial" left-c.e. real.

THEOREM 10.11 (Downey and Griffiths [31]). *There exist noncomputable left-c.e. reals α such that $\alpha \leqslant_{Sch} \emptyset$.*

Recently, Downey, Griffiths, and LaForte [32] proved that Schnorr trivial sets are quite different from K-trivial sets.

THEOREM 10.12 (Downey, Griffiths, and LaForte [32]).

(i) *There exist Turing complete c.e. sets that are Schnorr trivial.*
(ii) *No Schnorr trivial left-c.e. real is wtt-complete.*
(iii) *There exist nonzero c.e. degrees containing no Schnorr trivial sets.*

10.2. Computable randomness. It is not hard to prove that there is no computable enumeration of all computable martingales. Thus, as with Schnorr randomness, arguments about computable randomness need to deal with Π^0_2 behavior. We have already noted that there is a computably random set that is not Martin-Löf random. Wang [140] proved that there is also a Schnorr random set that is not computably random. Downey, Griffiths, and LaForte [32], and independently Nies, Stephan, and Terwijn [109], showed that this also holds for left-c.e. reals. The following theorem shows precisely how complex it is to separate these randomness notions.

THEOREM 10.13 (Nies, Stephan, and Terwijn [109]). *For every set A, the following are equivalent.*

(i) *A is high.*
(ii) *$\exists B \equiv_T A$ s.t. B is computably random but not Martin-Löf random.*
(iii) *$\exists C \equiv_T A$ s.t. C is Schnorr random but not computably random.*

Furthermore, if A is a left-c.e. real then B and C can also be chosen to be left-c.e. reals.

PROOF. We only prove that $\neg$(i) $\Rightarrow \neg$(ii) $\wedge \neg$(iii). For the other implications, see [109]. Let A be a nonhigh set that is not Martin-Löf random. Let $\{U_i\}_{i\in\omega}$ be the universal Martin-Löf test. We show that A is not Schnorr random, and hence also not computably random. Let f be the function that tells us when A is covered by the U_i. That is, $f(i)$ is the first stage at which an initial segment of A enters U_i. Notice that f is A-computable. Let g be a computable function that is infinitely often larger than f (which exists because A is not high). We can define a Schnorr test $\{V_n\}_{n\in\omega}$ by stopping the enumeration of each U_n after $g(n)$ many steps to obtain V_n. Then every V_n is finitely presented, so $\{V_n\}_{n\in\omega}$ is a Schnorr test, and A is in V_n for infinitely many n, which is sufficient to show that A is not Schnorr random (since we can convert the Schnorr test $\{V_n\}_{n\in\omega}$ into a Schnorr test $\{\tilde{V}_n\}_{n\in\omega}$ covering just as much and with the additional property that $\tilde{V}_{n+1} \subseteq \tilde{V}_n$, by letting $\tilde{V}_n = \bigcup_{m>n} V_m$). $\dashv$

We finish this section by mentioning that there is a measure-theoretic characterization (which could be turned into a machine characterization) of computable randomness.

DEFINITION 10.14 (Downey, Griffiths, and LaForte [32]). We say that a Martin-Löf test $\{V_n\}_{n\in\omega}$ is *computably graded* if there exists a computable map $f : 2^{<\omega} \times \omega \to \mathbb{R}$ such that, for any $n \in \omega$, any $\sigma \in 2^{<\omega}$, and any finite prefix-free set of strings $\{\sigma_i\}_{i\leqslant I}$ with $\bigcup_{i=0}^{I}[\sigma_i] \subseteq [\tau]$, the following conditions are satisfied:

(i) $\mu(V_n \cap [\sigma]) \leqslant f(\sigma, n)$
(ii) $\sum_{i=0}^{I} f(\sigma_i, n) \leqslant 2^{-n}$
(iii) $\sum_{i=0}^{I} f(\sigma_i, n) \leqslant f(\tau, n)$

From conditions (i) and (ii) it immediately follows that $\mu(V_n) \leqslant 2^{-n}$ for all n. Furthermore, if condition (ii) holds for any finite prefix-free set $\{\sigma_i\}_{i\leqslant I}$ then it also holds for any infinite prefix-free set of strings: the infinite sum is just the supremum of the associated finite sums, and so is also no greater than 2^{-n}. Similarly, since (iii) holds for finite prefix-free sets it also holds for infinite prefix-free sets. If $\bigcup_i[\sigma_i] = [\tau]$ then we can summarize conditions (i)–(iii) by the following:

$$\mu(V_n \cap [\tau]) \leqslant \sum_{i=0}^{I} f(\sigma_i, n) \leqslant f(\tau, n) \leqslant 2^{-n}.$$

The computably graded tests give an alternative to the martingale characterization of computable randomness:

THEOREM 10.15 (Downey, Griffiths, and LaForte [32]). *A set is computably random iff it passes all computably graded tests.*

A similar characterization was found by Merkle, Mihailović, and Slaman [92].

Hirschfeldt, Nies, and Stephan [51] have investigated *bases for computable randomness*, that is, sets A for which there is a $B >_{\mathrm{T}} A$ that is computably random relative to A (cf. Theorem 8.10). They obtained a partial characterization in terms of two well-known classes of degrees. The *PA-degrees* are the degrees of complete extensions of Peano Arithmetic. This is an important class of degrees, with many equivalent definitions. For instance, a degree $\mathbf{d}$ is a PA-degree iff every computable infinite binary tree has a $\mathbf{d}$-computable infinite path. (More generally, we say that $\mathbf{d}$ is a PA-degree relative to A if every A-computable infinite binary tree has a $\mathbf{d}$-computable infinite path.) The PA-degrees are also those that contain $\{0,1\}$-valued total functions f that are *diagonally noncomputable*, in the sense that $\forall e\,[\Phi_e(e) \neq f(e)]$. The class of PA-degrees is strictly contained in the class of *diagonally noncomputable (DNC) degrees*, which are those that contain (not necessarily $\{0,1\}$-valued) diagonally noncomputable functions.

THEOREM 10.16 (Hirschfeldt, Nies, and Stephan [51]).
(i) *If a Δ_2^0 set does not have DNC degree then it is a basis for computable randomness.*
(ii) *No set of PA-degree is a basis for computable randomness.*

The proof of the second part of Theorem 10.16 uses a lemma of independent interest: If A has PA-degree relative to B and X is computably random relative to A, then X is 1-random relative to B.

Let A be an n-c.e. set. If A is Turing incomplete then A does not have diagonally noncomputable degree, by Jockusch, Lerman, Soare and Solovay [56] (which extends Arslanov's Completeness Criterion). So A is a basis for computable randomness. On the other hand, if A is Turing complete then A has PA-degree, and hence is not a basis for computable randomness. Thus we have the following result.

COROLLARY 10.17 (Hirschfeldt, Nies, and Stephan [51]). *An n-c.e. set is a basis for computable randomness iff it is Turing incomplete.*

It would be interesting to investigate bases for other notions such as Schnorr and Kurtz randomness.

10.3. Kurtz randomness. In [73], Kurtz introduced a new notion of randomness which looks at the idea from another perspective. Namely, instead of thinking of a set as random if it *avoids* all effectively given null sets, Kurtz suggested that a set should be considered random if it *obeys* every effectively given test of measure 1.

DEFINITION 10.18 (Kurtz [73]). (i) A *Kurtz (positive) test* is a c.e. open set U such that $\mu(U) = 1$.
(ii) A set is called *Kurtz random* (or *weakly 1-random*) if $A \in U$ for all Kurtz tests U.

Kurtz originally called this notion *weak randomness*, and it is a weak notion in that, as shown by Wang [139], it is not *stochastic* in the sense of Church.[5] It is nevertheless a very interesting concept, especially in its relativized form. As we will see in Section 12, Kurtz 2-randomness, which means being in every Σ_2^0 open set of measure 1, is equivalent to passing every "generalized" Martin-Löf test $\{U_n\}_{n\in\omega}$, where we still have $\mu(U_n) \to 0$, but there may be no decreasing computable upper bound on the measures.

[5]For more on stochasticity, see Ambos-Spies and Kučera [1]. That a notion is not stochastic might even disqualify it from being called a *randomness* notion. Kurtz randomness is called so mainly because of the analogy with other definitions of randomness, but we note here that it is in fact more like a notion of genericity.

Most of the definitions of tests so far have been negative. There is an equivalent formulation of Kurtz randomness in terms of a negative test.

DEFINITION 10.19 (Wang [139]). A *Kurtz null test* is a sequence of c.e. open sets $\{V_n\}_{n\in\omega}$ such that
(i) $\mu(V_n) \leqslant 2^{-n}$ and
(ii) there is a computable function $f : \mathbb{N} \to (2^{<\omega})^{<\omega}$ such that $V_n = \bigcup_{\sigma\in f(n)}[\sigma]$.

THEOREM 10.20 (Wang [139], after Kurtz [73]). *A set A is Kurtz random iff it passes all Kurtz null tests.*

PROOF. We show how Kurtz positive tests correspond to Kurtz null tests. Given a c.e. open set W with $\mu(W) = 1$, define a Kurtz null test $\{V_n\}_{n\in\omega}$ as follows. To define V_n, enumerate W until a stage s is found with $\mu(W_s) > 1 - 2^{-n}$, then let $V_n = \overline{W_s}$. Note that V_n is of the correct form to be able to define a function f as in Definition 10.19, and $\bigcap_n V_n = \overline{W}$.

For the converse, given a Kurtz null test $\{V_n\}_{n\in\omega}$, let $W = \bigcup_n \overline{V_n}$. Then W is a c.e. open set of measure 1, and $\overline{W} = \bigcap_n V_n$. ⊣

There is a martingale definition of Kurtz randomness (cf. the martingale characterization of Schnorr randomness given by Theorem 10.5):

THEOREM 10.21 (Wang [139]). *A set A is not Kurtz random iff there exist a computable martingale d and a nondecreasing unbounded computable function h such that $d(A \restriction n) > h(n)$ for all n.*

Because of this result we easily see that Schnorr randomness implies Kurtz randomness. No Kurtz random set can be a c.e. set. In fact:

THEOREM 10.22 (Jockusch, see Kurtz [73]). *If A is Kurtz random then it is bi-immune; that is, neither A nor $\overline{A}$ contains an infinite computable subset. Hence, by Jockusch* [54], *there are $2^{\aleph_0}$ degrees that contain no Kurtz random sets.*

While no c.e. set can be Kurtz random, as with Martin-Löf randomness, the same is not true for left-c.e. reals. Kurtz [73, Corollary 2.3a] proved that every nonzero c.e. degree contains a Kurtz random set. The following improves this result to left-c.e. reals.

THEOREM 10.23 (Downey, Griffiths, and Reid [33]). *Each nonzero c.e. degree contains a Kurtz random left-c.e. real.*

No characterization of the degrees containing Kurtz random sets is known. Nies and Yu (unpublished) have shown that the conclusion of the previous theorem can be strengthened: Each nonzero c.e. degree contains a weakly 1-generic left-c.e. real. Here A is weakly 1-generic if A is in each dense c.e. open set.

Downey, Griffiths, and Reid [33] gave a machine characterization of Kurtz randomness in the style of Theorem 10.7.

DEFINITION 10.24 (Downey, Griffiths, and Reid [33]). We say a prefix-free machine M is *computably layered* if there is a computable function $f : \omega \to (2^{<\omega})^{<\omega}$ such that

(i) $\bigcup_i f(i) = \mathrm{dom}(M)$.
(ii) If $\gamma \in f(i+1)$, then $\exists \tau \in f(i)$ such that $M(\tau) \preccurlyeq M(\gamma)$.
(iii) If $\gamma \in f(i)$, then $|M(\gamma)| = |\gamma| + i + 1$.

The idea of a computably layered machine is that each layer $f(i)$ of the domain provides a layer of the range, and the range elements become more refined as i increases.

THEOREM 10.25 (Downey, Griffiths, and Reid [33]). *A set A is Kurtz random iff $K_M(A \upharpoonright n) \geqslant n - O(1)$ for each computably layered machine M.*

Interestingly, there is yet another machine characterization of Kurtz randomness, this one in terms of computable prefix-free machines (cf. Theorem 10.7).

THEOREM 10.26 (Downey, Griffiths, and Reid [33]). *A set A is* not *Kurtz random iff there are a computable prefix-free machine M and a computable function $f : \mathbb{N} \to \mathbb{N}$ such that*

$$\forall d\, [K_M(A \upharpoonright f(d)) < f(d) - d].$$

It is also possible to come up with suitable Solovay type characterizations of Kurtz randomness, as per Wang [139] and Downey, Griffiths, and Reid [33]. Using such a characterization, Downey, Griffiths, and Reid [33] provided a characterization of Schnorr randomness in terms of Kurtz randomness.

Kurtz positive tests can also be used to define 1-randomness, if the speed of enumeration is also taken into account.

THEOREM 10.27 (Davie [28]). *A set A is 1-random iff there is a constant c such that for each p, if the p-th computable sequence of intervals $I_1, I_2, \ldots$ is such that $\mu(\bigcup_j I_j) = 1$, then there is an n with $A \in I_n$ and $\mu(\bigcup_{j \leqslant n} I_j) < 1 - 2^{-|p|-c}$.*

10.4. Kolmogorov-Loveland randomness. The computable betting strategies (martingales) used to define computable randomness are monotonic, in the sense that they bet on the bit positions in their natural order. Dropping this monotonicity condition yields a more powerful notion of betting strategy.

We give an informal version of the definition in Muchnik, Semenov, and Uspensky [101]. See also Merkle, Miller, Nies, Reimann, and Stephan [93] for a more formal definition. A nonmonotonic betting strategy behaves as follows. Given a set A, at stage s suppose the previously scanned bit positions are $n_0, \ldots, n_{s-1}$ and the corresponding values are $r_i = A(n_i)$. From $n_0, \ldots, n_{s-1}$ and $r_0, \ldots, r_{s-1}$, the strategy determines a new position $n_s \neq n_0, \ldots, n_{s-1}$, or may choose to be undefined. If defined, it also chooses an $i < 2$ and makes a bet $v \in \mathbb{Q}$ with $0 \leqslant v \leqslant V$ on $A(n_s)$ being equal to i, where V is the current capital (if $s = 0$, then $V > 0$ is the initial capital). If the bet turns out right, then v is added to the capital; otherwise it is subtracted. A set A is *Kolmogorov-Loveland random* (*KL-random*) if no computable nonmonotonic betting strategy succeeds on A, in the sense that the limsup of the capital it obtains by betting on A is infinity. In [101] such sets are called *unpredictable*, and they have also been called *nonmonotonically random*. Clearly, KL-random sets are computably random, since computable martingales are a particular kind of nonmonotonic betting strategy. In fact, if A is KL-random then no *partial* computable martingale succeeds on A. However, by results in Muchnik, Semenov, and Uspensky [101], the converse is not true; that is, there are non-KL-random sets on which no partial computable martingale succeeds.

Muchnik, Semenov, and Uspensky [101] showed that every 1-random set is KL-random. Whether the converse holds is a major open problem.

QUESTION 10.28. Is every KL-random set 1-random?

While we allow partial betting strategies, we could as well require them to be total: Merkle (see [93]) proved that for each nonmonotonic betting strategy M there exist total (even primitive recursive) nonmonotonic betting strategies L_0, L_1 such that, if M succeeds on A, then one of L_0, L_1 succeeds on A. Thus if the answer to Question 10.28 is affirmative, then one might argue that Schnorr's critique of 1-randomness ceases to apply, as we will then have a characterization of 1-randomness based on computable strategies.[6] However, presently we do not even know whether the inclusion holds for left-c.e. reals. Several results have been obtained suggesting that KL-randomness is at least close to 1-randomness. Muchnik (see [101, Theorem 9.1]) showed that if there is an unbounded computable function d such that $K(A \upharpoonright n) \leqslant n - d(n)$ for all but finitely many n, then A is not KL-random. (In fact the weaker hypothesis that $\forall n\, [K(A \upharpoonright g(n)) \leqslant g(n) - n]$, where g is an unbounded computable function, suffices.)

Merkle, Miller, Nies, Reimann, and Stephan [93] showed that if $A = A_0 \oplus A_1$ is KL-random, then at least one of A_0, A_1 is 1-random, and in fact both are if A is Δ^0_2. Extending this argument shows that $\liminf_n (K(A \upharpoonright$

[6] This is debatable, however, since Schnorr believed that even computable martingales are not effective enough, because their rates of success may not be computable.

$n)/n) = 1$ (i.e., the effective Hausdorff dimension[7] of A is 1); see [93]. On the other hand, Merkle [90] showed that the effective Hausdorff dimension of a set on which no partial computable martingale succeeds is not necessarily 1. All of these results on KL-randomness are proved by constructing several strategies, one of which succeeds. The problem in answering Question 10.28 seems to be to understand the interplay between such strategies. A possible approach to answering this question might be to first investigate (apparently) weaker versions of KL-randomness, such as permutation randomness and injective randomness. For the definitions and a discussion of these notions, see Miller and Nies [96].

It follows from Theorem 11.12 below that every set that is low for KL-randomness is K-trivial.

10.5. Finite randomness. The notions above would seem to indicate that all of the randomness notions are linearly ordered in strength. We briefly mention a further randomness notion[8] which shows that this may not always be the case. A *finite test* $\{U_n\}_{n\in\omega}$ is a Martin-Löf test where each U_n is a finitely presented open set. For example, a Kurtz null test is finite. We say that A is *finitely random* if it passes all finite tests. It is not hard to see that a Δ^0_2 set is Martin-Löf random iff it is finitely random. We say that a finite test $\{U_n\}_{n\in\omega}$ is *computably bounded* if the U_n are presented by sets P_n for which there is a computable function g such that $|P_n| < g(n)$. We say that A is *computably finitely random* if it passes all computably bounded tests.

THEOREM 10.29 (Downey, Miller, and Reimann [43]).

(i) *Martin-Löf randomness implies finite randomness, but not conversely. However, finite randomness and computably finite randomness neither imply nor are implied by either Schnorr or computable randomness.*
(ii) *On the left-c.e. reals, (i) remains true, except that finite randomness coincides with Martin-Löf randomness.*

Once again we see a connection with traceability and array noncomputability:

THEOREM 10.30 (Downey, Miller, and Reimann [43]). *If a left-c.e. real is computably finitely random then it is array noncomputable.*

§11. Lowness properties revisited.

Recall that in Section 7 a set A was called low for a class $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}^A$. When discussing lowness for randomness notions, one has two options. For Schnorr randomness, for instance, one can look at sets A that are *low for the Schnorr null sets*

[7] See Section 15 for more on effective Hausdorff dimension.

[8] The notion of finite randomness discussed here is not stochastic, so the same remarks apply as in the case of Kurtz randomness; see footnote 5.

(called S_0-*low* in [1]), meaning that every set that is Schnorr null relative to A is Schnorr null, or one can look at the potentially larger class of sets A that are *low for Schnorr randomness*, meaning that every Schnorr random set is Schnorr random relative to A. In the case of 1-randomness, there is no difference between these notions because there is a universal Martin-Löf test. Ambos-Spies and Kučera [1, Problem 4.5] asked whether the two notions are different for Schnorr randomness. It will follow from Theorem 11.10 below that the answer is no, despite the absence of a universal Schnorr test. For computable randomness, the answer is even easier: the only sets that are low for computable randomness are the computable ones! (See Theorem 11.14.)

11.1. Lowness for Schnorr and Kurtz null sets. One nice aspect of Schnorr randomness is that there is a complete characterization of the sets that are low for Schnorr randomness. As usual, let D_n denote the n-th canonical finite set.

DEFINITION 11.1 (Terwijn and Zambella [137]; c.f. Definition 7.2). A set A is *computably traceable* if there is a computable function p (called a *bound*) such that, for each function $g \leqslant_T A$, there is a computable function h satisfying, for all n,

(i) $|D_{h(n)}| \leqslant p(n)$ and
(ii) $g(n) \in D_{h(n)}$.

The following proposition shows that it does not matter what bound p we choose.

PROPOSITION 11.2 (Terwijn and Zambella [137]). *Let A be computably traceable and let p be an unbounded nondecreasing computable function such that $p(0) > 0$. Then A is computably traceable with bound p.*

A degree is *hyperimmune-free* if each function of that degree is majorized by some computable function. If A is computably traceable then each function $g \leqslant_T A$ is majorized by the function $f(n) = \max D_{h(n)}$, where h is as in Definition 11.1. Thus every computably traceable set has hyperimmune-free degree. One may think of computable traceability as a uniform version of hyperimmune-freeness. Terwijn and Zambella [137] showed that a simple variation of the standard construction of hyperimmune-free sets by Miller and Martin [99] produces continuum many computably traceable sets.

Kjos-Hanssen and Nies (unpublished) have recently characterized the computably traceable sets within the class of sets of hyperimmune-free degree using prefix-free complexity.

DEFINITION 11.3 (Kjos-Hanssen and Nies (unpublished)).

(i) A set A is *weakly c.e. traceable* if Definition 7.2 holds for the computably bounded A-computable functions; that is, if there is a bound

p such that for every $f \leqslant_{\mathrm{T}} A$ that is majorized by some computable function, there is a c.e. trace for f with bound p (as before, the choice of bound does not matter).

(ii) A set X is *facile* if for every nondecreasing unbounded computable function h, for almost all n we have $K(X \upharpoonright n \mid n) \leqslant h(n)$.

PROPOSITION 11.4 (Kjos-Hanssen and Nies (unpublished)). *A set A is weakly c.e. traceable iff every set $X \leqslant_T A$ is facile.*

Thus every computably traceable set is facile. Conversely, suppose that A has hyperimmune-free degree and is facile. It is not hard to see that the facile sets are closed downwards under wtt-reducibility, but Turing reducibility implies wtt-reducibility within the hyperimmune-free degrees, so every A-computable set is facile. Thus by Proposition 11.4, A is weakly c.e. traceable, and hence c.e. traceable (since every A-computable function is computably bounded). But as shown below in step 2 of the proof of part (ii) of Theorem 11.10, we can use the fact that A has hyperimmune-free degree to convert a c.e. trace into a computable trace, so A is computably traceable. Thus we have the following corollary.

COROLLARY 11.5. *Suppose that A has hyperimmune-free degree. Then A is computably traceable iff A is facile.*

Remarkably, the class of sets that are low for the Schnorr null sets is characterized by the purely computability-theoretic property of computable traceability.

THEOREM 11.6 (Terwijn and Zambella [137]). *A set is low for the Schnorr null sets iff it is computably traceable.*

One direction of the proof of Theorem 11.6 relies on ideas of Raisonnier [113] on rapid filters for the "mathematical" proof of Shelah's theorem that the inaccessible cardinal cannot be removed from Solovay's [128] construction of a model of set theory where every set of reals is Lebesgue measurable.

Interestingly, we have the following fact about the degrees of sets that are low for Schnorr randomness (which was proved by Terwijn and Zambella [137] for the sets that are low for the Schnorr null sets, before it was known that these two classes are the same).

THEOREM 11.7 (Terwijn and Zambella [137]). *The degrees of sets that are low for Schnorr randomness are a proper subclass of the hyperimmune-free degrees, and hence, except for $\mathbf{0}$, none of them are Δ^0_2. In particular, the degrees of noncomputable sets that are low for 1-randomness and the degrees of noncomputable sets that are low Schnorr randomness are* disjoint.

Recently, an easier result relating a lowness notion to the computably traceable and hyperimmune-free degrees was obtained by Downey, Griffiths, and Reid [33]. We give the proof below since it is representative of the much more difficult proof of Theorem 11.6. A *Kurtz null set* is any subset of the intersection of a Kurtz null test (see Definition 10.19). A set A is *low for the Kurtz null sets* if every Kurtz null set relative to A is Kurtz null. It is currently not known whether being low for the Kurtz null sets is equivalent to being low for Kurtz randomness.[9]

THEOREM 11.8 (Downey, Griffiths, and Reid [33]).

(i) *If a set is computably traceable then it is low for the Kurtz null sets.*

(ii) *If a set is low for the Kurtz null sets then it has hyperimmune-free degree.*

PROOF. (i) Let A be computably traceable and let $\{V_n\}_{n\in\omega}$ be a Kurtz null test relative to A. We build a (computable) Kurtz null test $\{U_n\}_{n\in\omega}$ such that $\bigcap_n U_n \supseteq \bigcap_n V_n$.

Let $E_0, E_1, \dots$ be an effective listing of all finite subsets of $2^{<\omega}$. Let g be an A-computable function such that $V_n = \bigcup_{\sigma\in E_{g(n)}}[\sigma]$ for all n. Let h be as in Definition 11.1, with the bound p being defined by $p(n) = n+1$. (We are free to choose this bound by Proposition 11.2.) For each n, define the open set F_n as follows. For each $i \in D_{h(n)}$, if $\mu(\bigcup_{\sigma\in E_i}[\sigma]) \leqslant 2^{-n}$ then add $\bigcup_{\sigma\in E_i}[\sigma]$ to F_n. Since $g(n) \in D_{h(n)}$ and $\mu(\bigcup_{\sigma\in E_{g(n)}}[\sigma]) = \mu(V_n) \leqslant 2^{-n}$, we have $V_n \subseteq F_n$, and since $|D_{h(n)}| \leqslant n+1$, we have $\mu(F_n) \leqslant (n+1)2^{-n}$.

So if we let $U_n = F_{2n}$ then $\{U_n\}_{n\in\omega}$ is a Kurtz null test and $\bigcap_n U_n \supseteq \bigcap_n V_n$.

(ii) Let A be low for the Kurtz null sets, and let $g \leqslant_{\mathrm{T}} A$. We show that g is dominated by a computable function. Define $\{U_n\}_{n\in\omega}$ by letting U_n be the union of all basic clopen sets of the form $[\gamma_1 1\gamma_2 1\dots\gamma_n 1]$, where $|\gamma_i| = g(i)$. Clearly, $\mu(U_n) = 2^{-n}$, so this is a Kurtz null test relative to A. Since A is low for the Kurtz null sets, there is a (computable) Kurtz null test $\{V_n\}_{n\in\omega}$ such that $\bigcap_n V_n \supseteq \bigcap_n U_n$.

We use the V_n to build a function f dominating g. The Kurtz null set $\bigcap_n U_n$ contains all sets with a 1 at bits $g(1)+1$, $g(1)+g(2)+2$, etc., so

[9] Stephan and Yu have recently announced the following results, where a set A is *weakly* 1*-generic* if every dense Σ^0_1 subset of $2^{<\omega}$ contains an initial segment of A: (i) *A set is low for weak* 1*-genericity iff it is hyperimmune-free and not of diagonally noncomputable degree.* (ii) *There is a set of hyperimmune-free degree that is neither computably traceable nor of diagonally noncomputable degree.* (iii) *If a set is low for weak* 1*-genericity then it is low for Kurtz randomness.* (iv) *Thus lowness for weak* 1*-genericity, and hence lowness for Kurtz randomness, are not the same as lowness for Schnorr randomness.* Also, Johanna Franklin in her forthcoming PhD Dissertation has a number of interesting results concerning Schnorr triviality. In particular, she has shown that if a set is low for Schnorr randomness, then it is Schnorr trivial.

V_1 must also contain all such sets. Since $\mu(V_1) \leqslant 1/2$, there must be some k such that V_1 does not contain all sets with k-th bit 0, and this k must be one of $g(1)+1, g(1)+g(2)+2, \dots$. Notice that because $\{V_n\}_{n\in\omega}$ is a Kurtz null test, we can effectively find such a k. Now letting $f(1) = k$, we have $f(1) > g(1)$.

Since $\mu(V_2) \leqslant 1/4$, there must be $k_1 < k_2$ such that V_2 does not contain all sets with k_1-th bit 0, and also does not contain all sets with k_1-th bit 1 and k_2-th bit 0. Again, k_1 and k_2 are among $g(1)+1, g(1)+g(2)+2, \dots$, so letting $f(2) = k_2$, we have $f(2) > g(2)$. We can continue this process to define a computable function f dominating g. ⊣

11.2. Lowness for pairs of randomness notions. A more complete view of lowness arises when we consider lowness for any pair of randomness notions $\mathcal{C} \subseteq \mathcal{D}$. Since relativizing $\mathcal{D}$ usually makes it smaller, one would expect that in general $\mathcal{C} \not\subseteq \mathcal{D}^A$. The following class consists of the sets A for which the inclusion still holds.

DEFINITION 11.9. A set A is in $\mathrm{Low}(\mathcal{C},\mathcal{D})$ if $\mathcal{C} \subseteq \mathcal{D}^A$.

Clearly, if $\mathcal{C} \subseteq \widetilde{\mathcal{C}} \subseteq \widetilde{\mathcal{D}} \subseteq \mathcal{D}$ are randomness notions, then $\mathrm{Low}(\widetilde{\mathcal{C}},\widetilde{\mathcal{D}}) \subseteq \mathrm{Low}(\mathcal{C},\mathcal{D})$. That is, we make the class $\mathrm{Low}(\mathcal{C},\mathcal{D})$ larger by decreasing $\mathcal{C}$ or increasing $\mathcal{D}$.

Let $\mathsf{MLRand}, \mathsf{CRand}$ and SRand denote the classes of 1-random, computably random, and Schnorr random sets, respectively. Then, for instance, $\mathrm{Low}(\mathsf{MLRand},\mathsf{CRand})$ is the class of sets A such that every 1-random set is computably random relative A. We want to characterize lowness for any pair of randomness notions.

Recall from Definition 7.2 that a set A is *c.e.-traceable* if there is a computable p such that for every function $f \leqslant_{\mathrm{T}} A$, there is a computable function h such that for all n, we have $|W_{h(n)}| \leqslant p(n)$ and $f(n) \in W_{h(n)}$. We have seen that both c.e.-traceability and computable traceability, defined in the previous subsection, are deeply related to lowness notions. The following result expands on this relationship.

THEOREM 11.10 (Kjos-Hanssen, Nies, and Stephan [63]).
(i) *A set is in* $\mathrm{Low}(\mathsf{MLRand},\mathsf{SRand})$ *iff it is c.e.-traceable.*
(ii) *A set is in* $\mathrm{Low}(\mathsf{CRand},\mathsf{SRand})$ *iff it is computably traceable.*

PROOF SKETCH. (i) Notice that $A \in \mathrm{Low}(\mathsf{MLRand},\mathsf{SRand})$ iff every Schnorr null set relative to A is contained in the intersection of the universal Martin-Löf test. This already looks quite similar to "there is a c.e. trace for the functions computable in A". We obtain (i) by modifying the methods in Terwijn and Zambella [137] to the case of c.e. traces instead of computable ones.

(ii) Because of Theorem 11.6, it only remains to be shown that every set in $\mathrm{Low}(\mathsf{CRand},\mathsf{SRand})$ is computably traceable.

1. The first step was made by Bedregal and Nies [11, 63], who proved that every set in Low$(\mathsf{CRand}, \mathsf{SRand})$ has hyperimmune-free degree. To see this, assume that A has hyperimmune degree, so that there is a function $g \leqslant_{\mathrm{T}} A$ not dominated by any computable function. Use g to define an A-computable martingale that succeeds in the sense of Schnorr, with the computable lower bound $n/4$, on some $Z \in \mathsf{CRand}$. The construction of this martingale uses the fact that g is infinitely often above the running time of each computable martingale. Special care has to be taken with partial martingales, which results in a set Z that is only Δ^0_3.

2. Next we use the fact that if A has hyperimmune-free degree and is c.e.-traceable, then A is computably traceable. To see this, let $f \leqslant_{\mathrm{T}} A$ and let h be as in the definition of c.e.-traceability. Let $g(n)$ be the least s such that $f(n) \in W_{h(n),s}$. Then $g \leqslant_{\mathrm{T}} A$ and so, since A is hyperimmune-free, g is majorized by a computable function r. So if we choose a computable function $\widetilde{h}$ such that $D_{\widetilde{h}(n)} = W_{h(n),r(n)}$ for all n, then we obtain a computable trace for A.

3. By part (i), if $A \in$ Low$(\mathsf{CRand}, \mathsf{SRand})$ then A is c.e.-traceable, which by 1 and 2 imply that A is computably traceable. ⊣

If a set is low for the Schnorr null sets then it is obviously low for Schnorr randomness. Conversely, if A is low for Schnorr randomness then $A \in$ Low$(\mathsf{CRand}, \mathsf{SRand})$, and hence by part (ii) of Theorem 11.10, A is computably traceable. But then by Theorem 11.6, A is low for the Schnorr null sets. Thus we have the following result.

Corollary 11.11 (Kjos-Hanssen, Nies, and Stephan [63]). *A set is low for Schnorr randomness iff it is low for the Schnorr null sets.*

We have seen that K-triviality is equivalent to lowness for 1-randomness. Nies [105] showed that K-triviality already follows from the weaker hypothesis of being in Low$(\mathsf{MLRand}, \mathsf{CRand})$. (One consequence of this result is that every set that is low for Kolmogorov-Loveland randomness is K-trivial. Nothing about this class of sets is known beyond that.)

Theorem 11.12 (Nies [105]). *A set is* Low$(\mathsf{MLRand}, \mathsf{CRand})$ *iff it is K-trivial.*

Proof Sketch. The "if" direction has been discussed in Section 8. For the remaining direction, suppose that $A \in$ Low$(\mathsf{MLRand}, \mathsf{CRand})$. Then for every set Z, if N is an A-computable martingale that succeeds on Z, then $Z \notin \mathsf{MLRand}$. We first show that this fact implies a certain condition on finite strings. Let R be any c.e. open set such that $\mu(R) < 1$ and all sets in the complement of R are 1-random (for instance, let $R = \{\tau : \exists\sigma \preccurlyeq \tau\,[K(\sigma) \leqslant |\sigma| - 1]\}$). The following condition expresses the failure to build a set $\notin R$ on which N succeeds.

LEMMA 11.13. *Let N be any martingale that succeeds only on sets not in* MLRand. *Then there are $\sigma \in 2^{<\omega}$ and $d \in \omega$ such that $[\sigma] \nsubseteq R$ and*

$$\forall \tau \succcurlyeq \sigma\, [N(\tau) \geqslant 2^d \Rightarrow [\tau] \subseteq R]. \tag{1}$$

PROOF. Otherwise one could inductively build a sequence of strings $\sigma_0 \prec \sigma_1 \prec \cdots$ such that $[\sigma_i] \nsubseteq R$ and $N(\sigma_i) \geqslant 2^i$, which would imply that N succeeds on the set $Z = \bigcup_n \sigma_n$, which is not in R, since R is open. ⊣

Note that $[\sigma] \nsubseteq R$ implies that the relative measure $\mu_\sigma(R) = 2^{|\sigma|}\mu(R \cap [\sigma])$ is less than 1 (otherwise let $X \notin R$ be a set extending σ; then X is a 1-random set in a Π^0_1-class of measure 0, which is impossible).

As in the proof of Theorem 7.6, we want to enumerate a Kraft-Chaitin set W showing that A is K-trivial, in the sense that for each n there is a request $\langle r, A \upharpoonright n\rangle \in W$ with $r \leqslant K(n) + O(1)$. Recall that in that proof, the Kraft-Chaitin set (which was called L_d) depended on a number d such that $\Omega \notin R^A_d$. This time the Kraft-Chaitin set W depends on a witness $\langle \sigma, d, u\rangle$ for Lemma 11.13, where u is a number such that $2^{-u} \leqslant 1 - \mu_\sigma(R)$. We build a Turing functional L such that L^X is a martingale for each set X. We first pretend that we know a witness $\langle \sigma, d, u\rangle$ for Lemma 11.13 with $N = L^A$.

We have to approximate the possible initial segments $A \upharpoonright n$ to make W c.e., and need to be careful not to make too many errors, since otherwise W will not be a Kraft-Chaitin set. Roughly speaking, we work with a computable sequence of finite trees $\{T_s\}_{s\in\omega}$, where at each stage s, strings γ on T_s represent the possible initial segments of A of length $\leqslant s$. The tree T_s checks whether condition (1) holds at stage s: if for some τ we have defined $L^\gamma(\tau) \geqslant 2^d$ for a string γ on a previous tree T_t with $t < s$, then γ is only allowed to be on the present tree T_s if $[\tau] \subseteq R_s$.

A *procedure* α is a pair $\langle \rho, \gamma\rangle$, where $\rho, \gamma \in 2^{<\omega}$. We start α when there is a stage s such that $\gamma \in T_s$ and $U_s(\rho) = |\gamma|$, where U is the universal prefix-free machine used to define K. Now α wants to put $\langle r+c, \gamma\rangle$ into W, where $r = |\rho|$ (and c is an appropriate constant), as this would result in $K(\gamma) \leqslant r + O(1)$. First, α causes a clopen set $C \subseteq [\sigma]$ of relative measure $\mu_\sigma(C) = 2^{-(r+c)}$ to enter R. Basically, α chooses a clopen set $C = C(\alpha)$ of that measure, which is disjoint from R_s and the sets chosen by other procedures, and causes (in a way to be specified below) $L^X(\tau) \geqslant 2^d$ for each $X \succcurlyeq \gamma$ and each string $\tau \in C$ of minimal length. If at a stage $t > s$ we once again have $\gamma \in T_t$, then $C \subseteq R_t$, and α now has permission to put $\langle r+c, \gamma\rangle$ into W. In short, the weight of requests put into W is accounted against the measure of new enumerations into R. If the sets belonging to different procedures are disjoint, then W is a Kraft-Chaitin set.

We discuss how to guarantee this disjointness. Suppose $\beta \neq \alpha$ is a procedure that wants to choose its set $C(\beta)$ at a stage $s' > s$. If γ is in some T_q with $s < q < s'$, then $C(\alpha) \subseteq R_{s'}$, so there is no problem, since β chooses its set disjoint from $R_{s'}$. However, if γ has not appeared in any such tree (and it possibly never will), then α wants to keep $C(\alpha)$ away from possible assignment to other procedures, which may cause a conflict because $C(\alpha)$ is relatively large. The solution to this problem is to build up the set $C(\alpha)$ in small portions D, whose measure is a fixed fraction of $2^{-(r+c)}$, and only assign a new set D once the old one is in R. If γ always reappears on a tree after such a set is assigned, then eventually $C(\alpha)$ reaches the required measure $2^{-(r+c)}$, in which case α is allowed to enumerate the request $\langle r+c, \gamma\rangle$ into W. Otherwise, α keeps away from assignment to other procedures only a single set D, whose measure is so small that the union (over all procedures) of the measures of sets kept away in this fashion is at most the small quantity 2^{-u-2}.

To ensure that $L^X(\tau) \geqslant 2^d$ for each $X \succcurlyeq \gamma$, the procedure $\alpha = \langle \rho, \gamma\rangle$ acts as follows. Once $U_s(\rho) = |\gamma|$, it claims $\varepsilon = 2^{-r}$ of the initial capital 1 of L at the root node λ (recall that $r = |\rho|$), and generally preserves it along both extensions of a string. It chooses its strings τ of the form $\nu 0^{r+d+1}$, where ν is a string where the capital claimed by α is still available. At ν it "withdraws" this capital, by defining $L^\gamma(\nu 0) = L^\gamma(\nu) + \varepsilon$ and, to maintain the martingale property, $L^\gamma(\nu 1) = L^\gamma(\nu) - \varepsilon$. From $\nu 0$ on, it doubles the capital along τ, always betting all the capital on 0, thus eventually reaching an increase of 2^d at τ.

Different procedures $\langle \rho, \gamma\rangle$ and $\langle \rho', \gamma'\rangle$ have to choose their sets D to be disjoint, but there is no conflict as far as the capital is concerned: if γ, γ' are incompatible, then they refer to different martingales L^X. Otherwise they claim different amounts of the initial capital. For any L^X, the total capital claimed is at most Ω, as a procedure $\langle \rho, \gamma\rangle$ with $\gamma \prec X$ claims $2^{-|\rho|}$ much of the capital only once $U(\rho)$ converges, and there is at most one such procedure for each ρ.

Finally, as a witness for Lemma 11.13 is not actually known, we do the above for each possible witness $\langle \sigma, d, u\rangle$. Let $\{\langle \sigma_m, d_m, u_m\rangle\}_{m\in\omega}$ be an effective listing of such witnesses. For each m we build a martingale functional L^X_m as above, but now with initial capital 2^{-m}, and make it eventually constant if it turns out that $\mu_{\sigma_m}(R_s) > 1-2^{-u_m}$ for some s. We now have to choose τ of the form $\nu 0^{m+r+d+1}$ to make up for the smaller capital. Now simply let $L = \sum_m L_m$. Since $A \in \mathsf{Low}(\mathsf{MLRand}, \mathsf{CRand})$, Lemma 11.13 holds for $N = L^A$, via a witness $\langle \sigma_m, d_m, u_m\rangle$. Now (1) holds for $N = L^A_m$, since $L^A \geqslant L^A_m$. Thus each $\gamma \prec A$ reappears infinitely often on the trees T_s, and so the "accounting against" trick outlined in the proof of Theorem 7.6 allows us to define the desired Kraft-Chaitin set W (where the constant c is $d_m + m + u_m + 3$). $\dashv$

We have seen characterizations of lowness for 1-randomness and Schnorr randomness. These results raise the question of characterizing lowness for computable randomness. In [1, Problem 4.8], Ambos-Spies and Kučera asked whether there is a noncomputable set that is low for computable randomness. Downey conjectured that the answer is negative, and this conjecture was confirmed by Nies [105].

THEOREM 11.14 (Nies [105]). *A set is low for computable randomness iff it is computable.*

The proof in [105] is a direct argument similar to but preceding the proof of Theorem 11.12 discussed above. But we can also use Theorem 11.12: Suppose that A is low for computable randomness. Then $A \in \mathrm{Low}(\mathsf{MLRand}, \mathsf{CRand})$, and hence A is K-trivial, and thus Δ^0_2. On the other hand, by a result of Bedregal and Nies [11], A has hyperimmune-free degree. But the only Δ^0_2 sets of hyperimmune-free degree are the computable ones, by Miller and Martin [99].

The original proof has the advantage of being extendible to the resource bounded setting, and also to show that each set in $\mathrm{Low}(\mathsf{PrecRand}, \mathsf{CRand})$ is computable. Here $\mathsf{PrecRand}$ is the class of sets on which no partial computable martingale succeeds (i.e., no martingale whose values are uniformly computable, but that may choose to be undefined on strings off the given set).

§12. Relativized randomness. We have so far focused on 1-randomness and weaker notions. We can also obtain stronger notions of randomness by increasing the complexity of tests in terms of the arithmetical hierarchy. These notions are of particular interest when we study the relationship between randomness and Turing degrees. The basic result connecting 1-randomness to Turing reducibility is the celebrated one often attributed only to Gács, but actually first proved by Kučera.

THEOREM 12.1 (Kučera [67], Gács [47]). *Every set is wtt-reducible to a* 1*-random set.*

The easiest proof of Theorem 12.1 is the recent one of Merkle and Mihailović [91]. That proof shows that the bound on the wtt-reduction can be taken to be $n + o(n)$. However, this bound cannot be improved to $n + O(1)$; that is, there are sets that are not sw-reducible to a 1-random set (see [34] for a proof). It is an open question whether every set is rK-reducible (or even K-reducible) to a 1-random set.

Other results on the Turing degrees of 1-random sets include Kučera's theorem [67] that all degrees above $\mathbf{0}'$ contain 1-random sets. As noted in Section 4, for each c, the collection of sets with initial segment prefix-free complexity $\geqslant n - c$ for all n is a Π^0_1-class, so there are 1-random sets of low Turing degree. Kučera [67] proved that the Δ^0_2 degrees containing 1-random sets are not closed upwards. On the other hand, their jumps are better behaved: using a new basis theorem for Π^0_1-classes with no computable members, Downey and Miller [42] showed that for every S that is c.e. in and above $\emptyset'$, there is a Δ_2^0 1-random set A with $A' \equiv_{\mathrm{T}} S$. This result was stated earlier by Kučera [68], who constructed a high incomplete Δ_2^0 1-random set using a similar technique.

Kučera also observed that 1-randomness is connected to the PA-degrees, which were discussed in Section 10.2. This connection was recently clarified by Stephan [133], who proved the following.

THEOREM 12.2 (Stephan [133]). *If X is 1-random and has PA-degree, then $\emptyset' \leqslant_T X$.*

The following is another result demonstrating the computational weakness of the 1-random sets that cannot compute $\emptyset'$.

THEOREM 12.3 (Hirschfeldt, Nies, and Stephan [51]). *Suppose A is a c.e. set and $X \geqslant_T A$ is 1-random and such that $\emptyset' \nleqslant_T X$. Then A is K-trivial.*

Theorems 12.2 and 12.3 establish that there are two kinds of 1-random sets. The first are those that are computationally rich and can compute the halting problem. The second are those that are computationally feeble and cannot even compute a $\{0,1\}$-valued diagonally noncomputable function or a non-K-trivial c.e. set. As we see below, this means that all 2-random sets (defined below) are computationally weak.

The basic definition of 1-randomness can be generalized quite easily. We will use the following definitions, noted by several researchers, such as Solovay [129] and Kurtz [73].

DEFINITION 12.4. (i) A Σ_n^0-*test* is a sequence $\{V_k\}_{k\in\omega}$ of uniformly Σ_n^0-classes such that $\mu(V_k) \leqslant 2^{-k}$. A set A passes this test if $A \notin \bigcap_k V_k$.
(ii) A set is Σ_n^0-*random* or n-*random* if it passes all Σ_n^0 tests.
(iii) One can similarly define Π_n^0, Δ_n^0, etc. tests and randomness.
(iv) A set is *arithmetically random* if it is n-random for all n.

These definitions can be relativized in the same way as 1-randomness, to yield notions such as n-randomness relative to a set X.

We have identified Σ_1^0-classes of sets with c.e. sets of strings, since every Σ_1^0-class is equivalent to $\bigcup\{[\sigma] : \sigma \in W\}$ for some (prefix-free) c.e. set of strings W. However, we cannot do the same at higher levels of the arithmetical hierarchy. For example, consider the Σ_2^0-class consisting of those sets that are zero from some point onwards. This Σ_2^0-class is *not* equivalent to one of the form $\bigcup\{[\sigma] : \sigma \in W\}$ for some Σ_2^0 set of strings W.

The use of open sets is basic in many arguments involving 1-randomness. Fortunately, this technique can be resurrected for higher-order randomness, as we now see. We denote the n-th jump of $\emptyset$ by $\emptyset^{(n)}$.

THEOREM 12.5 (Kurtz [73], Kautz [61]).

(i) *From the index of a* Σ^0_n*-class* S *and* $q \in \mathbb{Q}$*, we can compute the index of a* $\Sigma^{\emptyset^{(n-1)}}_1$*-class* $U \supseteq S$ *that is also an open* Σ^0_n*-class and such that* $\mu(U) - \mu(S) < q$*.*

(ii) *From the index of a* Π^0_n*-class* T *and* $q \in \mathbb{Q}$*, we can compute the index of a* $\Pi^{\emptyset^{(n-1)}}_1$*-class* $V \subseteq T$ *that is also a closed* Π^0_n*-class and such that* $\mu(T) - \mu(V) < q$*.*

(iii) *From the index of a* Σ^0_n*-class* S *and* $q \in \mathbb{Q}$*, we can* $\emptyset^{(n)}$*-compute the index of a closed* Π^0_{n-1}*-class* $V \subseteq S$ *such that* $\mu(S) - \mu(V) < q$*. Moreover, if* $\mu(S)$ *is computable from* $\emptyset^{(n-1)}$ *then the index of* V *can be found computably from* $\emptyset^{(n-1)}$*.*

(iv) *From the index of a* Π^0_n*-class* T *and* $q \in \mathbb{Q}$*, we can* $\emptyset^{(n)}$*-compute the index of an open* Σ^0_{n-1}*-class* $U \supseteq T$ *such that* $\mu(U) - \mu(T) < q$*. Moreover, if* $\mu(S)$ *is computable from* $\emptyset^{(n-1)}$ *then the index of* U *can be found computably from* $\emptyset^{(n-1)}$*.*

Using the above result, we can easily show, for instance, that $(n+1)$-randomness coincides with 1-randomness relative to $\emptyset^{(n)}$, which is a theorem of Kurtz [73].

Let U be the standard universal prefix-free machine. Then U^X will be a universal prefix-free machine relative to any X, and we obtain the following natural $(n+1)$-random sets.

$$\Omega^{(n)} := \sum_{U^{\emptyset^{(n)}}(\sigma)\downarrow} 2^{-|\sigma|}.$$

See Section 14 for more on relativizing Ω. There are other natural examples of n-random sets, defined without the use of relativization; see for instance Becher, Daicz, and Chaitin [9]; Becher and Chaitin [8]; and Becher and Grigorieff [10].

There is a very interesting intertwining of plain Kolmogorov complexity and relativized randomness.

DEFINITION 12.6. A set A is *Kolmogorov random* if for some c,

$$\exists^\infty n\,[C(A \upharpoonright n) \geqslant n - c]. \tag{2}$$

We say that A is *time-bounded Kolmogorov random* with time bound t if (2) holds with C^t instead of C, where C^t is the time-t-bounded Kolmogorov complexity. (For more on time-bounded complexity, see Li and Vitányi [81].)

While we have seen in Section 3.3.1 that no set can satisfy (2) with $\forall$ in place of $\exists^\infty$, the class of Kolmogorov random sets has measure 1. The next theorem shows that *Kolmogorov randomness is equivalent to* 2-*randomness*. Yu, Ding, and Downey [144] proved that every 3-random set is Kolmogorov random. They also observed that there is no Δ_2^0 Kolmogorov random set. This fact is also implied by the following result, since 2-random sets cannot be Δ_2^0.

THEOREM 12.7 (Nies, Stephan, and Terwijn [109]). *Let g be a computable time bound such that $g(n) \geqslant n^2 - O(1)$. The following are equivalent for any set Z:*

(i) *Z is 2-random.*
(ii) *Z is Kolmogorov random.*
(iii) *Z is Kolmogorov random with time bound g.*

The implication (i) $\Rightarrow$ (ii) in Theorem 12.7 was proved independently and earlier by Miller [94].

A set A is *strongly Chaitin random* if there is a c such that $\exists^\infty n\, [K(A \upharpoonright n) \geqslant n + K(n) - c]$. Solovay [129] showed that (up to additive constants) if a string has maximal prefix-free Kolmogorov complexity then it has maximal plain Kolmogorov complexity, so by Theorem 12.7, strong Chaitin randomness implies 2-randomness. It is also known that 3-randomness implies strong Chaitin randomness (see Theorem 13.11 below). It is not known whether strong Chaitin randomness is equivalent to either 2-randomness or 3-randomness.

Another characterization of 2-randomness can be given by considering sets that are low for Ω (see Definition 7.5).

THEOREM 12.8 (Nies, Stephan, and Terwijn [109]). *A set is 2-random iff it is 1-random and low for Ω.*

PROOF. By Corollary 12.18 below, for any two sets A and B, if A is 1-random and B is 1-random relative to A, then A is 1-random relative to B. Thus if A is 1-random, then A is 2-random $\Leftrightarrow$ A is 1-random relative to $\Omega \Leftrightarrow \Omega$ is 1-random relative to $A \Leftrightarrow A$ is low for Ω. Since any 2-random set is 1-random, the equivalence follows. $\dashv$

Thus every 2-random set is low for Ω, which by Corollary 7.8 gives us the following result.

COROLLARY 12.9 (Sacks and Stillwell, see Kautz [61, Thm. IV.2.4]). *Every 2-random set is GL_1.*

It is straightforward to define Kurtz, Schnorr, and computably n-random sets for all n by analogy with the above. It is not difficult to see that being Kurtz 2-random coincides with passing all generalized Martin-Löf tests $\{U_n\}_{n\in\omega}$, where we have $\mu(U_n) \to 0$, but there may be no computable decreasing bound on $\mu(U_n)$. Thus every Kurtz 2-random set is 1-random. Relativizing this observation and the fact that every 1-random set is Kurtz 1-random (see Section 10.3), we see that every n-random set is Kurtz n-random, and every Kurtz $(n+1)$-random set is n-random. Neither implication can be reversed.

THEOREM 12.10 (Kurtz [73]). *For every $n \geqslant 1$, there is an n-random set that cannot be computed by any Kurtz $(n+1)$-random set.*

PROOF. By relativizing the proof that there are 1-random sets below $\emptyset'$, we see that there is an n-random set $A \leqslant_{\mathrm{T}} \emptyset^{(n)}$. For each e, let

$$P_e = \{B : \Phi_e^B = A\} = \{B : \forall x\, \exists s\, [\Phi_{e,s}^B(x)\downarrow] \;\wedge\; \forall x\, \forall s\, [\Phi_{e,s}^B(x)\downarrow \rightarrow \Phi_{e,s}^B(x) = A(x)]\}.$$

By Theorem 8.11, if A is not computable then $\mu(\{B : A \leqslant_{\mathrm{T}} B\}) = 0$, so each P_e is a $\Pi_1^{\emptyset^{(n)}}$-class of measure 0, and hence each $\overline{P_e}$ is a Σ_{n+1}^0-class of measure 1. So every Kurtz $(n+1)$-random set must be in each $\overline{P_e}$, and hence cannot compute A. ⊣

COROLLARY 12.11 (to the proof of Theorem 12.10).
(i) *No Kurtz $(n+1)$-random set is computable from $\emptyset^{(n)}$.*
(ii) *For $n \geqslant 1$, there is an n-random set computable from $\emptyset^{(n)}$.*

The following result was first proved by Kautz, although it was stated without proof by Gaifman and Snir [48]. Kautz's proof was fairly complicated, but we can obtain a simpler proof using relativizations of results mentioned above.

THEOREM 12.12 (Kautz [61], Kurtz [73] for $n = 1$). *Let $n \geqslant 1$. There is a Kurtz n-random set that is not n-random.*

PROOF. Let X be a $\Sigma_1^{\emptyset^{(n-1)}}$ set such that $\emptyset^{(n-1)} <_{\mathrm{T}} X <_{\mathrm{T}} \emptyset^{(n)}$. By Theorem 10.23 relativized to $\emptyset^{(n-1)}$, there is a Kurtz n-random set A such that $A \oplus \emptyset^{(n-1)} \equiv_{\mathrm{T}} X$. On the other hand, by Theorem 4.1 relativized to $\emptyset^{(n-1)}$, A cannot be n-random, since that would imply that $X \equiv_{\mathrm{T}} \emptyset^{(n)}$. ⊣

Kurtz [73], Kautz [61], and van Lambalgen [74] examined the relationship between relativized randomness and the Turing degrees. They proved a number of classic results. We give a sample, along with some more recent related results, and include a few proofs as examples.

THEOREM 12.13 (van Lambalgen [74], Kautz [61]).
(i) *If $A \oplus B$ is n-random, then so are A and B.*
(ii) *If A is n-random, then so is $A^{[n]}$, the n-th column of A.*

THEOREM 12.14 (van Lambalgen [74]). *If $A \oplus B$ is n-random, then A is n-random relative to B.*

COROLLARY 12.15 (Kučera (see [61]), van Lambalgen [74]). *If $A \oplus B$ is 1-random, then $A \mid_T B$.*

Thus every 1-random set X splits into two Turing incomparable halves, both of which are computable in X. So we have the following result.

COROLLARY 12.16 (Kurtz [73]). *No 1-random set has minimal degree.*

Using Theorem 10.13, Yu [141] has recently shown that Theorem 12.14 fails for Schnorr and computable randomness, even for $n = 1$. Thus this important tool in the theory of 1-randomness is not available in the study of these notions.

The following is a converse to Theorem 12.14 (which, as pointed out by Yu [141], does also hold for Schnorr and computable randomness).

THEOREM 12.17 (van Lambalgen [74]). *If B is n-random and A is n-random relative to B, then $A \oplus B$ is n-random.*

PROOF. We give a proof due to Nies. Suppose $A \oplus B$ is not n-random. We show that either B is not n-random or A is not n-random relative to B. By Theorem 12.5, we can choose a sequence $V_0 \supseteq V_1 \supseteq \cdots$ of uniformly Σ^0_n open sets such that $A \oplus B \in \bigcap_i V_i$ and $\mu(V_i) \leqslant 2^{-2i}$.

Let $\emptyset$ be the empty string. We write $[\sigma \oplus \tau]$ for the collection of sets $X = X_0 \oplus X_1$ such that $\sigma \prec X_0$ and $\tau \prec X_1$.

Let
$$S_i = \bigcup\{[\sigma] : \mu(V_i \cap [\emptyset \oplus \sigma]) \geqslant 2^{-i-|\sigma|}\}.$$
Clearly, $S_{i+1} \subseteq S_i$, and the S_i are uniformly Σ^0_n open sets. We show that $\mu(S_i) \leqslant 2^{-i}$. Let $\sigma_0, \sigma_1, \ldots$ be a listing of the strings σ that are minimal (under the substring relation) such that $\mu(V_i \cap [\emptyset \oplus \sigma]) \geqslant 2^{-i-|\sigma|}$. Then $S_i = \bigcup_j [\sigma_j]$. Since the sets $V_i \cap [\emptyset \oplus \sigma_j]$ are pairwise disjoint and $\mu(V_i) \leqslant 2^{-2i}$, we see that $\sum_j 2^{-i-|\sigma_j|} \leqslant 2^{-2i}$, and hence $\mu(S_i) \leqslant \sum_j 2^{-|\sigma_j|} \leqslant 2^{-i}$.

If $B \in \bigcap_i S_i$, then B is not n-random. Otherwise, there is a j such that $B \notin S_i$ for all $i > j$. For such i, let
$$R^k_i = \bigcup\{[\sigma] : |\sigma| = k \ \wedge \ [\sigma \oplus B \restriction k] \subseteq V_i\}.$$
Then $\mu(R^k_i) \leqslant 2^{-i}$, since $B \notin S_i$. Moreover, since V_i is open, $R^k_i \subseteq R^{k+1}_i$. Let $R_i = \bigcup_k R^k_i$. The R_i are open and uniformly Σ^0_n relative to B, and $\mu(R_i) = \sup\{\mu(R^k_i) : k \in \omega\} \leqslant 2^{-i}$ for $i > j$. Furthermore, $A \in R_i$ for each $i > j$, so A is not n-random relative to B. ⊣

Combining Theorems 12.14 and 12.17, we have the following corollary, which will be useful several times below.

COROLLARY 12.18 (van Lambalgen [74]). *If B is n-random and A is n-random relative to B, then B is n-random relative to A.*

The following is an application of this result.

THEOREM 12.19 (Miller and Yu [97]). *If A is n-random and $B \leqslant_T A$ is 1-random, then B is n-random.*

PROOF. We can assume that $n \geqslant 2$. Let $X \equiv_T \emptyset^{(n-1)}$ be 1-random. Since A is n-random, it is 1-random relative to X. So X is 1-random relative to A, and therefore relative to B. Hence B is 1-random relative to X, and thus is n-random. ⊣

By different means, Miller and Yu [97] proved the stronger result that for any X, if A is 1-random relative to X and $B \leqslant_{\mathrm{T}} A$ is 1-random, then B is 1-random relative to X.

THEOREM 12.20 (Kautz [61]). *If A and B are 2-random relative to each other, then their degrees form a minimal pair.*

PROOF. Suppose that $0 <_{\mathrm{T}} C \leqslant_{\mathrm{T}} A, B$. Let e be such that $\Phi_e^A = C$. It is easy to check that $\{X : \Phi_e^X = C\}$ is a Π_2^C-class, and by Theorem 8.11, it has measure 0. So A is not 2-random relative to C, and hence relative to B, contrary to hypothesis. ⊣

Thus, by Theorem 12.14, every 2-random set is the join of a minimal pair.

Theorem 12.20 cannot be extended to the $n = 1$ case. Indeed, Kučera [67] showed that no two Δ_2^0 1-random sets form a minimal pair. However, we have the following consequence of Theorem 8.10.

THEOREM 12.21 (Hirschfeldt, Nies, and Stephan [51]). *If A and B are 1-random relative to each other, then any $X \leqslant_T A, B$ is K-trivial.*

PROOF. Since A is 1-random relative to B, it is 1-random relative to X. So X is a basis for 1-randomness, and thus is K-trivial. ⊣

Randomness is linked to properties of "almost all" degrees. Classically, Kolmogorov's 0-1 law states that any class of sets closed under finite translations has measure 0 or 1 (see e.g. Oxtoby [112]). There is also an effective 0-1 law.

LEMMA 12.22 (Kurtz [73], Kautz [61], Kučera for $n = 1$). *Let X be a set, let $n \geqslant 1$, and let T be a Π_n^X-class of positive measure. If $A \in 2^\omega$ is n-random relative to X, then there are $\sigma \in 2^{<\omega}$ and $B \in T$ such that $A = \sigma B$. Thus, for every set A that is n-random relative to X, the class T contains a member Turing equivalent to A.*

A class $\mathcal{C} \subseteq 2^\omega$ is *degree invariant* if $A \in \mathcal{C}$ and $B \equiv_{\mathrm{T}} A$ implies $B \in \mathcal{C}$. It is *closed under translations* if $A \in \mathcal{C}$ implies $\sigma A \in \mathcal{C}$ for every string σ.

COROLLARY 12.23 (Kurtz [73], Kautz [61]). *Let $\mathcal{C}$ be a Σ_{n+1}^0- or Π_{n+1}^0-class that is degree invariant, or even just closed under translations. Then $\mathcal{C}$ contains either all n-random sets or no n-random sets.*

For example, Martin (unpublished; see Kurtz [73, Theorem 3.3] for a proof) showed that the class $\{A : \deg(A) \text{ is hyperimmune}\}$ has measure 1. By analyzing Martin's proof and applying the effective 0-1 law, Kautz [61] showed that this class includes every 2-random set. This result cannot be extended to the 1-random sets, since, by Jockusch and Soare [58], every Π_1^0-class has a member of hyperimmune-free degree.

Kautz [61] also noted that the effective 0-1 law has an easy but interesting consequence for the theory of the Turing degrees. For a degree $\mathbf{a}$, let $\mathrm{Th}(\mathcal{D}_{\leqslant \mathbf{a}})$ be the theory of the degrees less than or equal to $\mathbf{a}$. For any sentence ψ in the language of degree theory, the class of all A such that $\psi \in \mathrm{Th}(\mathcal{D}_{\leqslant \deg(A)})$ is arithmetical, so if $\mathbf{a}$ and $\mathbf{b}$ are degrees of arithmetically random sets, then $\mathrm{Th}(\mathcal{D}_{\leqslant \mathbf{a}}) = \mathrm{Th}(\mathcal{D}_{\leqslant \mathbf{b}})$. A more careful analysis can be made to calculate the level of randomness necessary to ensure the equality of the n-quantifier fragments of these theories.

There is a wealth of material in this area, and not enough room to present it all. We finish this section by looking at the initial segment complexity of n-random sets, a topic which will also be addressed in the next section. Of course, there is a natural characterization in terms of relativized machines; namely, A is $(n+1)$-random iff $K^{\emptyset^{(n)}}(A \upharpoonright k) \geqslant k - O(1)$. However, one would naturally expect n- and $(n+1)$-random sets to have different unrelativized initial segment complexities. This is true for $\Omega^{(n)}$.

Theorem 12.24 (Yu, Ding, and Downey [144]). *For all c and $n < m$,*

$$\exists^\infty k\, [K(\Omega^{(n)} \upharpoonright k) < K(\Omega^{(m)} \upharpoonright k) - c].$$

For $n = 0$ and $m = 1$, Theorem 12.24 was proved by Solovay [129], using totally different methods.

In contrast to this result, Miller and Yu [97] showed that the different relativizations of Ω have incomparable K-degrees. Indeed, they proved the following stronger result, which will be further discussed in the next section.

Theorem 12.25 (Miller and Yu [97]). *For all $m \neq n$, the K-degrees of $\Omega^{(m)}$ and $\Omega^{(n)}$ have no upper bound.*

§13. Results of Miller and Yu, and van Lambalgen reducibility. Recently, Joe Miller and Liang Yu [95, 97, 98] have proved some remarkable results on the initial segment complexities of random sets, which highlight both the strengths and limitations of initial segment complexity as a measure of relative randomness. Motivated by van Lambalgen's Theorems 12.14 and 12.17, they introduced the following measure of relative randomness.

Definition 13.1 (Miller and Yu [97]). We say that A is *van Lambalgen reducible* to B, and write $A \leqslant_{\mathrm{vL}} B$, if for all $C \in 2^\omega$, if $A \oplus C$ is 1-random then $B \oplus C$ is 1-random.

This notion is closely related to one introduced by Nies. In [105, Section 8], Nies defined $A \leqslant_{\mathrm{LR}} B$ if every set that is 1-random relative to B is 1-random relative to A. (So, for instance, $A \leqslant_{\mathrm{LR}} \emptyset$ iff A is low for 1-randomness.) Notice that this relation is implied by Turing reducibility.

Nies [105] studied the monotone Σ^0_3 operator $\mathcal{LR}(B) := \{A : A \leqslant_{\mathrm{LR}} B\}$. He also showed that if A and B are c.e., then $A \leqslant_{\mathrm{LR}} B$ implies $A' \leqslant_{\mathrm{tt}} B'$. Moreover, applying the technique of pseudo-jumps from [57] to the c.e. operator given by the construction of a set that is low for 1-randomness, he showed that there is a c.e. set that is Turing incomplete but LR-complete.

If A and B are both 1-random then Theorems 12.14 and 12.17 imply that $A \leqslant_{\mathrm{LR}} B$ iff $B \leqslant_{\mathrm{vL}} A$. (Notice the inverse relationship.) If A is not 1-random, then $A \oplus C$ is never 1-random, no matter what C is, so the least vL-degree consists of all sets that are not 1-random. Thus vL-reducibility is interesting only on the 1-random sets. It might be fruitful to explore extensions of vL-reducibility that behave nontrivially on the non-1-random sets.

The following result summarizes some of the basic properties of vL-reducibility and the resulting vL-degrees.

Theorem 13.2 (Miller and Yu [97]).

(i) *If A is n-random and $A \leqslant_{vL} B$, then B is n-random.*
(ii) *If $A \oplus B$ is 1-random then the vL-degrees of A and B have no upper bound. Thus there is no join operator on the vL-degrees.*
(iii) *If $A \leqslant_T B$ and A is 1-random, then $B \leqslant_{vL} A$.*
(iv) *There are 1-random sets $A \equiv_{vL} B$ such that $A <_T B$.*
(v) *There are no maximal or minimal vL-degrees of 1-random sets.*
(vi) *If $A \oplus B$ is 1-random then $A \oplus B <_{vL} A, B$.*
(vii) *Every finite partial order can be embedded into the vL-degrees, and hence the Σ^0_1-theory of the vL-degrees is decidable.*

One of the attractive features of vL-reducibility is that it can be used to prove results about K- and C-reducibility, in ways that are often easier than dealing directly with these reducibilities. The following result is what allows for the transfer of results from vL-reducibility to K- and C-reducibility.

Theorem 13.3 (Miller and Yu [97]). *For any sets A and B,*

(i) $A \leqslant_K B \Rightarrow A \leqslant_{vL} B$ *and*
(ii) $A \leqslant_C B \Rightarrow A \leqslant_{vL} B$.

We state the following consequences for K-reducibility, but they also hold for C-reducibility.

Corollary 13.4 (Miller and Yu [97]).

(i) *If $A \leqslant_K B$ and A is n-random, then B is n-random.*
(ii) *If $A \oplus B$ is 1-random, then $A \mid_K B$, and the K-degrees of A and B have no upper bound. Thus there is no join operator on the K-degrees.*

According to Miller and Yu [97], R. Rettinger independently announced that if $A \oplus B$ is 1-random then $A \mid_K B$.

Theorem 12.25 follows from the second part of the above result, since if $m \neq n$ then $\Omega^{(m)}$ and $\Omega^{(n)}$ are 1-random relative to each other, and hence $\Omega^{(m)} \oplus \Omega^{(n)}$ is 1-random.

Thus we see that, while there is no direct correlation between increasing levels of randomness and increasing K-degrees, there *is* a relationship between levels of randomness and initial segment complexity.

Although certain results on vL-reducibility can be transfered to K-reducibility, there are some important differences between these notions. For instance, by part (iii) of Theorem 13.2, every Δ^0_2 1-random set is $\geqslant_{\mathrm{vL}} \Omega$. On the other hand, we have the following result, which shows that if $\Omega = \Omega_0 \oplus \Omega_1$ then Ω_0 is an example of a Δ^0_2 1-random set $\not\geqslant_K \Omega$.

THEOREM 13.5 (Miller and Yu [97]). *If $A \oplus B$ is 1-random then $A \mid_K A \oplus B$.*

An even more significant difference between the vL-degrees and the K-degrees is that the former are invariant under computable permutations (by part (iii) of Theorem 13.2 and the closure of the notion of 1-randomness under computable permutations), but the latter are not.

THEOREM 13.6 (Miller and Yu [97]). *There is a computable permutation $f : \omega \to \omega$ such that for every 1-random set A, the K-degrees of A and $f(A)$ have no upper bound.*

This result suggests that K-reducibility may be too strong as a measure of relative randomness on the 1-random sets, and that vL-reducibility may be a better measure in this context.

As pointed out by Miller and Yu [97], it follows from part (ii) of Theorem 13.2 that if we let Ω_n be the n-th column of Ω, then $\{\Omega_n : n \in \omega\}$ is a vL-antichain, and hence a K-antichain. So we have a concrete example of infinitely many pairwise incomparable K-degrees of 1-random sets. It was a vexing open question whether there are any *comparable* K-degrees of 1-random sets. Miller and Yu [98] recently answered this question, making use of a converse to part (ii) of Theorem 3.10, which shows that the Ample Excess Theorem (part (i) of Theorem 3.10) is in a sense tight.

THEOREM 13.7 (Miller and Yu [98]). *Let f be any function such that $\sum_n 2^{-f(n)} < \infty$. There is a 1-random set A such that*

$$K(A \upharpoonright n) \leqslant n + f(n) + O(1).$$

COROLLARY 13.8 (Miller and Yu [98]). *Let B be 1-random. There is a 1-random set $A <_K B$. In fact, A can be chosen so that $\lim_n K(B \upharpoonright n) - K(A \upharpoonright n) = \infty$.*

PROOF. Let $g(n) = K(B \upharpoonright n) - n$. By the Ample Excess Theorem, $\sum_n 2^{-g(n)} < \infty$, so there is a function f such that $\lim_n g(n) - f(n) = \infty$

and $\sum_n 2^{-f(n)} < \infty$. Let A be as in Theorem 13.7. Then $\lim_n K(B \upharpoonright n) - K(A \upharpoonright n) = \infty$. ⊣

Miller and Yu [98] showed that it is also possible to ensure additionally that $A \oplus C <_K B$ for all C. As they pointed out in [97], if we take a C that is 1-random relative to A, then $A \oplus C$ is 1-random, and the K-degree of B bounds the K-degrees of A and $A \oplus C$, which does away with a possible improvement of Theorem 13.5.

Using new techniques and extensions of the above methods, Miller proved the following.

THEOREM 13.9 (Miller [95]).

(i) *If A and B are* 1*-random and $A \equiv_K B$, then $A' \equiv_{tt} B'$. Thus every K-degree of* 1*-random sets is countable.*
(ii) *If A and B are* 3*-random and $A \leqslant_K B$, then $B \leqslant_T A \oplus \emptyset'$ and $B' \leqslant_T A'$.*

As noted by Miller [95], it follows from part (ii) that the upper cone above the K-degree of a 3-random set is always countable. On the other hand, Miller and Yu [98] showed that there is a 1-random set whose K-degree is below an antichain of K-degrees of size $2^{\aleph_0}$.

As we have seen in Theorem 3.5, Miller and Yu [97] gave a plain Kolmogorov complexity characterization of 1-randomness. They also gave the following "mixed" characterization.

THEOREM 13.10 (Miller and Yu [97]). *A set A is* 1*-random iff*

$$C(A \upharpoonright n) \geqslant n - K(n) - O(1).$$

Several results we have seen point to the computational weakness of random sets. Indeed, there are ways in which sufficiently random sets begin to resemble highly nonrandom sets such as K-trivial sets.[10] For instance, Miller [95] has shown that if A is 3-random, then it is often useless in lowering the prefix-free complexity of strings, so that A resembles sets that are low for K. We say that A is *weakly low for K* if $\exists^\infty [K(n) \leqslant K^A(n) + O(1)]$. That is, for infinitely many n, the information in A is so useless that it cannot help to compress n.

THEOREM 13.11 (Miller [95]).

(i) *If A is* 3*-random, then it is weakly low for K.*
(ii) *If A is weakly low for K and* 1*-random, then it is strongly Chaitin random (as defined in Section 12).*

[10] Miller has suggested that we should find ways to quantify the idea that, while highly random sets have a great deal of information, it is *useless information.* There are existing concepts of useful information (see Antunes and Fortnow [3] for several references), but none seem to have been successfully applied yet to the context of higher order randomness.

§14. Relativizing Ω. So far when we have looked at relativizations, we have always looked at the "standard way" of forming a universal prefix-free machine (as in the definition of $\Omega^{(n)}$ in Section 12). Relativization acts strangely on randomness notions, since we are dealing with c.e. operators, but not CEA (computably enumerable in *and above*) operators. We have already seen this (in Section 8) since, for instance, if A is not GL_1, or even just not K-trivial, then no set that is 1-random relative to A is Turing above A (Kučera [69]; Hirschfeldt, Nies, and Stephan [51]). In particular, whatever Ω^Ω is, $\Omega \not\leqslant_{\mathrm{T}} \Omega^\Omega$.

Notice that this means that if we could construe Ω as an invariant operator, meaning that $A \equiv_{\mathrm{T}} B$ implies $\Omega^A \equiv_{\mathrm{T}} \Omega^B$, then it would be a degree invariant operator that is not the jump or an iterate of the jump, thereby resolving a longstanding conjecture of Martin (see [62, p. 279]). As we will see below, this is not the case, but given the central role of Ω in this area, it is natural to try to understand it as an operator in the same way that we seek to understand the halting problem, and hence the jump operator, in classical computability theory.

The first thing we need to understand is what we actually mean when we talk about relativizing Ω. Clearly, any reasonable definition must ensure that Ω^X is an X-left-c.e. real and is 1-random relative to X. Furthermore, the definition should be relatively oracle-independent, in the sense that it should only involve machines U such that U^X is a universal prefix-free machine relative to X for every oracle X, and the coding constants of prefix-free oracle machines do not depend on the oracle. More precisely, we have the following definitions from [37]. An oracle machine M is a *prefix-free oracle machine* if M^A is prefix-free for every $A \in 2^\omega$. A prefix-free oracle machine U is *universal* if for every prefix-free oracle machine M there is a $\tau \in 2^{<\omega}$ such that

$$\forall A \in 2^\omega\, \forall \sigma \in 2^{<\omega}\, [U^A(\tau\sigma) = M^A(\sigma)].$$

In other words, U can simulate any prefix-free oracle machine in a way that does not depend on the oracle. Given such a machine, we can define the halting probability

$$\Omega_U^A := \sum_{U^A(\sigma)\downarrow} 2^{-|\sigma|},$$

which can be thought of as an operator on 2^ω; we call such an operator an *Omega operator.*

Recall Theorems 4.5 and 4.6, which together imply that the only 1-random left-c.e. reals are versions of Ω. We would like to relativize these results. The relativization of Theorem 4.6 is straightforward, but the same is not true of Theorem 4.5. We can relativize the notion of Solovay reducibility in the natural way, and hence talk about X-Solovay complete X-left-c.e. reals. The proof of Theorem 4.5 given above does relativize in the sense that if a real is X-left-c.e. and X-Solovay complete then it is the halting probability of some prefix-free oracle machine M^X that is universal with respect to prefix-free machines with oracle X. However, there is no reason to expect that M should be a universal prefix-free oracle machine, since it is unclear how M^Y should behave for $Y \neq X$.

However, Downey, Hirschfeldt, Miller, and Nies [37] did show that the relativization of Theorem 4.5 holds. Together with the relativization of Theorem 4.6, this result yields the following theorem.

THEOREM 14.1 (Downey, Hirschfeldt, Miller, and Nies [37]). *The following are equivalent.*

(i) *α is an X-left-c.e. real and is 1-random relative to X.*
(ii) *α is an X-left-c.e. real and is X-Solovay complete.*
(iii) *$\alpha = \Omega_U^X$ for some universal prefix-free oracle machine U.*

This result has some rather counterintuitive consequences for the possible values of Ω^X for various oracles X. Let us reconsider n-randomness, that is, 1-randomness relative to $\emptyset^{(n-1)}$. Since every 2-random set is 1-random relative to Ω, it follows from Corollary 12.18 that Ω is 1-random relative to any 2-random set. Furthermore, Ω is a left-c.e. real relative to any oracle. Since any version of Ω^Ω is 2-random, we see that Ω is a possible value of Ω relative to Ω^Ω.

Another interesting property of Omega operators is that the class of low 1-random sets is closed under their action. Indeed, let U be a universal prefix-free oracle machine and let A be low and 1-random. Since Ω_U^A is 1-random relative to A, Corollary 12.18 implies that A is 1-random relative to Ω_U^A. So by Theorem 7.7, Ω_U^A is GL_1. But Ω_U^A is A-left-c.e., and hence is Δ_2^0. So Ω_U^A is low.

Theorem 14.1 is particularly interesting in light of the following result.

THEOREM 14.2 (Downey, Hirschfeldt, Miller, and Nies [37]). *Let $X \in 2^\omega$ be 2-random. Then there is an $A \in 2^\omega$ such that X is an A-left-c.e. real and is 1-random relative to A. Thus $X = \Omega_U^A$ for some universal prefix-free oracle machine U.*

So almost every real is a halting probability relative to some set. Furthermore, as shown in [37], for every universal prefix-free oracle machine U, the range of Ω_U has positive measure, which implies (by Kolmogorov's 0-1 law mentioned in Section 12) that for almost every X, there is an A such that $X =^* \Omega_U^A$, where we write $B =^* C$ to mean that B and C agree on a cofinite set.

As pointed out in [37], Theorem 14.2 cannot be extended to all 1-random sets, since if $1 - \Omega$ is an A-left-c.e. real then it is A-computable.

The case where Ω_U^A is a left-c.e. real is particularly interesting. The following result shows that this is not at all a rare occurrence.

THEOREM 14.3 (Downey, Hirschfeldt, Miller, and Nies [37]). *Let U be a universal prefix-free oracle machine.*

(i) $0 < \mu(\{A : \Omega_U^A \textit{ is a left-c.e. real}\}) < 1$.
(ii) *If* $\mu(\{A : \Omega_U^A = X\}) > 0$ *then X is a left-c.e. real.*

It is shown in [37] that A is low for Ω iff there is a universal prefix-free oracle machine U such that Ω_U^A is a left-c.e. real. By Theorem 12.8, every 2-random set is low for Ω, so almost every set is taken to a left-c.e. real by *some* Omega operator.

Another result in [37] is that for any universal prefix-free oracle machine U and any set X, the set of all B such that Ω_U^B is 1-random relative to X has positive measure. Together with the first part of Theorem 14.3, this gives a resoundingly negative solution to the question of the degree-invariance of Omega operators.

THEOREM 14.4 (Downey, Hirschfeldt, Miller, and Nies [37]). *Let U be a universal prefix-free oracle machine.*

(i) *For all $X \in 2^\omega$, there are $A, B \in 2^\omega$ with $A =^* B$ such that Ω_U^A is a left-c.e. real and Ω_U^B is 1-random relative to X.*
(ii) *There are $A, B \in 2^\omega$ such that $A =^* B$ and $\Omega_U^A \mid_T \Omega_U^B$ (and in fact, Ω_U^A and Ω_U^B are 1-random relative to each other).*

PROOF. (i) Let $\mathcal{S} = \{A : \Omega_U^A \text{ is a left-c.e. real}\}$ and $\mathcal{R} = \{B : \Omega_U^B \text{ is 1-random relative to } X\}$. Let $\widehat{\mathcal{R}} = \{A : \exists B \in \mathcal{R}\ (A =^* B)\}$. Since $\mathcal{R}$ has positive measure, Kolmogorov's 0-1 law implies that $\mu(\widehat{\mathcal{R}}) = 1$. Since $\mathcal{S}$ has positive measure, there is an $A \in \mathcal{S} \cap \widehat{\mathcal{R}}$.

(ii) By part 1, there are $A, B \in 2^\omega$ with $A =^* B$ such that Ω_U^A is a left-c.e. real and Ω_U^B is 2-random. Hence Ω_U^B is Ω_U^A-random and, by Corollary 12.18, Ω_U^A is Ω_U^B-random. This implies that $\Omega_U^A \mid_{\mathrm{T}} \Omega_U^B$. ⊣

In light of this result, one might wonder whether there are *any* degrees on which Omega operators are invariant. Once again, the answer is connected with K-triviality.

THEOREM 14.5 (Downey, Hirschfeldt, Miller, and Nies [37]). *Let $A \in 2^\omega$. the following are equivalent.*

(i) *A is K-trivial.*
(ii) *Every Omega operator takes A to a left-c.e. real.*
(iii) *Every Omega operator is degree invariant on the degree of A.*

There is an example in [37] of an Omega operator that is degree invariant only on the K-trivial degrees. It is not known whether every Omega operator has this property.

For further properties of Omega operators, including their interesting analytic behavior, see [37].

§15. Hausdorff dimension and partial randomness. In this section, we look at the following intuitive notion, which provides yet another way to calibrate randomness. Suppose that $\Omega = .a_1 a_2 \dots$. Then we would expect $.a_1 0 a_2 0 \dots$ to be "$\frac{1}{2}$-random". To make this notion of partial randomness precise, and describe some results that are very interesting in their own right, we need to detour through the theory of Hausdorff dimension.

15.1. Classical Hausdorff dimension. First we recall the definition of classical Hausdorff dimension [49]. For comments and discussion see for instance Falconer [45].

DEFINITION 15.1. (i) $C \subseteq 2^{<\omega}$ is an *n-cover* if $\sigma \in C \to |\sigma| \geqslant n$.
(ii) *C covers* $\mathcal{A} \subseteq 2^\omega$ if $\mathcal{A} \subseteq \bigcup_{\sigma \in C}[\sigma]$.
(iii) Define $H^\varepsilon_n(\mathcal{A}) := \inf\Big\{ \sum_{\sigma \in C} 2^{-\varepsilon|\sigma|} : C \text{ is an } n\text{-cover of } \mathcal{A}\Big\}$.
(iv) Define $H^\varepsilon(\mathcal{A}) := \lim_{n\to\infty} H^\varepsilon_n(\mathcal{A})$. This is the *ε-dimensional outer Hausdorff measure* of $\mathcal{A}$.

LEMMA 15.2. *Let* $\mathcal{A} \subseteq 2^\omega$. *There exists* $\varepsilon \in [0,1]$ *such that*
(i) $H^{\varepsilon'}(\mathcal{A}) = 0$ *for* $\varepsilon' > \varepsilon$ *and*
(ii) $H^{\varepsilon'}(\mathcal{A}) = \infty$ *for* $0 \leqslant \varepsilon' < \varepsilon$.

The ε in Lemma 15.2 is called the *Hausdorff dimension* of $\mathcal{A}$:

DEFINITION 15.3. $\dim(\mathcal{A}) := \inf\{\varepsilon : H^\varepsilon(\mathcal{A}) = 0\}$.

Hausdorff dimension has a number of basic properties:
(i) It gives a refinement of the notion of measure zero: If $\mu(X) \neq 0$, then $\dim(X) = 1$.
(ii) (monotonicity) If $X \subseteq Y$ then $\dim(X) \leqslant \dim(Y)$.
(iii) (countable stability) If I is countable, then

$$\dim\Big(\bigcup_{i\in I} Y_i\Big) = \sup_{i\in I}\big\{\dim(Y_i)\big\}.$$

In particular, $\dim(X \cup Y)$ is $\max\{\dim(X), \dim(Y)\}$.

15.2. Effective Hausdorff dimension. We now discuss effectivizations of Hausdorff dimension. There has been a large amount of research in effective dimension, and we only scratch the surface here. In particular, we do not discuss work in effectivizing other types of fractal dimension, such as packing dimension (see Athreya, Hitchcock, Lutz, and Mayordomo [4]). For more on effective dimension theory, see Reimann [115] or Lutz [87]; for results on the complexity of these and related notions, see Hitchcock, Lutz, and Terwijn [52].

Recall from Definition 10.4 the null sets of the form $S_h[d]$. Schnorr also addressed null sets of *exponential* order, which have the form $S_h[d]$

for $h(n) = 2^{\varepsilon n}$ with $\varepsilon \in (0,1]$. Although he did not make an explicit reference to Hausdorff dimension, it turns out that the theory of Hausdorff dimension can be cast precisely in terms of such null sets of exponential order.

Lutz constructivized Hausdorff dimension in [85, 86], using what he called *s-gales* (a generalization of martingales). Let $s \in [0,\infty)$. An *s-gale* is a function $d : 2^{<\omega} \to \mathbb{R}^+$ that satisfies the averaging condition

$$2^s d(\sigma) = d(\sigma 0) + d(\sigma 1) \tag{3}$$

for every $\sigma \in 2^{<\omega}$. (Notice that 1-gales are the same as martingales.) Similarly, d is an *s-supergale* if (3) holds with $\geqslant$ instead of equality. The success set $S[d]$ is defined exactly as was done for martingales in Section 2.2.

Although the following theorem shows that we do not really need s-gales for the treatment of Hausdorff dimension, it is sometimes convenient to use them.

THEOREM 15.4. (Lutz [85], Ambos-Spies, Merkle, Reimann, and Stephan [2], Calude, Staiger, and Terwijn [19]). *For any $\mathcal{A} \subseteq 2^\omega$ and $r \in [0,1]$, the following are equivalent:*

(i) *$\mathcal{A}$ has Hausdorff dimension r,*
(ii) *$r = \inf\{s \in \mathbb{Q} :$ there is an s-(super)gale d s.t. $\mathcal{A} \subseteq S[d]\}$.*
(iii) *$r = \inf\{s < 1 :$ there is a (super)martingale d s.t. $\mathcal{A} \subseteq S_{2^{(1-s)n}}[d]\}$, or $r = 1$ if this set is empty.*

So we see that the theory of (effective) Hausdorff dimension falls out as a special case of Schnorr's treatment of effective measure theory.

Theorem 15.4 motivates the following definition:

DEFINITION 15.5. Let $\mathcal{C}$ be a complexity class. A class $\mathcal{A} \subseteq 2^\omega$ has *$\mathcal{C}$-dimension r* if

$$r = \inf\{s < 1 : \exists d \in \mathcal{C}\,[d \text{ is a supermartingale and } \mathcal{A} \subseteq S_{2^{(1-s)n}}[d]\,]\},$$

or $r = 1$ if this set is empty.

The $\mathcal{C}$-dimension of a set A is the $\mathcal{C}$-dimension of the singleton $\{A\}$.

There is an important connection between Σ^0_1-dimension and Kolmogorov complexity, which was established in the form given below by Mayordomo [89] and prefigured by Ryabko [117, 118], Staiger [130, 131], and Cai and Hartmanis [13] (see Staiger [132] for a discussion of these and other related papers).

THEOREM 15.6 (Mayordomo [89]). *For any set A, the Σ^0_1-dimension of A is equal to*

$$\liminf_n \frac{K(A \upharpoonright n)}{n}.$$

Since plain and prefix-free Kolmogorov complexity are equal up to a log factor, this theorem also holds with C in place of K.

15.3. The picture of implications. The following relationships hold between various notions of effective randomness and dimension, where Δ^0_2-randomness and Schnorr Δ^0_2-randomness are the relativizations to $\emptyset'$ of computable randomness and Schnorr randomness, respectively.

$$
\begin{array}{ccc}
\Delta^0_2\text{-random} & & \\
\Downarrow & & \\
\text{Schnorr } \Delta^0_2\text{-random} & \Longrightarrow & \Delta^0_2\text{-dimension } 1 \\
\Downarrow & & \Downarrow \\
\text{1-random} & \Longrightarrow & \Sigma^0_1\text{-dimension } 1 \\
\Downarrow & & \\
\text{computably random} & & \Big\Downarrow \\
\Downarrow & & \\
\text{Schnorr random} & \Longrightarrow & \text{computable dimension } 1
\end{array}
$$

No other implications hold than the ones indicated. That these implications hold follows easily from the definitions. The strictness of the implications in the first column was discussed in Section 10 (except for the fact that there is a Schnorr Δ^0_2-random set that is not 1-random, which follows immediately from the fact that no Δ^0_2 set can be Schnorr Δ^0_2-random). That there are no more implications between the first and the second column follows from the next proposition. The strictness of the two implications in the second column follows by similar means. (It is easy to show (see [86]) that the class of computable sets has Σ^0_1-dimension 0, but is not computably null, so in particular the class of computable sets has computable dimension 1. Also, Lutz [86] has shown that for every Δ^0_2 real $r \in [0, 1]$, there is a Δ^0_2 set of Σ^0_1-dimension r, but it is obvious that every Δ^0_2 set has Δ^0_2-dimension 0.)

PROPOSITION 15.7 (Terwijn [136]). *There are sets that are not Schnorr random but have Δ^0_2-dimension 1.*

An important related open question, formulated independently by Reimann and by Terwijn (see [96]), is whether every set of positive Σ^0_1-dimension computes a 1-random set. Even if we strengthen the hypothesis to Σ^0_1-dimension 1, the question is still open.

15.4. Partial randomness. There are at least two possible versions of partial Martin-Löf randomness, as we now discuss. First we might base the definition upon a straightforward generalization of the original definition. A natural variation on s-gales is *s-measure*:

$$\mu_s(V) := \sum_{\sigma \in V} 2^{-s|\sigma|}$$

for an open set V. Here it is important to think of V as a prefix-free collection of strings, since if $s \neq 1$ then different presentations of the same open set can have different s-measures.

We can use this notion to define partial Martin-Löf randomness as follows.

DEFINITION 15.8 (Tadaki [134]). (i) A *weak Martin-Löf s-test* is a computable collection of c.e. open sets $\{V_k\}_{k\in\omega}$ such that $\mu_s(V_k) \leqslant 2^{-k}$ for all k.

(ii) We say that A is *weakly Martin-Löf s-random* if $A \notin \bigcap_k V_k$ for every weak Martin-Löf s-test $\{V_k\}_{k\in\omega}$.

We can also define a set A to be weakly Levin-Chaitin s-random if $K(A \upharpoonright n) \geqslant sn - O(1)$ for all n. The analog of Schnorr's Theorem 3.8 that Levin-Chaitin random is the same as Martin-Löf random can be established with a similar proof.

THEOREM 15.9 (Tadaki [134]). *A set A is weakly Martin-Löf s-random iff A is weakly Levin-Chaitin s-random.*

Armed with this result, we can emulate the proof that Ω is Levin-Chaitin random to show the following:

THEOREM 15.10 (Tadaki [134]). *Let $0 < s \leqslant 1$ be a computable real and define*

$$\Omega^s := \sum_{U(\sigma)\downarrow} 2^{-\frac{|\sigma|}{s}},$$

where U is a universal prefix-free machine. Then Ω^s is weakly Martin-Löf s-random.

Similarly, we can construct a universal weak Martin-Löf s-test, and establish similar analogues to results on Martin-Löf randomness.

This notion squares with our intuition that if $A = a_1 a_2 \ldots$ is random then $B = a_1 0 a_2 0 a_2 \ldots$ should be "somewhat" random. Indeed B is weakly Martin-Löf $\frac{1}{2}$-random. To see this suppose that for each d there are infinitely many n such that $K(B \upharpoonright n) < \frac{1}{2}n - d$. Consider the prefix-free machine M that simulates the universal prefix-free machine U and, when it finds that $U(\sigma)\downarrow$ is of the form $b_1 0 b_2 0 \ldots b_n 0$, outputs $b_1 \ldots b_n$. Then $K_M(A \upharpoonright n) = K(B \upharpoonright 2n)$, so for each c there are infinitely many n such that $K_M(A \upharpoonright n) < n - c$, and hence A is not 1-random.

We would like to prove the analogs of our basic results that martingale randomness, test set randomness, and incompressibility all coincide. Unfortunately, the proof breaks down for the martingale case. Consider the proof that if a set is Martin-Löf random then no c.e. martingale succeeds on it. We are given a c.e. martingale d, and when we see $d(\sigma) > 2^k$ we put σ into U_k. Now imagine we are following the same proof method for the $s < 1$ case. The problem is that d is only c.e. We might see that $d_t(\sigma 0) > 2^k$ and put $\sigma 0$ into V_k. At some later stage $u > t$, we might see that $d_u(\sigma) > 2^k$. We would like to put σ into V_k, but need to keep the set prefix-free. In the $s = 1$ case we can do this by putting $\sigma 1$ into V_k. But in the $s < 1$ case, $2(2^{-s(|\sigma|+1)})$ might be much bigger than $2^{-s|\sigma|}$.

Another approach, taken by Lutz [85, 86], is to define partial randomness using martingales and orders, as in the development of Hausdorff dimension. Recall that Schnorr proved that a set is 1-random iff no c.e. martingale succeeds on A. Now we want to say that no c.e. martingale *quickly* succeeds on A.

DEFINITION 15.11 (Calude, Staiger, and Terwijn [19, 136]).

(i) A *strong Martin-Löf s-test* is a computable collection of c.e. sets of strings $\{V_k\}_{k\in\omega}$ (*not necessarily prefix-free*) such that for all prefix-free subsets $\widehat{V}_k \subseteq V_k$,

$$\sum_{\sigma\in\widehat{V}_k} 2^{-s|\sigma|} \leqslant 2^{-k}.$$

(ii) We say that A is *strongly Martin-Löf s-random* if $A \notin \bigcap_k \bigcup_{\sigma\in V_k}[\sigma]$ for every strong Martin-Löf s-test $\{V_k\}_{k\in\omega}$.

(iii) We say that A is *Schnorr s-random* if for any c.e. (super)martingale d,

$$\limsup_{n\to\infty} \frac{d(A \upharpoonright n)}{2^{(1-s)n}} < \infty.$$

(iv) We say that A is *Lutz s-random* if for any c.e. s-(super)gale d, we have $A \notin S[d]$.

THEOREM 15.12 (Calude, Staiger, and Terwijn [19, 136]). *Fix a computable s with $0 < s \leqslant 1$. Then a set A is strongly Martin-Löf s-random iff A is Schnorr s-random iff A is Lutz s-random.*

It is also possible to give a machine characterization of strong Martin-Löf s-randomness. We say that a machine M is *s-measurable* if for all prefix-free $V \subseteq \mathrm{dom}(M)$, we have $\mu_s(\bigcup_{\sigma\in V}[\sigma]) \leqslant 1$.

THEOREM 15.13 (Downey, Reid, and Terwijn (see [114])). *A set A is strongly Martin-Löf s-random iff $K_M(A \upharpoonright n) \geqslant n - O(1)$ for all s-measurable machines M.*

Reimann and Stephan (unpublished; see [116]) have recently shown that weak Martin-Löf s-randomness does not imply strong Martin-Löf s-randomness.

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