Erratum to "On the structure of the Medvedev lattice"
S. A. Terwijn, April 28, 2009.

This is an erratum to Lemma 2.3 of [4]. We thank Paul Shafer and Richard Shore for pointing out a mistake at the end of the proof of Lemma 2.3, where it is falsely concluded that from the incomparability of the $\mathcal{C}_I$ the same follows for the $\mathcal{B}\times\mathcal{C}_I$. Below we fix the proof of Lemma 2.3. The rest of the paper is not affected, and in particular Theorem 2.10 that depends on Lemma 2.3 remains valid. We use the same notation as in [4]. In particular, for any $f\in\omega^\omega$ we let $f^-(x)=f(x+1)$. 0^ω denotes the all zero sequence. For any mass problem $\mathcal{C}$, instead of $\{0^\omega\}+\mathcal{C}$ we simply write $0^\omega+\mathcal{C}$.

Lemma A. *Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems such that*

$$\forall \mathcal{C}\subseteq\mathcal{A} \text{ finite } \;(\mathcal{B}\times\mathcal{C}\not\leqslant_M \mathcal{A}), \tag{1}$$

and let $\mathcal{D}_0$, $\mathcal{D}_1\subseteq 2^\omega$ be dense (or even just dense along 0^ω). Then there exists a pair $\mathcal{C}_0$, $\mathcal{C}_1\geqslant_M\mathcal{A}$ such that

$$\big(\mathcal{D}_0+\mathcal{C}_0\big)\cup\big(0^\omega+\mathcal{C}_1\big)\not\geqslant_M \mathcal{B}\times\mathcal{C}_1,$$
$$\big(\mathcal{D}_1+\mathcal{C}_1\big)\cup\big(0^\omega+\mathcal{C}_0\big)\not\geqslant_M \mathcal{B}\times\mathcal{C}_0.$$

Proof. Instead of $\mathcal{A}$ we work with the set

$$\textstyle\prod_{n\in\omega}\mathcal{A}=\big\{n\hat{\ }f : n\in\omega\wedge f\in\mathcal{A}\big\}.$$

This set gives us infinitely many disjoint copies of $\mathcal{A}$, so that we can pick witnesses from fresh copies of $\mathcal{A}$. We will build the $\mathcal{C}_i\subseteq\prod_n\mathcal{A}$ as unions of finite sets $\bigcup_s\mathcal{C}_{i,s}$. We want to satisfy the following requirements for all $e\in\omega$:

R^0_e : $\quad\Phi_e\big(\mathcal{D}_0+\mathcal{C}_0\cup 0^\omega+\mathcal{C}_1\big)\not\subseteq\mathcal{B}\times\mathcal{C}_1.$

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The construction proceeds as follows.

Stage s=0. Let $\mathcal{C}_{0,0}=\mathcal{C}_{1,0}=\emptyset$. All $n\in\omega$ are declared fresh. At every stage there will be only finitely many numbers that are not fresh.

Stage s+1=2e+1. We satisfy R^0_e. Pick $n\in\omega$ fresh. Consider the computable functional $\Phi(f)=\Phi_e(0^\omega\oplus n\hat{\ }f)$. By condition (1) there is $f\in\mathcal{A}$ such that $\Phi(f)\notin\mathcal{B}\times\mathcal{C}_{1,s}$. (Either by being undefined or by not being an element of $\mathcal{B}\times\mathcal{C}_{1,s}$.) If $\Phi(f)(0)\neq 1$ or $\Phi(f)$ is not total (i.e. $\Phi(f)(x)\uparrow$ for some x) then put $n\hat{\ }f$ into $\mathcal{C}_1$. Then $0^\omega\oplus n\hat{\ }f$ is a witness for R^0_e. Otherwise $\Phi(f)$ is total and $\Phi(f)(0)=1$, and $\Phi(f)\notin 1\hat{\ }\mathcal{C}_{1,s}$. Since $\mathcal{C}_{1,s}$ is *finite* it follows that there is a finite initial segment 0^k such that $\Phi_e(0^k\oplus n\hat{\ }f)(x)\downarrow=\Phi(f)(x)$ for $x\in\{0,1\}$ and for all $g\in\mathcal{C}_{1,s}$ there is an x such that $\Phi_e(0^k\oplus n\hat{\ }f)(x)\downarrow\neq 1\hat{\ }g(x)$. Hence for all $h\sqsupseteq 0^k$ it holds that $\Phi_e(h\oplus n\hat{\ }f)\notin 1\hat{\ }\mathcal{C}_{1,s}$. By density of $\mathcal{D}_0$, choose $h\in\mathcal{D}_0$ with $h\sqsupseteq 0^k$ and put $n\hat{\ }f$ into $\mathcal{C}_0$. Then $h\oplus n\hat{\ }f$ is a witness for R^0_e, provided that future elements of $\mathcal{C}_1$ are kept different from $\Phi_e(h \oplus n\widehat{\ }f)^-$. This will be by freshness: We declare $m = \Phi(f)(1)$ to be *nonfresh*, which ensures that at later stages only elements $n\widehat{\ }f$ with $n \neq m$ will enter $\mathcal{C}_1$.

Stage s+1=2e+2. The construction to satisfy R^1_e is symmetric to the one for R^0_e, switching the roles of $\mathcal{C}_1$ and $\mathcal{C}_0$. This ends the construction.

The construction succeeds in satisfying the requirements: Either the element $0^\omega \oplus n\widehat{\ }f$ that is put into $0^\omega + \mathcal{C}_1$ at stage $s + 1 = 2e + 1$ is a witness to R^0_e or there is an element $h \oplus n\widehat{\ }f \in \mathcal{D}_0 + \mathcal{C}_0$ with $\Phi_e(h \oplus n\widehat{\ }f) \notin 1\widehat{\ }\mathcal{C}_{1,s}$. In the latter case, since $m = \Phi_e(h \oplus n\widehat{\ }f)(1) = \Phi(f)(1)$ is declared nonfresh, all witnesses put into $\mathcal{C}_1$ at a later stage are different from $\Phi_e(h \oplus n\widehat{\ }f)^-$ (because at the beginning of every stage we pick a fresh n), hence $\Phi_e(h \oplus n\widehat{\ }f) \notin 1\widehat{\ }\mathcal{C}_1$ and R^0_e is satisfied. The case of R^1_e is again symmetric. □

Lemma B. *Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems such that*

$$\forall \mathcal{C} \subseteq \mathcal{A} \text{ finite } \ (\mathcal{B} \times \mathcal{C} \not\leqslant_M \mathcal{A}), \tag{1}$$

and let $\mathcal{D}_\alpha \subseteq 2^\omega$ be dense (or even just dense along 0^ω) for every $\alpha \in 2^\omega$. Then there exist countable mass problems $\mathcal{C}_\alpha \geqslant_M \mathcal{A}$, $\alpha \in 2^\omega$, such that

$$\big(\mathcal{D}_\alpha + \mathcal{C}_\alpha\big) \cup \big(0^\omega + \mathcal{C}_\beta\big) \not\geqslant_M \mathcal{B} \times \mathcal{C}_\beta$$

for all $\alpha \neq \beta$.

Proof. As in Sacks' construction of an antichain of size $2^{\aleph_0}$ in the Turing degrees [2], [1, p. 462] we construct a tree of $\mathcal{C}_\alpha$, $\alpha \in 2^\omega$, but now with the basic strategies from the proof of Lemma A. We build finite sets $\mathcal{C}_\sigma \subseteq \prod_{n\in\omega} \mathcal{A}$, $\sigma \in 2^{<\omega}$, and thus obtain for every path $\alpha \in 2^\omega$ a set $\mathcal{C}_\alpha = \bigcup_{\sigma \sqsubset \alpha} \mathcal{C}_\sigma$. Given two sets $\mathcal{C}_\sigma$ and $\mathcal{C}_\tau$, $|\sigma| = |\tau| = s$, at stage $s = e$, we want to ensure that

$\Phi_e\big(\mathcal{D}_\alpha + \mathcal{C}_\alpha \cup 0^\omega + \mathcal{C}_\beta\big) \not\subseteq \mathcal{B} \times \mathcal{C}_\beta$ and

$\Phi_e\big(\mathcal{D}_\beta + \mathcal{C}_\beta \cup 0^\omega + \mathcal{C}_\alpha\big) \not\subseteq \mathcal{B} \times \mathcal{C}_\alpha$

for all $\alpha \sqsupset \sigma$ and $\beta \sqsupset \tau$. The basic strategy for doing this is exactly the same as in Lemma A, and the way in which the strategies are put together on a tree is the same as in Sacks' construction. □

Lemma C. *Let $\mathcal{C}_\alpha$, $\alpha \in 2^\omega$, be as in Lemma B. There is a perfect set of indices $\mathcal{T} \subseteq 2^\omega$ such that*

$$(\forall \alpha, \beta \in \mathcal{T})(\forall f \in \mathcal{C}_\alpha)(\forall g \in \mathcal{C}_\beta)\ [\alpha \neq \beta \to \alpha \oplus f \,|_T\, \beta \oplus g]. \tag{2}$$

Proof. The reason that it is possible to construct such a set $\mathcal{T}$ is that every $\mathcal{C}_\alpha$ is countable, and if $f \in \mathcal{C}_\alpha$ then f in its totality is put into $\mathcal{C}_\alpha$ at some finite stage of the construction in Lemma B. We construct $\mathcal{T}$ as the set of paths in a (noncomputable) tree $T \subseteq 2^{<\omega}$.

Construction of T. Let $\mathcal{C}_\sigma$, $\sigma \in 2^{<\omega}$, refer to the finite approximations of the $\mathcal{C}_\alpha$ from the proof of Lemma B. At stage s of the construction we have defined $T(\sigma) \in 2^{<\omega}$ for all $\sigma \in 2^{<\omega}$ of length $< s$, and we define extensions $T(\sigma)$ for every σ of length s. For every $\sigma \neq \tau$ of length $s = e$ we guarantee

$$(\forall \alpha \sqsupset T(\sigma))(\forall \beta \sqsupset T(\tau))(\forall f \in \mathcal{C}_\sigma)(\forall g \in \mathcal{C}_\tau)\ \big[\Phi_e(\alpha \oplus f) \neq \beta \oplus g \wedge \\ \Phi_e(\beta \oplus g) \neq \alpha \oplus f\big].$$

This can be realized in a standard finite extension construction à la Sacks, because the sets $\mathcal{C}_\sigma$ and $\mathcal{C}_\tau$ are finite. Given f and g (possibly equal), the basic strategy for constructing α and β with $\alpha \oplus f \,|_T\, \beta \oplus g$ is the same as in the Kleene-Post construction of two incomparable sets. This concludes the construction of T.

The construction of T guarantees that its set of paths $\mathcal{T}$ satisfies (2): Given $\alpha \neq \beta$ in $\mathcal{T}$ and $f \in \mathcal{C}_\alpha$, $g \in \mathcal{C}_\beta$, the construction guarantees that $\Phi_e(\alpha \oplus f) \neq \beta \oplus g$ and $\Phi_e(\beta \oplus g) \neq \alpha \oplus f$ for all e larger than the point in T where α and β split and larger than the stage where f has entered $\mathcal{C}_\alpha$ and g has entered $\mathcal{C}_\beta$. Since we have this for almost every e, by padding we have $\alpha \oplus f \,|_T\, \beta \oplus g$. $\square$

Lemma 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems satisfying the condition*

$$\forall \mathcal{C} \subseteq \mathcal{A} \text{ finite } \ (\mathcal{B} \times \mathcal{C} \not\leqslant_M \mathcal{A}). \tag{1}$$

Then there exists an antichain $\mathcal{C}_I$, $I < 2^{2^{\aleph_0}}$, of mass problems such that $\mathcal{C}_I \geqslant_M \mathcal{A}$ for every I and such that the elements $\mathcal{B} \times \mathcal{C}_I$ are also pairwise M-incomparable. In particular, none of the $\mathcal{C}_I$ is above $\mathcal{B}$.

Proof. Let $\mathcal{C}_\alpha$, $\alpha \in 2^\omega$, be as in Lemma B, and let $\mathcal{T}$ be as in Lemma C. Fix a single element $\alpha_0 \in \mathcal{T}$, and for every $\alpha \neq \alpha_0$ define

$$\mathcal{G}_\alpha = \big(\deg_T(\alpha) + \mathcal{C}_\alpha\big) \cup \big(0^\omega + \mathcal{C}_{\alpha_0}\big).$$

For every subset $I \subseteq \mathcal{T} - \{\alpha_0\}$ define

$$\mathcal{C}_I = \bigcup_{\alpha \in I} \mathcal{G}_\alpha.$$

Note that $\mathcal{C}_I \geqslant_M \mathcal{A}$ for every I via the mapping $h \oplus n\widehat{\ }f \mapsto f$.

Now consider any family $\mathcal{I}$ of cardinality $2^{2^{\aleph_0}}$ of pairwise incomparable (with respect to inclusion) subsets of $\mathcal{T} - \{\alpha_0\}$ (cf. Proposition 3.1). We claim that if I and J are incomparable subsets of $\mathcal{T} - \{\alpha_0\}$ then $\mathcal{B} \times \mathcal{C}_I \,|_M\, \mathcal{B} \times \mathcal{C}_J$. Namely, let $\alpha \in I - J$. Because $\deg_T(\alpha)$ is dense we have by Lemma B that $\mathcal{G}_\alpha \not\geqslant_M \mathcal{B} \times \mathcal{C}_{\alpha_0}$, hence also

$$\mathcal{G}_\alpha \not\geqslant_M \mathcal{B} \times \big(0^\omega + \mathcal{C}_{\alpha_0}\big). \tag{3}$$

It follows from (2) that no element of $\deg_T(\alpha) + \mathcal{C}_\alpha$ or of $\mathcal{C}_{\alpha_0}$ can compute any element of $\deg_T(\beta) + \mathcal{C}_\beta$ whenever $\beta \neq \alpha, \alpha_0$, $\beta \in \mathcal{T}$. Hence from (3) it follows that

$$\mathcal{G}_\alpha \not\geqslant_M \mathcal{B} \times \bigcup_{\beta \neq \alpha, \alpha_0} \mathcal{G}_\beta.$$

Since $\mathcal{G}_\alpha \subseteq \mathcal{C}_I$ and $\mathcal{C}_J \subseteq \bigcup_{\beta \neq \alpha, \alpha_0} \mathcal{G}_\beta$ we have in particular that $\mathcal{C}_I \not\geqslant_M \mathcal{B} \times \mathcal{C}_J$, and hence $\mathcal{B} \times \mathcal{C}_I \not\geqslant_M \mathcal{B} \times \mathcal{C}_J$. So the sets $\mathcal{B} \times \mathcal{C}_I$, $I \in \mathcal{I}$, form an antichain of cardinality $2^{2^{\aleph_0}}$. $\square$

References

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