INTERMEDIATE LOGICS AND FACTORS OF THE MEDVEDEV LATTICE

ANDREA SORBI AND SEBASTIAAN A. TERWIJN

ABSTRACT. We investigate the initial segments of the Medvedev lattice as Brouwer algebras, and study the propositional logics connected to them.

1. INTRODUCTION

The Medvedev lattice $\mathfrak{M}$ was introduced by Medvedev [14] in order to provide a computational semantics for constructive (propositional) logic. $\mathfrak{M}$ is a rich structure that is interesting in its own right, for example it can be studied in connection with other structures from computability theory such as the Turing degrees, but certainly the connections with constructive logic add an extra flavour to it. There are of course many other approaches to the semantics for constructive logics, ranging from algebraic (McKinsey and Tarski [13]) to Kripke semantics, and from realizability (Kleene) to the Logic of Proofs (Artemov and others [1]), to name only a few of many possible references. Medvedev's approach, following informal ideas of Kolmogorov, provides a complete computational semantics for various intermediate propositional logics, that is, propositional logics lying in between intuitionistic logic and classical logic. The notion of Medvedev reducibility has recently been applied also in other areas of computability theory, e.g. in the study of Π_1^0-classes, cf. for example Simpson [18].

In this paper we study the logics connected to the factors (or equivalently, the initial segments) of $\mathfrak{M}$. This is natural from an algebraic viewpoint, but it is also motivated by Skvortsova's magnificent result that there is a factor of $\mathfrak{M}$ whose theory is the intuitionistic propositional logic, cf. Section 4 below. We start by briefly recalling some background material. For more extensive discussions about $\mathfrak{M}$ we refer to the survey paper by Sorbi [23]. Our computability theoretic notation is fairly standard and follows e.g. Odifreddi [17]. In particular, ω denotes the natural numbers, ω^ω is the set of all functions from ω to ω (Baire space),

and Φ_e is the eth partial Turing functional.[1] $\omega^{<\omega}$ is the set of all finite strings of natural numbers. $\sigma \sqsubseteq \tau$ denotes that the finite string σ is an initial segment of the (possibly infinite) string τ. $\sigma^{\frown}\tau$ denotes string concatenation (with τ possibly infinite). $[\sigma]$ denotes the set $\{f \in \omega^\omega : \sigma \sqsubseteq f\}$. $f{\upharpoonright}n$ denotes the finite string $f(0)f(1)\ldots f(n-1)$. We list some further notation according to theme:

Lattice theory: In order to avoid confusion when interpreting logical formulas on lattices we refrain from using the notation $\wedge$ and $\vee$ in the context of lattices, but rather use $\times$ and $+$ for meet and join, as in Balbes and Dwinger [2]. Given a finite set A of elements in a lattice, $\prod A$ denotes the meet of all the elements in A and $\sum A$ denotes the join.

A *Brouwer algebra* is a distributive lattice with a least element 0 and greatest element 1 and equipped with a binary operation $\rightarrow$ satisfying for all a and b

$$a \rightarrow b = \min\{c : a + c \geqslant b\}.$$

Given $\rightarrow$ one can also define the unary operation of negation by $\neg a = a \rightarrow 1$. If $\mathfrak{L}$ is a Brouwer algebra then $\mathrm{Th}(\mathfrak{L})$ denotes the set of propositional formulas that are valid in $\mathfrak{L}$, i.e. that evaluate to 1 under every valuation of the variables with elements from $\mathfrak{L}$, where $\wedge$ is interpreted by $+$, $\vee$ by $\times$, $\rightarrow$ by $\rightarrow$, and $\neg$ by $\neg$. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are Brouwer algebras we say that $\mathfrak{L}_1$ is *B-embeddable* in $\mathfrak{L}_2$ if there is a lattice-theoretic homomorphism $f : \mathfrak{L}_1 \longrightarrow \mathfrak{L}_2$, preserving 0 and 1, and the binary operation $\rightarrow$ as well. If $f : \mathfrak{L}_1 \longrightarrow \mathfrak{L}_2$ is a B-embedding then $\mathrm{Th}(\mathfrak{L}_2) \subseteq \mathrm{Th}(\mathfrak{L}_1)$, as is easily seen. If f is surjective then also $\mathrm{Th}(\mathfrak{L}_1) \subseteq \mathrm{Th}(\mathfrak{L}_2)$. For $a \in \mathfrak{L}$, if $\mathfrak{G}$ is the principal filter generated by a, the factorized lattice $\mathfrak{L}/\mathfrak{G}$ is again a Brouwer algebra, with the same operations as in $\mathfrak{L}$, except for $\neg$ which is defined in the factor as $\neg b = b \rightarrow a$. We recall that for elements b and c from $\mathfrak{L}$, it holds that $b \leqslant c$ in $\mathfrak{L}/\mathfrak{G}$ if there is $d \in \mathfrak{G}$ such that $b \times d \leqslant c$ in $\mathfrak{L}$. For notational simplicity we denote this Brouwer algebra by $\mathfrak{L}/a$. Note that $\mathfrak{L}/a$ is isomorphic, as a Brouwer algebra, to the initial segment $[0, a]$ in $\mathfrak{L}$, so that studying factors of $\mathfrak{L}$ amounts to the same as studying the initial segments of $\mathfrak{L}$. An element $a \in \mathfrak{L}$ is *join-reducible* if there are $b, c < a$ such that $a = b + c$, and a is *meet-reducible* if there are $b, c > a$ such that $a = b \times c$.

Medvedev degrees: A *mass problem* is a subset of ω^ω. One can think of such a subset as a "problem", namely the problem of producing an element of it, and so we can think of the elements of the mass problem as its set of solutions. Informally, a mass problem $\mathcal{A}$ *Medvedev reduces* to a mass problem $\mathcal{B}$ if there is an effective procedure of transforming solutions to $\mathcal{B}$ into solutions to $\mathcal{A}$. Formally, $\mathcal{A} \leqslant_M \mathcal{B}$ if there is a partial Turing functional $\Psi : \omega^\omega \rightarrow \omega^\omega$ such that for all $f \in \mathcal{B}$, $\Psi(f)$ is defined and $\Psi(f) \in \mathcal{A}$. The relation $\leqslant_M$ induces an equivalence relation on the mass problems: $\mathcal{A} \equiv_M \mathcal{B}$ if $\mathcal{A} \leqslant_M \mathcal{B}$ and $\mathcal{B} \leqslant_M \mathcal{A}$. The equivalence class of $\mathcal{A}$ is denoted by $\deg_M(\mathcal{A})$ and is called the *Medvedev degree* (abbreviated by M-degree) of $\mathcal{A}$ (or, following Medvedev [14], the *degree of difficulty* of $\mathcal{A}$). We use boldface letters $\mathbf{A}$ for M-degrees and calligraphic letters $\mathcal{A}$ for mass problems. The

---

[1]Usually in computability theory, Φ_e is a partial function $\omega^\omega \times \omega \rightarrow \omega$, where the first argument is called the oracle. By writing $\Phi_e(f)(x) = \Phi_e(f, x)$, we can of course also view Φ_e as a partial function $\omega^\omega \rightarrow \omega^\omega$, which is the view taken in this paper.

collection of all M-degrees is denoted by $\mathfrak{M}$, partially ordered by $\deg_M(\mathcal{A}) \leqslant_M \deg_M(\mathcal{B})$ if $\mathcal{A} \leqslant_M \mathcal{B}$. Note that there is a smallest Medvedev degree $\mathbf{0}$, namely the degree of any mass problem containing a computable function. There is also a largest degree $\mathbf{1}$, the degree of the empty mass problem. For functions f and g, as usual define the function $f \oplus g$ by $f \oplus g(2x) = f(x)$ and $f \oplus g(2x+1) = g(x)$. Let $n^\frown\mathcal{A} = \{n^\frown f : f \in \mathcal{A}\}$, where $n^\frown f$ stands for $\langle n\rangle^\frown f$, i.e. string concatenation, with $\langle n\rangle$ being the string consisting of the unique number n. The join operator

$$\mathcal{A} + \mathcal{B} = \big\{f \oplus g : f \in \mathcal{A} \wedge g \in B\big\}$$

and the meet operator

$$\mathcal{A} \times \mathcal{B} = 0^\frown\mathcal{A} \cup 1^\frown\mathcal{B}.$$

make $\mathfrak{M}$ a distributive lattice, as is easy to check. Finally, given mass problems $\mathcal{A}$ and $\mathcal{B}$, let us define

$$\mathcal{A} \to \mathcal{B} = \{e^\frown f : \forall g \in \mathcal{B}(\Phi_e(g \oplus f) \in \mathcal{A})\}.$$

(Recall that Φ_e is the eth Turing functional.) Then, by Medvedev [14], the binary operation $\to$ on mass problems generates a well-defined binary operation $\to$ on M-degrees that turns $\mathfrak{M}$ into a Brouwer algebra.

An important mass problem is $0' = \big\{f \in \omega^\omega : f \text{ noncomputable}\big\}$. The boldface version $\mathbf{0}'$ denotes the M-degree of $0'$. It is the unique nonzero minimal element of $\mathfrak{M}$: if $\mathcal{A} \not\equiv_M 0$ then $0' \leqslant_M \mathcal{A}$. The join-irreducible mass problems

$$\mathcal{B}_f = \big\{g \in \omega^\omega : g \not\leqslant_T f\big\}$$

also play an important role in the study of $\mathfrak{M}$.

We will make an occasional reference to the nonuniform variant of the Medvedev lattice: the Muchnik lattice $\mathfrak{M}_w$. This is the structure resulting from the reduction relation on mass problems defined by

$$\mathcal{A} \leqslant_w \mathcal{B} \Leftrightarrow (\forall g \in \mathcal{B})(\exists f \in \mathcal{A})[f \leqslant_T g],$$

where $\leqslant_T$ denotes Turing reducibility. $\mathfrak{M}_w$ is a Brouwer algebra in the same way that $\mathfrak{M}$ is, with the same lattice theoretic operations, and the operation $\to$ given by

$$\mathcal{A} \to \mathcal{B} = \{f : \forall g \in \mathcal{B} \exists h \in \mathcal{A}(h \leqslant_T g \oplus f)\}.$$

An M-degree is a *Muchnik degree* if it contains a mass problem that is upwards closed under Turing reducibility $\leqslant_T$. The Muchnik degrees of $\mathfrak{M}$ form a substructure that is isomorphic to $\mathfrak{M}_w$ with respect to the operations $+$ and $\to$. That the Muchnik degrees are closed under $\to$ follows from Skvortsova [19, Lemma 5].

Using the algebraic framework defined above, we can now study factors of $\mathfrak{M}$: Given any mass problem $\mathcal{A}$ we can consider $\mathfrak{M}$ modulo the principal filter generated by the M-degree of $\mathcal{A}$. Using the notational convention from above, we denote this structure by $\mathfrak{M}/\deg_M(\mathcal{A})$, or simply by $\mathfrak{M}/\mathcal{A}$. In this paper we are interested in the theories of the form $\text{Th}(\mathfrak{M}/\mathcal{A})$.

To illustrate the above definitions we note the following simple result.

Proposition 1.1. *For every $\mathcal{A}$ we have* $\text{Th}(\mathfrak{M}/\mathcal{A}) \subseteq \mathsf{CPC}$.

Proof. The two element Brouwer algebra $\{0, 1\}$ is always B-embeddable into $\mathfrak{M}/\mathcal{A}$, hence we have $\mathrm{Th}(\mathfrak{M}/\mathcal{A}) \subseteq \mathrm{Th}(\{0, 1\})$. But the latter theory equals CPC. $\qquad\square$

The leading question that concerns us in this paper is the following:

Question 1.2. *What are the possible logics of the form* $\mathrm{Th}(\mathfrak{M}/\mathcal{A})$ *?*

In Section 2 we summarize what we know about this question, including some of the results of this paper.

Notation for logics: IPC is the intuitionistic propositional calculus and CPC is the classical propositional calculus. A logic L is called *intermediate* if $\mathsf{IPC} \subseteq L \subseteq \mathsf{CPC}$. A very useful source for what is known about intermediate logics up to 1983 is the annotated bibliography by Minari [16]. For a logic L we denote the positive part (i.e. the negation-free fragment) by L^+. Jan is Jankov's logic $\mathsf{IPC} + \neg p \vee \neg\neg p$ consisting of the closure of IPC and the weak law of the excluded middle, sometimes also called De Morgan logic. Other logical principles considered in this paper are the Kreisel-Putnam formula

$$(\neg p \rightarrow q \vee r) \rightarrow (\neg p \rightarrow q) \vee (\neg p \rightarrow r) \qquad\qquad \text{(KP)}$$

and the Scott formula

$$((\neg\neg p \rightarrow p) \rightarrow (\neg p \vee p)) \rightarrow (\neg\neg p \vee p). \qquad\qquad \text{(Sc)}$$

If it cannot cause confusion we will sometimes also use KP and Sc to denote the logics corresponding to these principles, i.e. $\mathsf{IPC} + \mathsf{KP}$ and $\mathsf{IPC} + \mathsf{Sc}$. LM denotes the *Medvedev logic (of finite problems)*, cf. page 8.

2. Questions and summary of results

We summarize what we know about Question 1.2 in the following list. As the theories of the factors are determined by the structural properties of the factors, results and questions about them go hand in hand.

- For every $\mathcal{A}$, $\mathsf{IPC} \subseteq \mathrm{Th}(\mathfrak{M}/\mathcal{A})$. This follows from the fact that $\mathfrak{M}/\mathcal{A}$ is a Brouwer algebra.

- For every $\mathcal{A}$, $\mathrm{Th}(\mathfrak{M}/\mathcal{A}) \subseteq \mathsf{CPC}$. (Cf. Proposition 1.1.) So we see from this and the previous item that indeed all logics of the form $\mathrm{Th}(\mathfrak{M}/\mathcal{A})$ are intermediate.

- (Medvedev [15], Jankov [7], Sorbi [21]) $\mathrm{Th}(\mathfrak{M}) = \mathsf{Jan}$. This result can be seen as a failure to obtain a semantics for intuitionistic logic along the lines of Kolmogorov and Medvedev. However, see the next item.

- (Skvortsova [19]) There exists $\mathcal{A}$ such that $\mathrm{Th}(\mathfrak{M}/\mathcal{A}) = \mathsf{IPC}$. (We discuss this result in more detail in Section 4.) This showed that even if the logic of $\mathfrak{M}$ is not IPC (by the previous item), the program envisaged by Kolmogorov and Medvedev can be made to succeed after all by considering suitable factors of $\mathfrak{M}$.

- Recall the mass problem $0'$ from above. The factor corresponding to this element gives classical logic: $\mathrm{Th}(\mathfrak{M}/0') = \mathsf{CPC}$, and $0'$ is the *only* such element. Note that $\mathfrak{M}/0'$ consists precisely of two elements, playing the

role of classical true and false. In all other factors we have at least three elements, and hence we can refute $p \vee \neg p$ by taking for p an element different from 0 and 1, i.e. the least and greatest elements of the factor, respectively.

- If $\mathcal{A}$ is a closed mass problem then we always have $\text{Th}(\mathfrak{M}/\mathcal{A}) \subsetneq \mathsf{Jan}$. (Cf. Theorem 6.1 and the remark after Theorem 7.2.)

- (Sorbi [22, Theorem 4.3]) $\neg p \vee \neg\neg p \in \text{Th}(\mathfrak{M}/\mathcal{A})$ if and only if $\mathcal{A}$ is join-irreducible. For any join-irreducible $\mathcal{A} >_M 0'$, we always have $\text{Th}(\mathfrak{M}/\mathcal{A}) = \mathsf{Jan}$.

- If $\mathcal{A}$ bounds a join-irreducible mass problem $>_M 0'$ it holds that $\text{Th}(\mathfrak{M}/\mathcal{A}) \subseteq \mathsf{Jan}$ (cf. Theorem 7.1). Does *every* $\mathcal{A} >_M 0'$ bound a join-irreducible degree $>_M 0'$? Not every $\mathcal{A}$ bounds a $\mathcal{B}_f$, f noncomputable (cf. Theorem 7.6). However, every *closed* $\mathcal{A}$ bounds a $\mathcal{B}_f$, f noncomputable (cf. Theorem 7.2).

- (Skvortsova [19, p.138]) If $\mathcal{A}$ is a Muchnik degree then $\text{Th}(\mathfrak{M}/\mathcal{A})$ contains the Kreisel-Putnam formula KP (cf. Proposition 7.4). In particular for such $\mathcal{A}$ the theory $\text{Th}(\mathfrak{M}/\mathcal{A})$ is strictly larger than IPC. Muchnik degrees play an important role in Skvortsova's result that there is a factor with theory IPC. It is however still an open problem (raised by Skvortsova [19, p.134]) whether there is an M-degree $\mathcal{A}$, which is the infimum of finitely many Muchnik degerees, and such that the theory of $\text{Th}(\mathfrak{M}/\mathcal{A})$ coincides with IPC.

- If $\mathcal{A} >_M 0'$ is Muchnik then $\text{Th}(\mathfrak{M}/\mathcal{A}) \subseteq \mathsf{Jan}$. This is because every Muchnik bounds a $\mathcal{B}_f$ (cf. Proposition 7.3).

- Let $\mathcal{A}$ be a join-reducible Muchnik degree. Then

$$\mathsf{IPC} \subsetneq \text{Th}(\mathfrak{M}/\mathcal{A}) \subsetneq \mathsf{Jan}.$$

The first inclusion is strict because $\mathfrak{M}/\mathcal{A}$ satisfies the Kreisel-Putnam formula KP, and the second inclusion follows by the previous item and is strict since $\mathcal{A}$ is join-reducible.

- If $\mathcal{A} >_M 0'$ then the linearity axiom $(p \to q) \vee (q \to p)$ is not in the theory $\text{Th}(\mathfrak{M}/\mathcal{A})$. (Cf. Corollary 8.5.) In particular it is not possible to realize the Gödel logics G_n and the Gödel-Dummett logic G_∞ (cf. Hájek [5]) by a factor of $\mathfrak{M}$.

- There are infinitely many intermediate logics of the form $\text{Th}(\mathfrak{M}/\mathcal{A})$. (Cf. Corollary 5.8.)

Questions:

- Are all $\text{Th}(\mathfrak{M}/\mathcal{S})$ the same for $\mathcal{S}$ solvable? ($\mathcal{S}$ is called *solvable* if it contains a singleton mass problem. The solvable M-degrees form a substructure of $\mathfrak{M}$ isomorphic to the Turing degrees.) If so, what are they? By Sorbi [20, Theorem 4.7] all solvable $\mathcal{S}$ are join-reducible, so $\mathsf{Jan} \not\subseteq \text{Th}(\mathfrak{M}/\mathcal{S})$.

- Does $\mathrm{Th}(\mathfrak{M}/\mathcal{A}) \subseteq \mathsf{Jan}$ hold for all $\mathcal{A} >_M 0'$? This is connected to the question quoted above whether every $\mathcal{A} >_M 0'$ bounds a join-irreducible degree $>_M 0'$. At the moment it is not clear whether any intermediate logic lying strictly between Jan and CPC can be obtained. Cf. also the discussion in Section 7 below.

3. Lattice theoretic preliminaries

We begin with some definitions and basic results. In particular we review some well known constructions that are relevant to our purposes.

Given a poset $\mathfrak{P} = \langle P, \leqslant \rangle$, for every $a \in P$ let

$$[a) = \{b \in P : a \leqslant b\},$$

and for $A \subseteq P$ let $[A) = \bigcup_{a \in A}[a)$. By definition $[\emptyset) = \emptyset$. A subset $O \subseteq P$ is called *open* if it is of the form $[A)$. We denote by $\mathrm{Op}(\mathfrak{P})$ the collection of open subsets of $\mathfrak{P}$.

Definition 3.1. If $\mathfrak{P} = \langle P, \leqslant \rangle$ is a poset we define $\mathfrak{B}(\mathfrak{P}) = \langle \mathrm{Op}(\mathfrak{P}), \supseteq \rangle$.

Lemma 3.2. $\mathfrak{B}(\mathfrak{P})$ *is a Brouwer algebra.*

Proof. The lattice theoretic operations $+$ and $\times$ are given by set theoretic $\cap$ and $\cup$, respectively. The least element is P, and the greatest element is $\emptyset$. Finally, for U, V open,

$$U \to V = \{a \in P : [a) \cap U \subseteq V\}. \qquad \square$$

Definition 3.3. Given a (not necessarily distributive) upper semilattice $\mathfrak{U}$, let $\mathrm{Fr}_\times(\mathfrak{U})$ be the free distributive lattice generated by it, i.e. $\mathfrak{U}$ embeds into $\mathrm{Fr}_\times(\mathfrak{U})$ as an upper semilattice, and for every distributive lattice $\mathfrak{L}$, if $f : \mathfrak{U} \longrightarrow \mathfrak{L}$ is a homomorphism of upper semilattices, then the embedding of $\mathfrak{U}$ into $\mathrm{Fr}_\times(\mathfrak{U})$ extends to a unique lattice theoretic homomorphism of $\mathrm{Fr}_\times(\mathfrak{U})$ into $\mathfrak{L}$, which commutes with f.

It is a consequence of general category theory that $\mathrm{Fr}_\times(\mathfrak{U})$ exists for every upper semilattice $\mathfrak{U}$, and in fact $\mathrm{Fr}_\times$ provides a functor from the category of upper semilattices to the category of distributive lattices, which is left adjoint to the corresponding forgetful functor. For more on these notions, see e.g. [11, Chapter IV]. A direct construction of $\mathrm{Fr}_\times(\mathfrak{U})$ (modeled on the notion of the *hull* of an upper semilattice described in [19, p. 134]) is the following: the universe of $\mathrm{Fr}_\times(\mathfrak{U})$ is given by

$$\{[A) : A \subseteq \mathfrak{U}, \ A \neq \emptyset, \ A \text{ finite}\}$$

with lattice theoretic operations $+$ and $\times$ given by $\cap$ and $\cup$, respectively. It is easy to check that $\mathrm{Fr}_\times(\mathfrak{U})$ has the universal property characterizing free objects.

Lemma 3.4. *In every finite distributive lattice, for each element a there exists a unique antichain A of meet-irreducible elements such that $a = \prod A$.*

Proof. See any standard textbook on distributive lattices, for instance [2]. $\qquad \square$

Corollary 3.5. *For an upper semilattice $\mathfrak{U}$, if $a \in \mathrm{Fr}_\times(\mathfrak{U})$, then there is a unique antichain A of elements of $\mathfrak{U}$ such that $a = \prod A$.*

Proof. By the previous lemma, since the meet-irreducible elements of $\mathrm{Fr}_\times(\mathfrak{U})$ are exactly the elements of $\mathfrak{U}$. $\square$

Lemma 3.6. *If $\mathfrak{U}$ is an implicative upper semilattice with implication operation $\rightarrow$ (i.e. $a \rightarrow b = \min\{c : a + c \geqslant b\}$ where $+$ is the binary operation making $\mathfrak{U}$ an upper semilattice) then $\mathfrak{U}$ embeds into $\mathrm{Fr}_\times(\mathfrak{U})$ as an implicative structure as well.*

Proof. See Skvortsova [19, p. 135], or use an argument similar to Lemma 3.11 below. $\square$

In the following we also use n to denote the set $\{0, \ldots, n-1\}$.

Definition 3.7. Given a set X let $\mathrm{Fr}(X)$ denote the free distributive lattice on X, and let $0 \oplus \mathrm{Fr}(X)$ denote the free bottomed distributive lattice on X, which is simply $\mathrm{Fr}(X)$ with an extra bottom element added. We define $\mathfrak{F}_n = 0 \oplus \mathrm{Fr}(n)$.

Clearly every finite distributive lattice is a Brouwer algebra. Hence $\mathfrak{F}_n$ is a Brouwer algebra.

Definition 3.8. For every $n \geqslant 1$ let

$$2^n = \langle \mathcal{P}(n), \supseteq \rangle$$

denote the poset of subsets of $\{0, \ldots, n-1\}$ ordered by $\supseteq$. Let $\mathfrak{B}_n = \mathfrak{B}(2^n - \{\emptyset\})$.

Theorem 3.9. *We have:*

    (i) $\mathfrak{B}_n$ *is isomorphic with $\mathfrak{F}_n$;*
    (ii) (Skvortsova [19, Lemma 3]) $\mathfrak{B}_n$ *is isomorphic with $\mathrm{Fr}_\times(2^n)$.*

Proof. We distinguish the two cases in order:

    (i) Let $\mathfrak{F}_n$ be as in Definition 3.7, i.e. the free bottomed distributive lattice with n generators. Let 0 be the bottom of $\mathfrak{F}_n$ and let $\mathrm{Gen}_n = \{a_1, \ldots, a_n\}$ be the set of generators. The set $\mathrm{Irr}_\times(\mathfrak{F}_n)$ of meet-irreducible elements of $\mathfrak{F}_n$ is exactly the set

$$\left\{ \textstyle\sum_{i \in I} a_i : I \subseteq n \right\}$$

(where it is understood that $\sum \emptyset = 0$). As a poset, ordered by $\leqslant$, $\mathrm{Irr}_\times(\mathfrak{F}_n)$ is isomorphic with 2^n, under the mapping

$$\textstyle\sum_{i \in I} a_i \longmapsto n - I.$$

On the other hand the set $\mathrm{Irr}_\times(\mathfrak{B}_n)$ of meet-irreducible elements of $\mathfrak{B}_n$ is easily seen to consist exactly of the basic open sets, i.e. the sets of the form $[J]$, $J \subseteq n$, which is again order-theoretically isomorphic to 2^n. Therefore, as a poset, $\mathrm{Irr}_\times(\mathfrak{F}_n)$ is isomorphic to $\mathrm{Irr}_\times(\mathfrak{B}_n)$. Using Lemma 3.4, it follows that $\mathfrak{F}_n$ is isomorphic to $\mathfrak{B}_n$.

    (ii) In view of Corollary 3.5, one can see that the function F that maps $\prod A$ to $\bigcup_{a \in A}[a]$ for every antichain $A \subseteq 2^n$ is an order-theoretic isomorphism of $\mathrm{Fr}_\times(2^n)$ with $\mathfrak{B}_n$. More generally, if $\mathfrak{U}$ is a finite upper semilattice with top 1, then $\mathrm{Fr}_\times(\mathfrak{U}) \simeq \mathfrak{B}(\mathfrak{U} - \{1\})$. $\square$

Notice that by duality the set $\mathrm{Irr}_+(\mathfrak{F}_n)$ of nonzero join-irreducible elements of $\mathfrak{F}_n$ is given by

$$\{\textstyle\prod_{i\in I} a_i : I \subseteq n \text{ nonempty}\}.$$

By definition, $\prod\emptyset = 1$. Note that in $\mathfrak{F}_n$ the top 1 is always join-reducible, except for the case $n = 1$, whereas by definition of $\mathfrak{F}_n$ the bottom 0 always is.

Lemma 3.10 (Representation Lemma). *For every element $a \in \mathfrak{F}_n$ there exists a function $\rho_a : \alpha_a \longrightarrow \mathcal{P}(\mathrm{Gen}_n) - \{\emptyset\}$ with α_a a finite ordinal, such that, letting $\rho_a(j) = A_j$ one has*

$$a = \textstyle\sum_{j\in\alpha_a}\prod A_j.$$

Moreover, if we require that $\{\prod A_j : j \in \alpha_a\}$ be an antichain, and for every $j \in \alpha_a$ we additionally require that A_j be an antichain, then the function ρ_a exists and is unique. We call $a = \sum_{j\in\alpha_a}\prod A_j$ a representation of a.

Proof. This is immediate from Lemma 3.4, and its dual, and the characterization of the join-irreducible elements of $\mathfrak{F}_n$ given above. $\qquad\square$

Notice that $0 = \sum\emptyset$ and $1 = \sum_{a\in\mathrm{Gen}_n}\{a\}$, are representations.

The following lemma allows one to compute $\to$ in $\mathfrak{F}_n$.

Lemma 3.11. *If $a = \sum_{j\in\alpha_a}\prod A_j$ and $b = \sum_{j\in\alpha_b}\prod B_j$ are representations of elements of $\mathfrak{F}_n$ then*

$$a \to b = \textstyle\sum\{\prod B_j : j \in \alpha_b \wedge \prod B_j \not\leqslant a\}.$$

Proof. This follows from the fact that each $\prod B_j$ is join-irreducible for B_j nonempty, cf. [2, Theorem V.3.7]. $\qquad\square$

4. A SKETCH OF SKVORTSOVA'S PROOF

In [19] Skvortsova proved that there is a factor $\mathfrak{M}/\mathbf{E}$ of the Medvedev lattice such that $\mathrm{Th}(\mathfrak{M}/\mathbf{E}) = \mathsf{IPC}$. Skvortsova's analysis also has other interesting consequences. For this reason we give here a brief account of the main ingredients of the proof.

1. If $a, b \in \mathfrak{L}$, with $\mathfrak{L}$ a Brouwer algebra, and $a < b$ then $\mathfrak{L}([a,b])$, i.e. the interval $[a,b]$ in $\mathfrak{L}$, is still a Brouwer algebra, with $u \to_{[a,b]} v = (u \to v) + a$. This includes the case $a = 0$, and we denote $\mathfrak{L}(\leqslant b) = \mathfrak{L}([0,b])$.

2. If $a, b, c \in \mathfrak{L}$ with $c + a = b$ then $f(u) = u + a$ is a B-homomorphism from $\mathfrak{L}(\leqslant c)$ onto $\mathfrak{L}([a,b])$. Thus $\mathrm{Th}(\mathfrak{L}(\leqslant c)) \subseteq \mathrm{Th}(\mathfrak{L}([a,b]))$.

3. If $\mathrm{Th}(\mathfrak{L})^+ = \mathsf{IPC}^+$ then $\bigcap_{a\in\mathfrak{L}} \mathrm{Th}(\mathfrak{L}(\leqslant a)) = \mathsf{IPC}$.

4. Let $\mathfrak{F}_\omega$ be the algebra of finite and cofinite subsets of ω (ordered by $\supseteq$; $\mathfrak{F}_\omega$ is also an implicative lattice), and let $\mathfrak{B}_\omega = \mathrm{Fr}_\times(\mathfrak{F}_\omega)$.

5. Consider the interval $\mathfrak{B}_\omega([n,\emptyset])$ of $\mathfrak{B}_\omega$, for $n \geqslant 1$. It follows from Theorem 3.9 (ii) that $\mathfrak{B}_\omega([n,\emptyset]) \simeq \mathfrak{B}_n$, and thus by 2., $\mathrm{Th}(\mathfrak{B}_\omega) \subseteq \mathrm{Th}(\mathfrak{B}_n)$. Thus $\mathrm{Th}(\mathfrak{B}_\omega) \subseteq \mathsf{LM}$, where $\mathsf{LM} = \bigcap_{n\geqslant 1} \mathrm{Th}(\mathfrak{B}_n)$ is called the *Medvedev logic (of finite problems)*. It is known that $\mathsf{LM}^+ = \mathsf{IPC}^+$, cf. [12, 15].

Similar to item 3 we have that

$$\bigcap_{n \geqslant 1} \bigcap_{b \in \mathfrak{B}_n} \mathrm{Th}\big(\mathfrak{B}_n(\leqslant b)\big) = \mathsf{IPC}.$$

Then in view of the fact that $\mathfrak{B}_\omega([n, \emptyset]) \simeq \mathfrak{B}_n$, one can choose in $\mathfrak{B}_\omega$ intervals $[\alpha_n, \beta_n]$ with the α_n's disjoint and finite, such that

$$\bigcap_{n \geqslant 1} \mathrm{Th}\big(\mathfrak{B}_\omega([\alpha_n, \beta_n])\big) = \mathsf{IPC}.$$

Each β_n is freely $\times$-generated by subsets of α_n, so is of the form $\prod_{1 \leqslant i \leqslant k_n} \beta_n^i$, with $\beta_n^i \subseteq \alpha_n$.

Lemma 4.1. $\mathfrak{B}_\omega$ *is embeddable in* $\mathfrak{M}$.

Proof. The proof uses Lachlan's theorem that every countable upper semilattice with a least element 0 can be embedded as an initial segment of the Turing degrees, cf. [17, p.528]. (It suffices here: every countable implicative upper semilattice with 0 can be order-theoretically embedded as an initial segment.) In more detail, let $\mathfrak{D} = \langle D, 0, +, \rightarrow, \leqslant \rangle$ be a countable implicative semilattice. Embed $\mathfrak{D}$ as an initial segment of the Turing degrees, mapping, say, a generic $a \in D$ to $\deg_T(f_a)$. Then one can check that the assignment, for every $a \in D$,

$$a \mapsto \deg_M \big(\{f : f_a \leqslant_T f\} \cup \{f : (\forall b \in D)[f \not\equiv_T f_b]\}\big)$$

is an embedding into $\mathfrak{M}$ preserving $0, +, \rightarrow$, and also preserves freely generated infima.

In our case it suffices to embed $\mathfrak{F}_\omega$ as an initial upper semilattice of the Turing degrees. Notice that the range of such an embedding consists of Muchnik degrees. □

At this point consider the Medvedev degrees $\mathbf{A}_n, \mathbf{B}_n$ that correspond to α_n, β_n under the embedding of $\mathfrak{B}_\omega$ into $\mathfrak{M}$, with $\mathbf{B}_n = \prod_{1 \leqslant i \leqslant k_n} \mathbf{B}_n^i$ (where $\mathbf{B}_n^i$ corresponds to β_n^i). The final step of the proof is:

Lemma 4.2. *There exists a Medvedev degree* $\mathbf{E}$ *such that* $\mathbf{E} + \mathbf{A}_n = \mathbf{B}_n$ *for every* n.

Proof. Let $\mathcal{A}_n, \mathcal{B}_n^i$ be representatives in $\mathbf{A}_n, \mathbf{B}_n^i$. Define

$$\mathcal{E} = \bigcup_{\substack{n \geqslant 1 \\ 1 \leqslant i \leqslant k_n}} n \widehat{} i \widehat{} \mathcal{B}_n^i.$$

It can then be shown that the degree $\mathbf{E} = \deg_M(\mathcal{E})$ satisfies the lemma. □

From item 2 it follows that for this $\mathbf{E}$ it holds that $\mathrm{Th}(\mathfrak{M}/\mathbf{E}) = \mathsf{IPC}$.

5. Intermediate logics contained in the logic of the weak law of excluded middle

Next we show that there are infinitely many intermediate logics one can get from initial segments determined by Muchnik degrees. Some of the results exhibited below can be obtained as corollaries of Skvortsova's theorem recalled above (cf.

Remark 5.12.) If nothing else, the proofs below are less demanding from the point of view of computability theory, since they do not require embeddings of upper semilattices as initial segments, but only an embedding of a countable antichain in the Turing degrees.

Theorem 5.1. *For every $n \geqslant 1$ there exists a Medvedev degree $\mathbf{B}_n$ such that $\mathfrak{B}_n$ is B-embeddable in $\mathfrak{M}/\mathbf{B}_n$.*

Proof. Let $\mathcal{F} = \{f_i : i \in \omega\}$ be a collection of functions whose T-degrees are pairwise incomparable, and let $\mathbf{B}_{f_i} = \deg_M(\mathcal{B}_{f_i})$. (Recall the mass problems $\mathcal{B}_f$ which were defined in the introduction.) We will make use of the following lemma:

Lemma 5.2. (Sorbi [21]) *Each $\mathbf{B}_{f_i}$ is both join-irreducible and meet-irreducible in the Medvedev lattice. Moreover for every $I \subseteq \omega$, $\sum_{i \in I} \mathbf{B}_{f_i} = \deg_M(\bigcap_{i \in I} \mathcal{B}_{f_i})$, and*

$$\forall i, I \, (i \notin I \Rightarrow \mathbf{B}_{f_i} \not\leqslant_M \textstyle\sum_{j \in I} \mathbf{B}_{f_j}).$$

We now claim that the degree

$$\mathbf{B}_n = \textstyle\sum_{i < n} \mathbf{B}_{f_i}$$

has the desired properties. We embed $\mathfrak{F}_n$ into $\mathfrak{M}/\mathbf{B}_n$. We identify Gen_n with n, thus for every generator $i \in \mathrm{Gen}_n$ choose the function $f_i \in \mathcal{F}$ and define

$$F(i) = \mathbf{B}_{f_i}.$$

By freeness, F extends to a (unique) lattice theoretic homomorphism $F : \mathfrak{F}_n \longrightarrow \mathfrak{M}$, which is $0, 1$ preserving. We claim that F is a B-embedding as well.

Lemma 5.3. *For every $a, b \in \mathfrak{F}_n$, one has*

$$F(a \to b) = F(a) \to F(b).$$

Proof. Let $a = \sum_{i \in \alpha_a} \prod A_i$ and $b = \sum_{j \in \alpha_b} \prod B_j$ be elements of $\mathfrak{F}_n$, given through their representations.

A warning on the notation employed throughout this proof and also later proofs: If a is a generator of $\mathfrak{F}_n$ then let $\mathbf{B}_a$ denote $F(a) = \mathbf{B}_{f_a}$, and let $\mathcal{B}_a = \mathcal{B}_{f_a}$; if $A \subseteq \mathrm{Gen}_n$ then let $\mathbf{B}_A = \{\mathbf{B}_a : a \in A\}$ and $\mathcal{B}_A = \{\mathcal{B}_a : a \in A\}$. Moreover via identification of Gen_n with n, for every $A \subseteq \mathrm{Gen}_n$ we may also identify

$$\textstyle\prod_{a \in A} \mathcal{B}_a = \bigcup_{a \in A} a \widehat{\,} \mathcal{B}_a.$$

Let us now go back to the proof of Lemma 5.3. In view of Lemma 3.11 it is sufficient to show that

$$F(a) \to F(b) = \textstyle\sum \{\prod \mathbf{B}_{B_j} : j \in \alpha_b \wedge \prod \mathbf{B}_{B_j} \not\leqslant_M F(a)\}.$$

This amounts to showing that for every mass problem $\mathcal{X}$ and any $j \in \alpha_b$ such that

$$\textstyle\prod \mathcal{B}_{B_j} \not\leqslant_M \sum_{i \in \alpha_a} \prod \mathcal{B}_{A_i}, \tag{1}$$

and

$$\textstyle\prod \mathcal{B}_{B_j} \leqslant_M \left(\sum_{i \in \alpha_a} \prod \mathcal{B}_{A_i}\right) + \mathcal{X} \tag{2}$$

one has $\prod \mathcal{B}_{B_j} \leqslant_M \mathcal{X}$. Let us therefore fix $\mathcal{X}$ and $j \in \alpha_b$ satisfying (1) and (2). From (1) it follows that

$$\forall i \in \alpha_a \, \exists a_i \in A_i \, \forall y \in B_j (a_i \neq y), \tag{3}$$

for otherwise we would have $\exists i \in \alpha_a \, \forall x \in A_i \, \exists y \in B_j (x = y)$, from which it would follow that

$$\prod \mathcal{B}_{B_j} \leqslant_M \prod \mathcal{B}_{A_i} \leqslant_M \sum_{i \in \alpha_a} \prod \mathcal{B}_{A_i},$$

contrary to assumption (1). Thus, given i choose a_i as in (3). Assume that the reduction in (2) is via the functional Ψ. Let $f \in \mathcal{X}$ be given. Simply by searching, and by density of the $\mathcal{B}_{a_i}$'s, we can effectively find $\sigma = \oplus_{i \in \alpha_a} \sigma_i$ such that $y = \Psi(\sigma \oplus f)(0)$ is defined, i.e. Ψ decides which $\mathcal{B}_y$ to map $\sigma \oplus f$ to. Since by (3) we have $f_y \not\leqslant_T f_{a_i}$ and $\sigma_i \widehat{} f_y \equiv_T f_y$, it holds that $\oplus_i (\sigma_i \widehat{} f_y) \in \bigoplus_i \mathcal{B}_{a_i}$, and since Ψ has to map $\oplus_i(\sigma_i \widehat{} f_y) \oplus f$ into $\mathcal{B}_y$, this is only possible if $f \in \mathcal{B}_{f_y}$. Thus we see that for every $f \in \mathcal{X}$ we can effectively find $y \in B_j$ with $f \in \mathcal{B}_y$, hence $\prod \mathcal{B}_{B_j} \leqslant_M \mathcal{X}$ as desired. This concludes the proof of Lemma 5.3. $\qquad\square$

Thus the proof of Theorem 5.1 is complete. $\qquad\qquad\qquad\qquad\qquad\square$

Notice that for $n = 1$ we could also have taken $\mathbf{B}_1 = \mathbf{0}'$. In fact $\mathfrak{M}/\mathbf{0}'$ is isomorphic to the two-element Boolean algebra.

We have a number of corollaries to the proof of Theorem 5.1:

Corollary 5.4. $\mathfrak{B}_{n+1}$ *is not embeddable in* $\mathfrak{M}/\mathbf{B}_n$.

Proof. The top element of $\mathfrak{M}/\mathbf{B}_n$ is the join

$$\mathbf{B}_n = \mathbf{B}_{f_1} + \ldots + \mathbf{B}_{f_n}$$

of an antichain of n join-irreducible elements, whereas the top element of $\mathfrak{B}_{n+1}$ is the join of an independent set of $n + 1$ of elements by Lemma 5.2. Thus if $\mathfrak{B}_{n+1}$ were embeddable in $\mathfrak{M}/\mathbf{B}_n$ we would have that

$$\mathbf{B}_{f_1} + \ldots + \mathbf{B}_{f_n} = \mathbf{X}_1 + \ldots + \mathbf{X}_{n+1}$$

where the family $\{\mathbf{X}_1, \ldots, \mathbf{X}_{n+1}\}$ forms an independent set. By join-irreducibility of each $\mathbf{B}_{f_i}$, it follows that for every i, there exists j_i such that $\mathbf{B}_{f_i} \leqslant_M \mathbf{X}_{j_i}$. Thus

$$\mathbf{X}_1 + \ldots + \mathbf{X}_{n+1} \leqslant_M \mathbf{X}_{j_1} + \ldots + \mathbf{X}_{j_n}$$

contradicting that the $\mathbf{X}_i$'s form an independent set. $\qquad\qquad\qquad\square$

Corollary 5.5. *There exists a Muchnik degree* $\mathbf{B}_\omega$ *such that every* $\mathfrak{B}_n$ *is B-embeddable in* $\mathfrak{M}/\mathbf{B}_\omega$.

Proof. Let $\mathbf{B}_\omega = \sum_{i \in \omega} \mathbf{B}_{f_i}$, where $\{f_i : i \in \omega\}$ is as in the proof of Theorem 5.1. First of all, by Lemma 5.2 we have that $\sum_{i \in \omega} \mathbf{B}_{f_i} = \deg_M(\bigcap_i \mathcal{B}_{f_i})$, from which we see that $\mathbf{B}_\omega$ is a Muchnik degree. Now let $n \geqslant 1$ and for every $a \in \mathrm{Gen}_n = \{a_1, \ldots, a_n\}$, let

$$\mathbf{B}'_a = \begin{cases} \mathbf{B}_{f_i} & \text{if } a = a_i, i < n, \\ \sum_{j \geqslant n} \mathbf{B}_{f_j} & \text{if } a = a_n. \end{cases}$$

We claim that virtually the same proof as in Theorem 5.1 works, upon replacing each $\mathbf{B}_{a_j}$ with $\mathbf{B}'_{a_j}$, and consequently each $\mathbf{B}_A = \{\mathbf{B}_a : a \in A\}$ with $\mathbf{B}'_A = \{\mathbf{B}'_a : a \in A\}$, where $A \subseteq \mathrm{Gen}_n$. Similar notation is employed for mass problems $\mathcal{B}'_{a_j}$ and $\mathcal{B}'_A$. The proof hinges on the fact that the mass problem $\mathcal{B}'_{a_n} = \sum_{j \geqslant n} \mathcal{B}_{f_j}$ is completely independent of the $\mathcal{B}_{f_j}$, $j < n$, in the sense of Lemma 5.2. $\qquad\square$

Corollary 5.6. $\mathsf{KP} \subseteq \mathrm{Th}(\mathfrak{M}/\mathbf{B}_\omega) \subseteq \mathsf{LM}$.

Proof. The first inclusion follows from the fact that $\mathbf{B}_\omega$ is a Muchnik degree, so one can use Proposition 7.4 below. The other inclusion follows from the fact that every $\mathfrak{B}_n$ is B-embeddable in $\mathfrak{M}/\mathbf{B}_\omega$, and the fact that $\mathsf{LM} = \bigcap_{n \geqslant 1} \mathrm{Th}(\mathfrak{B}_n)$. $\qquad\square$

Corollary 5.7. *For every $n \geqslant 1$, and for every $1 \leqslant j \leqslant n$, $\mathfrak{B}_j$ is Brouwer-embeddable in $\mathfrak{M}/\mathbf{B}_n$, but $\mathfrak{B}_{n+1}$ is not Brouwer-embeddable in $\mathfrak{M}/\mathbf{B}_n$.*

Proof. To embed $\mathfrak{B}_j$ with $j \leqslant n$, consider

$$\mathbf{B}'_{f_i} = \begin{cases} \mathbf{B}_{f_i} & \text{if } i < j, \\ \mathbf{B}_{f_j} + \cdots + \mathbf{B}_{f_n} & \text{if } i = j. \end{cases}$$

The argument employed in the proof of Theorem 5.1 allows to conclude that the lattice-theoretic homomorphism extending by freeness the mapping $F(a_i) = \mathbf{B}'_{f_i}$, where a_i is the $i-$th generator, is a Brouwer-embedding of $\mathfrak{F}_j$ into $\mathfrak{M}/\mathbf{B}_n$. $\qquad\square$

Corollary 5.8. *There is an ascending sequence $\mathbf{B}_1 <_M \mathbf{B}_2 <_M \mathbf{B}_3 <_M \ldots$ of Muchnik degrees such that*

$$\mathrm{Th}(\mathfrak{M}/\mathbf{B}_1) \supset \mathrm{Th}(\mathfrak{M}/\mathbf{B}_2) \supset \mathrm{Th}(\mathfrak{M}/\mathbf{B}_3) \supset \ldots$$

and for every $i \geqslant 1$, $\mathsf{LM} \subseteq \mathrm{Th}(\mathfrak{M}/\mathbf{B}_i)$, thus the class of logics

$$\{\mathrm{Th}(\mathfrak{M}/\mathbf{D}) : \mathsf{LM} \subseteq \mathrm{Th}(\mathfrak{M}/\mathbf{D})\} \tag{4}$$

is infinite.

Proof. This follows from that fact that

$$\mathrm{Th}(\mathfrak{B}_1) \supset \mathrm{Th}(\mathfrak{B}_2) \supset \mathrm{Th}(\mathfrak{B}_3) \supset \ldots$$

To obtain a formula that separates $\mathrm{Th}(\mathfrak{B}_{n+1})$ from $\mathrm{Th}(\mathfrak{B}_n)$ consider e.g. the maximal length of antichains. If a maximal antichain in $\mathfrak{B}$ has length $\leqslant k$ then $\mathfrak{B}$ satisfies the formula

$$\forall x_1 \forall x_2 \ldots \forall x_k \forall x_{k+1} \ \phi(x_1, \ldots, x_{k+1}),$$

where ϕ expresses that there is at least one dependency between the x_i. Note that since by Theorem 3.9 we have that $\mathfrak{B}_n \simeq \mathfrak{F}_n$, a maximal antichain in $\mathfrak{B}_{n+1}$ is at least one longer than in $\mathfrak{B}_n$.[2] $\qquad\square$

Consider the degree $\mathbf{B}_\omega = \sum_i \mathbf{B}_{f_i}$ as defined above. Let Sc denote the Scott logic, i.e.

$$\mathsf{Sc} = \mathsf{IPC} + ((\neg\neg p \to p) \to (\neg p \lor p)) \to (\neg\neg p \lor p).$$

Although we know that

$$\mathsf{KP} \subseteq \mathrm{Th}(\mathfrak{M}/\mathbf{B}_\omega) \subseteq \mathsf{LM}$$

we have:

Corollary 5.9. $\mathsf{Sc} \nsubseteq \mathrm{Th}(\mathfrak{M}/\mathbf{B}_\omega)$.

---

[2]Interestingly, it is not possible to separate the theories $\mathrm{Th}(\mathfrak{B}_n)$ all by one-variable formulas, because the Scott formula Sc, and hence almost all of the formulas in the Rieger-Nishimura lattice, holds in all of them (cf. the proof of Corollary 5.10).

Proof. Consider the degree of difficulty

$$\mathbf{X} = \big((\neg\neg\mathbf{B}_{f_0} \to \mathbf{B}_{f_0}) \to (\neg\mathbf{B}_{f_0} \times \mathbf{B}_{f_0})\big) \to (\neg\neg\mathbf{B}_{f_0} \times \mathbf{B}_{f_0})$$

which is obtained by replacing the variable in Scott's formula by $\mathbf{B}_{f_0}$ and the $\vee$'s by meets.

Using that each $\mathbf{B}_{f_j}$ is join-irreducible, see Lemma 5.2, and that these degrees form an independent set of elements, one can show that in $\mathfrak{M}/\mathbf{B}_\omega$,

$$\neg\mathbf{B}_{f_0} = \sum\nolimits_{i>0}\mathbf{B}_{f_i}$$

and

$$\neg\neg\mathbf{B}_{f_0} = \mathbf{B}_{f_0}.$$

Thus

$$\begin{aligned}
\mathbf{X} &= \big((\mathbf{B}_{f_0} \to \mathbf{B}_{f_0}) \to (\sum\nolimits_{i>0}\mathbf{B}_{f_i} \times \mathbf{B}_{f_0})\big) \to (\mathbf{B}_{f_0} \times \mathbf{B}_{f_0}) \\
&= \big(\mathbf{0} \to (\sum\nolimits_{i>0}\mathbf{B}_{f_i} \times \mathbf{B}_{f_0})\big) \to \mathbf{B}_{f_0} \\
&= (\sum\nolimits_{i>0}\mathbf{B}_{f_i} \times \mathbf{B}_{f_0}) \to \mathbf{B}_{f_0}.
\end{aligned}$$

Hence $\mathbf{X} \neq \mathbf{0}$, as $\sum_{i>0}\mathbf{B}_{f_i} \times \mathbf{B}_{f_0} <_M \mathbf{B}_{f_0}$.   □

Corollary 5.10. Th$(\mathfrak{M}/\mathbf{B}_\omega)$ *is strictly included in* LM.

Proof. This follows from Corollary 5.9 and the fact that Sc is true in every finite free distributive lattice $\mathfrak{F}_n$, as is fairly straightforward to check. It follows from Theorem 3.9 that Sc holds in LM.   □

Remark 5.11. As an easy remark we observe that if $\mathbf{A}, \mathbf{B}$ are incomparable and join-irreducible degrees then by an argument similar to the one in the proof of Corollary 5.9 we have that in $\mathfrak{M}/\mathbf{A}+\mathbf{B}$ it holds that $\neg\mathbf{A} = \mathbf{B}$ and $\neg\mathbf{B} = \mathbf{A}$. Thus

$$\mathfrak{M}/\mathbf{A} + \mathbf{B} \not\models \mathsf{Sc}.$$

Remark 5.12. We finally show how one can derive some of the above results as consequences of Skvortsova's theorem:

If one takes as $\mathbf{B}_\omega$ the Muchnik degree $\mathbf{D}$ corresponding to the image of the top element of Skvortsova's embedding of $\mathfrak{B}_\omega$ into $\mathfrak{M}$, then by item 5. of Skvortsova's proof in Section 4 one obtains Corollary 5.5 and Corollary 5.6.

Inspection of Skvortsova's proof shows also that each $\mathfrak{B}_n$ can be embedded in such a way that the top element is a Muchnik degree which is the join of an antichain of n degrees, but not the join of any finite antichain of bigger cardinality. So one also obtains in this way the infinity of the set described in (4).

6. Closed sets

In this section we examine factors of the form $\mathfrak{M}/\mathcal{F}$ where $\mathcal{F}$ is a nonempty closed subset of ω^ω, in the usual Baire topology. Our conclusions follow from two simple observations that can be summarized as follows:

First observation: Let $\mathcal{F}$ be a nonempty closed mass problem and let $\mathcal{D}$ be dense. Let $\mathbf{F} = \deg_M(\mathcal{F})$ and $\mathbf{D} = \deg_M(\mathcal{D})$ be the respective M-degrees. Let $g : \mathfrak{B} \hookrightarrow [\mathbf{0}, \mathbf{D}]$ an embedding of a Brouwer algebra $\mathfrak{B}$ with meet-irreducible $\mathbf{0}$ and

join-irreducible 1 into the Medvedev degrees below $\mathbf{D}$ and such that $g(1) = \mathbf{D}$. Suppose further that $\mathcal{D} \leqslant_M \mathcal{F}$. If $\widehat{g} : \mathfrak{B} \hookrightarrow [0, \mathbf{F}]$ is identical to g except that $\widehat{g}(1) = \mathbf{F}$, then $\widehat{g}$ is again a lattice theoretic homomorphism preserving $\rightarrow$. To prove this, it suffices to check that negation is preserved. Suppose that $\mathcal{A} \rightarrow \mathcal{D} \equiv_M \mathcal{D}$. Then we have to prove that also $\mathcal{A} \rightarrow \mathcal{F} \equiv_M \mathcal{F}$. Suppose that $\mathcal{A} + \mathcal{C} \geqslant_M \mathcal{F}$. We prove that $\mathcal{C} \geqslant_M \mathcal{F}$. Since $\mathcal{A} \leqslant_M \mathcal{D}$ we have $\mathcal{D} + \mathcal{C} \geqslant_M \mathcal{F}$, via Ψ say. We inductively define a partial computable functional Φ mapping $\mathcal{C}$ into $\mathcal{F}$ as follows. Given $g \in \mathcal{C}$ look for any finite string $\sigma_0 \in \omega^{<\omega}$ such that $\Psi(\sigma_0 \oplus g)(0) \downarrow$. Given σ_n, look for $\sigma_{n+1} \sqsupset \sigma_n$ such that $\Psi(\sigma_{n+1} \oplus g)(n+1) \downarrow$. Finally define $\Phi(g)(n) = \Psi(\sigma_n \oplus g)(n)$ for every n. Then $\Phi(g) \in \mathcal{F}$. Namely, suppose otherwise, i.e. assume for a contradiction that $\Phi(g) \notin \mathcal{F}$. Then for some σ_n, $\Psi(\sigma_n \oplus g) \upharpoonright n+1$ is an initial segment of an element in the open complement of $\mathcal{F}$. By density of $\mathcal{D}$ we can choose $f \in \mathcal{D}$ with $f \sqsupset \sigma_n$. But then $\Psi(f \oplus g) \notin \mathcal{F}$, contradiction. So we have proved that every Brouwer embedding below $\mathcal{D}$ can be modified to one below $\mathcal{F}$.

Second observation: Let $\mathcal{J}$ be a join-irreducible mass problem $>_M 0'$. Then by Sorbi [22, Theorem 4.3] every finite Brouwer algebra with irreducible meet and join is embeddable below $\mathcal{J}$, with $\mathcal{J}$ as top.

As before let $\mathcal{B}_g = \{h : h \not\leqslant_T g\}$. Then the M-degree of $\mathcal{B}_g$ is join-irreducible. It follows that $\mathrm{Th}(\mathfrak{M}/\mathcal{B}_g) = \mathsf{Jan}$. Since $\mathcal{B}_g$ is dense, by the first observation above every embedding below $\mathcal{B}_g$ extends to any closed degree above it.

Now take any nonzero degree of solvability $\{f\}$, and choose g such that $f \not\leqslant_T g$, so that $\mathcal{B}_g \leqslant_M \{f\}$. Then by the above we have that $\mathrm{Th}(\mathfrak{M}/\{f\}) \subsetneqq \mathsf{Jan}$. The inclusion is strict since $\{f\}$ is join-reducible by Sorbi [20, Theorem 4.7].

This also works for any special (i.e. nonempty and without computable elements) Π_1^0-class: Given a special Π_1^0-class $\mathcal{C}$, by Jockusch and Soare [9, Theorem 2] there is a function g, of nonzero c.e. T-degree, such that g computes no elements in $\mathcal{C}$, so that $\mathcal{B}_g \leqslant_M \mathcal{C}$ via the identity. So again we have that $\mathrm{Th}(\mathfrak{M}/\mathcal{C}) \subseteq \mathsf{Jan}$. Also, the inclusion is strict, since by Binns [3] the Medvedev degree of any special Π_1^0-class is join-reducible.

Now every closed mass problem $\mathcal{F}$ is a $\Pi_1^{0,X}$ class for some set $X \subseteq \omega$. By relativizing the results of Jockusch and Soare and Binns we obtain the above result for *any* closed $\mathcal{F}$:

Theorem 6.1. *Let $\mathcal{F}$ be a nonempty and nonzero closed mass problem. Then* $\mathrm{Th}(\mathfrak{M}/\mathcal{F}) \subsetneqq \mathsf{Jan}$.

7. Bounding join-irreducible degrees

Recall the mass problems $\mathcal{B}_f$ from section 1. It is easy to check that for any f and any mass problem $\mathcal{A}$, either $\mathcal{B}_f \leqslant_M \mathcal{A}$ via the identity or $\mathcal{A} \leqslant_M \{f\}$. It follows in particular that $\mathcal{B}_f$ is join-irreducible for any f.

Theorem 7.1. *If $\mathbf{A}$ bounds a join-irreducible $\mathbf{J} >_M 0'$ then* $\mathrm{Th}(\mathfrak{M}/\mathbf{A}) \subseteq \mathsf{Jan}$.

Proof. Let $\mathbf{J} >_M 0'$ be join-irreducible, $\mathbf{A} \geqslant_M \mathbf{J}$, and let $\mathfrak{B}$ be a finite Brouwer algebra with irreducible top 1 and second largest element d. Let $F : \mathfrak{B} \hookrightarrow \mathfrak{M}/\mathbf{J}$

be an embedding of Brouwer algebras. Then $G : \mathfrak{B} \hookrightarrow \mathfrak{M}/\mathbf{A}$ defined by

$$G(a) = \begin{cases} F(a) & \text{if } a \leqslant d, \\ \mathbf{A} & \text{if } a = 1 \end{cases}$$

is a B-embedding as well. To see this it suffices to show that $G(a \to 1) = G(a) \to G(1)$ for every $a \leqslant d$, i.e. that $F(a) \to \mathbf{A} = G(a \to 1) = G(1) = \mathbf{A}$ for every $a \leqslant d$. Let $\mathbf{X} = F(a) \to \mathbf{A}$. Then $\mathbf{X} \leqslant_M \mathbf{A}$. Also, $\mathbf{A} \leqslant_M F(a) + \mathbf{X}$ and hence

$$\begin{aligned} F(a) + (\mathbf{J} \times \mathbf{X}) &= (F(a) + \mathbf{J}) \times (F(a) + \mathbf{X}) \\ &\geqslant_M \mathbf{J} \times \mathbf{A} \\ &= \mathbf{J} \end{aligned}$$

by distributivity. Hence $\mathbf{J} \times \mathbf{X} = \mathbf{J}$ by irreducibility of $\mathbf{J}$, and thus $\mathbf{X} \geqslant_M \mathbf{J}$. Therefore $\mathbf{X} \geqslant_M \mathbf{A}$ because $\mathbf{A} \leqslant_M F(a) + \mathbf{X} = \mathbf{X}$ since $F(a) <_M \mathbf{J}$. So $\mathbf{X} = \mathbf{A}$. $\square$

Theorem 7.2. *Every closed $\mathcal{A} \not\equiv_M 0$ bounds a join-irreducible $\mathcal{J} >_M 0'$.*

Proof. Let $\mathcal{A}$ be closed and nonzero. We prove that there is a noncomputable f such that $\mathcal{B}_f \leqslant_M \mathcal{A}$ via the identity. (Note that since $\mathcal{B}_f$ is Muchnik, for any reduction from $\mathcal{B}_f$ the identity is also a reduction.) As remarked above, every $\mathcal{B}_f$ is join-irreducible. The basic strategy to prevent f from computing something in $\mathcal{A}$ is to make f look computable. We use a finite extension construction (cf. Odifreddi [17]) to build $f = \bigcup_s f_s$ meeting the following requirements for every e:

$$P_e : \qquad \exists x \big(\varphi_e(x) \neq f(x) \big),$$

$$R_e : \qquad \Phi_e(f) \notin \mathcal{A}.$$

The requirements P_e make f noncomputable, and the R_e ensure that f does not compute any element of $\mathcal{A}$, so that $\mathcal{A} \subseteq \mathcal{B}_f$.

Stage s=2e. We satisfy P_e. Let x be the first number on which f_s is not defined. Let $f_{s+1}(x)$ be any value different from $\varphi_e(x)$ if $\varphi_e(x)$ converges, or simply $f_{s+1}(x) = 0$ if $\varphi_e(x)$ diverges.

Stage s=2e+1. We satisfy R_e. Suppose that

$$\big\{ \rho \in \omega^{<\omega} \ : \exists \tau \in \omega^{<\omega} \ \big(\tau \sqsupseteq f_s \wedge \rho \sqsubseteq \Phi_e(\tau) \big) \big\}$$

contains a string in the open complement $\overline{\mathcal{A}}$ of $\mathcal{A}$ (meaning that all extensions of it are in $\overline{\mathcal{A}}$). Then define f_{s+1} to be a string τ such that $\Phi_e(\tau)$ contains a string ρ with this property. Then f_{s+1} satisfies R_e. Otherwise, all strings $\rho \sqsubseteq \Phi_e(\tau)$, $\tau \in \omega^{<\omega}$, are consistent with a function in $\mathcal{A}$. If for all τ and x there were $\tau' \sqsupseteq \tau$ such that $\Phi_e(\tau')(x) \downarrow$ then since $\mathcal{A}$ is closed we could compute a path in $\mathcal{A}$, contradicting that $\mathcal{A}$ is of nonzero M-degree. So there are a string τ and a number x such that $\forall \tau' \sqsupseteq \tau \big(\Phi_e(\tau')(x) \uparrow \big)$. Define f_{s+1} to be such a τ. Then again f_{s+1} satisfies R_e. $\square$

Note that by combining Theorems 7.1 and 7.2 we obtain another proof of Theorem 6.1.

Proposition 7.3. *If $\mathcal{A} >_M 0'$ is Muchnik then $\mathrm{Th}(\mathfrak{M}/\mathcal{A}) \subseteq \mathsf{Jan}$.*

Proof. This is because every nonzero Muchnik M-degree bounds a $\mathcal{B}_f$ for some noncomputable f. Namely, suppose that $\mathcal{A}$ has Muchnik M-degree (i.e. we may assume that $\mathcal{A}$ satisfies: if $g \in \mathcal{A}$ and $g \leqslant_T f$ then $f \in \mathcal{A}$) and does not bound any $\mathcal{B}_f$, f noncomputable. Then $\mathcal{A} \leqslant_M 0'$: If f is noncomputable, then as $\mathcal{B}_f \not\leqslant_M \mathcal{A}$ there is $g \in \mathcal{A}$ such that $g \leqslant_T f$, but then $f \in \mathcal{A}$ since $\mathcal{A}$ is of Muchnik M-degree, giving that $0' \subseteq \mathcal{A}$. The result now follows from Theorem 7.1 and the join-irreducibility of $\mathcal{B}_f$. $\qquad\square$

Proposition 7.4 (Skvortsova [19]). *If* $\mathbf{D}$ *is a Muchnik degree then* $\mathfrak{M}/\mathbf{D} \models \mathsf{KP}$.

Proof. The proof rests on the fact that if $\mathbf{D}$ is a Muchnik degree then for every $\mathbf{B}$ the degree $\mathbf{B} \to \mathbf{D}$ is still a Muchnik degree ([19, Lemma 5]), and on the other hand, if $\mathbf{C}$ is Muchnik then it holds that

$$\mathbf{C} \to \mathbf{A} \times \mathbf{B} = (\mathbf{C} \to \mathbf{A}) \times (\mathbf{C} \to \mathbf{B})$$

because every Muchnik degree is effectively homogeneous (cf. [19], [21]). $\qquad\square$

Corollary 7.5. *If* $\mathbf{D} >_M 0'$ *is a Muchnik degree then*

$$\mathsf{KP} \subseteq \mathrm{Th}(\mathfrak{M}/\mathbf{D}) \subseteq \mathsf{Jan}.$$

Proof. Immediate from Propositions 7.3 and 7.4. $\qquad\square$

We do not know at this point whether there are mass problems $\mathcal{A} >_M 0'$ such that $\mathrm{Th}(\mathfrak{M}/\mathcal{A}) \not\subseteq \mathsf{Jan}$. By Theorem 7.1 such $\mathcal{A}$, if it exists at all, does not bound any join-irreducible degree $>_M 0'$. We do not know whether every $\mathcal{A} >_M 0'$ bounds a join-irreducible degree $>_M 0'$. We conjecture that this is not the case. All we know is that for our canonical examples of join-irreducible mass problems $\mathcal{B}_f$ we have the following:

Theorem 7.6. *There exists a mass problem* $\mathcal{A} >_M 0'$ *that does not bound any* $\mathcal{B}_f$, f *noncomputable.*

Proof. First note that if $\mathcal{B}_f \leqslant_M \mathcal{A}$ then, since $\mathcal{B}_f$ is Muchnik, it holds that $\mathcal{A} \subseteq \mathcal{B}_f$, i.e. $\mathcal{B}_f \leqslant_M \mathcal{A}$ via the identity. So it is enough to construct $\mathcal{A}$ such that

(I) $\forall f$ noncomputable $\exists g \in \mathcal{A}$ noncomputable $g \leqslant_T f$,

(II) $\forall e \exists h$ noncomputable $\Phi_e(h) \notin \mathcal{A}$,

where in (II), as before, $\Phi_e(h) \notin \mathcal{A}$ is by divergence or otherwise. (I) ensures that $\mathcal{A} \not\subseteq \mathcal{B}_f$ for f noncomputable, and (II) ensures that $\mathcal{A} \not\leqslant_M 0'$.

We construct $\mathcal{A}$ in stages, and we start the construction with $\mathcal{A}_0 = 0'$. Clearly at this stage (I) is satisfied. At stage $s > 0$ we have defined $\mathcal{A}_{s-1} = 0' - \{f_0, \ldots, f_{s-1}\}$, where the f_i's need not be distinct. Take h to be T-incomparable to the f_i's. If $\Phi_s(h) \downarrow$ let $f_s = \Phi_s(h)$ and let $\mathcal{A}_s = 0' - \{f_0, \ldots, f_s\}$. This concludes the construction of $\mathcal{A} = \bigcap_{s \in \omega} \mathcal{A}_s$. Clearly at stage s we satisfy (II). To see that at the end of the construction (I) is still satisfied it is enough to observe that $\mathcal{A}$ contains an element below f_s for every s. Since h at each stage is chosen to be incomparable to the previous f_i, the only things that can be deleted from $\mathcal{A}$ below

f_s after stage s must be strictly below f_s. Hence there is always an $f \equiv_T f_s$ to such that $f \in \mathcal{A}$. $\qquad\square$

8. Linearity

An M-degree is a *degree of solvability* if it contains a singleton mass problem. For a degree of solvability $\mathbf{S}$ there is a unique minimal M-degree $>_M \mathbf{S}$ that is denoted by $\mathbf{S}'$ (cf. [14]). If $\mathbf{S} = \deg_M(\{f\})$ then $\mathbf{S}'$ is the degree of the mass problem

$$\{f\}' = \left\{ n^\frown g : f <_T g \wedge \Phi_n(g) = f \right\}. \tag{5}$$

(Note however that $\mathbf{S}'$ has little to do with the Turing jump.) By Theorem 8.1 the degrees of solvability are precisely characterized by the existence of such an $\mathbf{S}'$. So we see that the Turing degrees form a first-order definable substructure of $\mathfrak{M}$. The empty intervals in $\mathfrak{M}$ are characterized by the following:

Theorem 8.1. (Dyment [4],[3] cf. [23, Theorem 4.7]) *For Medvedev degrees* $\mathbf{A}$ *and* $\mathbf{B}$ *with* $\mathbf{A} <_M \mathbf{B}$ *it holds that* $(\mathbf{A}, \mathbf{B}) = \emptyset$ *if and only if there is a degree of solvability* $\mathbf{S}$ *such that* $\mathbf{A} = \mathbf{B} \times \mathbf{S}$, $\mathbf{B} \not\leqslant_M \mathbf{S}$, *and* $\mathbf{B} \leqslant_M \mathbf{S}'$.

Next we show that the only linear intervals in $\mathfrak{M}$ are the empty ones (Theorem 8.4). Call a mass problem *nonsolvable* if its Medvedev-degree does not contain any singleton set, and say that is has *finite degree* if its M-degree contains a finite mass problem. We isolate the main construction in a lemma.

Lemma 8.2. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be mass problems such that*

$$\forall \mathcal{C} \subseteq \mathcal{A} \text{ finite } \ (\mathcal{B} \times \mathcal{C} \not\leqslant_M \mathcal{A}). \tag{6}$$

Then there exists a pair $\mathcal{C}_0$, $\mathcal{C}_1$ *of M-incomparable mass problems* $\mathcal{C}_0$, $\mathcal{C}_1 \geqslant_M \mathcal{A}$ *such that* $\mathcal{B} \times \mathcal{C}_0$ *and* $\mathcal{B} \times \mathcal{C}_1$ *are M-incomparable. (In particular neither of* $\mathcal{C}_0$ *and* $\mathcal{C}_1$ *is above* $\mathcal{B}$.)

Proof. The plan is to build $\mathcal{C}_0$ and $\mathcal{C}_1$ above $\mathcal{A}$ in a construction that meets the following requirements for all $e \in \omega$:

$$R_e^0 : \qquad \Phi_e(\mathcal{C}_0) \not\subseteq \mathcal{B} \times \mathcal{C}_1.$$

$$R_e^1 : \qquad \Phi_e(\mathcal{C}_1) \not\subseteq \mathcal{B} \times \mathcal{C}_0.$$

The $\mathcal{C}_i \subseteq \mathcal{A} \times \mathcal{A} \equiv_M \mathcal{A}$ will be built as unions of finite sets $\bigcup_s \mathcal{C}_{i,s}$, such that $\mathcal{C}_{i,s} \subseteq \mathcal{A} \times \mathcal{A}$ for each pair i, s. We start the construction with $\mathcal{C}_{i,0} = \emptyset$. The idea to meet R_e^0 is simple: By condition (6) we have at stage s of the construction that $\mathcal{B} \times \mathcal{C}_{1,s} \not\leqslant_M \mathcal{A}$, so there is a witness $f \in \mathcal{A}$ such that $\Phi_e(f) \notin \mathcal{B} \times \mathcal{C}_{1,s}$. (Either by being undefined or by not being an element of $\mathcal{B} \times \mathcal{C}_{1,s}$.) We put such a witness in $\mathcal{C}_0$. Now this f will be a witness to $\Phi_e(\mathcal{C}_0) \not\subseteq \mathcal{B} \times \mathcal{C}_1$ provided that we can keep future elements of $1^\frown \mathcal{C}_1$ distinct from $\Phi_e(f)$. The problem is that some requirement R_i^1 may want to put $\Phi_e(f)$ into $1^\frown \mathcal{C}_1$ because $\Phi_e(f)(0) = 1$ and the function $\Phi_e(f)^- = \lambda x. \Phi_e(f)(x+1)$ is the *only* witness that $\Phi_i(\mathcal{A}) \not\subseteq \mathcal{B} \times \mathcal{C}_0$. To resolve this conflict it suffices to complicate the construction somewhat by

[3]It may be informative to note that Dyment is the maiden name of Skvortsova.

prefixing all elements of $\mathcal{A}$ by an extra bit $x \in \{0,1\}$, that is, to work with $\mathcal{A} \times \mathcal{A}$ rather than $\mathcal{A}$. This basically gives us *two* versions of every potential witness, and we can argue that either choice of them will be sufficient to meet our needs, so that we can always keep them apart. We now give the construction in technical detail.

We use the following notation: We let f^- be the function such that $f^-(x) = f(x+1)$ (i.e. f with its first element chopped off) and we let $\mathcal{X}^- = \{f^- : f \in \mathcal{X}\}$. We build $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{A} \times \mathcal{A}$.

Stage s=0. Let $\mathcal{C}_{0,0} = \mathcal{C}_{1,0} = \emptyset$.

Stage s+1=2e+1. We take care of R_e^0. We claim that there is an $f \in \mathcal{A} - \mathcal{C}_{0,s}^-$ and an $x \in \{0,1\}$ such that

$$\exists h \in \mathcal{C}_{0,s} \cup \{x \widehat{\ } f\} \big(\Phi_e(h) \notin \mathcal{B} \times (\mathcal{C}_{1,s}^- \times \mathcal{C}_{1,s}^-)\big). \tag{7}$$

Namely, otherwise we would have that for all $f \in \mathcal{A} - \mathcal{C}_{0,s}^-$ and $x \in \{0,1\}$

$$\forall h \in \mathcal{C}_{0,s} \cup \{x \widehat{\ } f\} \big(\Phi_e(h) \in \mathcal{B} \times (\mathcal{C}_{1,s}^- \times \mathcal{C}_{1,s}^-)\big). \tag{8}$$

But then it follows that $\mathcal{A} \geqslant_M \mathcal{B} \times (\mathcal{C}_{1,s}^- \times \mathcal{C}_{1,s}^-)$, contradicting the assumption (6). To see this, assume (8) and let

$$\mathcal{D} = \mathcal{C}_{0,s} \cup \big\{x \widehat{\ } f : x \in \{0,1\} \wedge f \in \mathcal{A} - \mathcal{C}_{0,s}^-\big\}.$$

Then $\mathcal{B} \times (\mathcal{C}_{1,s}^- \times \mathcal{C}_{1,s}^-) \leqslant_M \mathcal{D}$ via Φ_e. But we also have $\mathcal{D} \leqslant_M \mathcal{A}$, so we have $\mathcal{B} \times \mathcal{C}_{1,s}^- \leqslant_M \mathcal{A}$, contradicting (6). To show that $\mathcal{D} \leqslant_M \mathcal{A}$, let $\mathcal{C}_{0,s}^- = \{f_1, \ldots, f_s\}$ and let $\tilde{f}_i$, $1 \leqslant i \leqslant s$, be finite initial segments such that the only element of $\mathcal{C}_{0,s}^-$ extending $\tilde{f}_i$ is f_i. (Note that such finite initial segments exist since $\mathcal{C}_{0,s}^-$ is finite.) Let x_i be such that $x_i \widehat{\ } f_i \in \mathcal{C}_{0,s}$. Then $\mathcal{D} \leqslant_M \mathcal{A}$ via

$$\Phi(f) = \begin{cases} x_i \widehat{\ } f & \text{if } \exists i \ \tilde{f}_i \sqsubseteq f, \\ 0 \widehat{\ } f & \text{otherwise.} \end{cases}$$

So we can choose h as in (7). Put h into $\mathcal{C}_{0,s+1}$. If $\Phi_e(h) = 1 \widehat{\ } y \widehat{\ } g$ for some $g \in \mathcal{A} - \mathcal{C}_{1,s}^-$ and $y \in \{0,1\}$ we also put $(1-y) \widehat{\ } g$ into $\mathcal{C}_{1,s+1}$.

Stage s+1=2e+2. The construction to satisfy R_e^1 is completely symmetric to the one for R_e^0, now using $\mathcal{C}_{1,s}$ instead of $\mathcal{C}_{0,s}$. This ends the construction.

We verify that the construction succeeds in meeting all requirements. At stage $s + 1 = 2e + 1$, the element h put into $\mathcal{C}_0$ is a witness for $\Phi_e(\mathcal{C}_0) \not\subseteq \mathcal{B} \times \mathcal{C}_{1,s+1}$. In order for h to be a witness for $\Phi_e(\mathcal{C}_0) \not\subseteq \mathcal{B} \times \mathcal{C}_1$ it suffices to prove that all elements $x \widehat{\ } f$ entering $\mathcal{C}_1$ at a later stage $t > 2e + 1$ are different from $\Phi_e(h)^-$.

If $\Phi_e(h)$ is not of the form $1 \widehat{\ } y \widehat{\ } g$ for $g \in \mathcal{A} - \mathcal{C}_{1,s}^-$ and $y \in \{0,1\}$ then this is automatic, since only elements of this form are put into $\mathcal{C}_1$ at later stages.

Suppose $\Phi_e(h)$ is of the form $1 \widehat{\ } y \widehat{\ } g$ for some $g \in \mathcal{A} - \mathcal{C}_{1,s}^-$ and $y \in \{0,1\}$. Then $(1-y) \widehat{\ } g$ was put into $\mathcal{C}_{1,s+1}$ at stage $s + 1$, if not earlier. By construction, this ensures that all elements $x \widehat{\ } f$ entering $\mathcal{C}_1$ at a later stage $t > s + 1$ satisfy $f \neq g$:

- If $x \widehat{\ } f$ enters $\mathcal{C}_{1,t+1}$ at $t = 2i+1$ then $x \widehat{\ } f = (1-y') \widehat{\ } g'$ for some $g' \in \mathcal{A} - \mathcal{C}_{1,t}^-$ and $y' \in \{0,1\}$. In particular $f \neq g$ since $g \in \mathcal{C}_{1,t}^-$.
- If $x \widehat{\ } f$ enters $\mathcal{C}_{1,t+1}$ at $t = 2i+2$ then $f \in \mathcal{A} - \mathcal{C}_{1,t}^-$, so again $f \neq g$.

Thus R_e^0 is satisfied. The verification of R_e^1 at stage $2e+2$ is again symmetric. $\square$

Lemma 8.3. *For any singleton mass problem $\mathcal{S}$, if $\mathcal{B} \not\leqslant_M \mathcal{S}'$ then $\mathcal{S}'$ and $\mathcal{B}$ satisfy condition (6) from Lemma 8.2.*

Proof. Suppose that $\mathcal{S} = \{f\}$ and that $\mathcal{C} \subseteq \mathcal{S}'$ is finite such that $\mathcal{B} \times \mathcal{C} \leqslant_M \mathcal{S}'$, via Φ say. We prove that $\mathcal{B} \leqslant_M \mathcal{S}'$.

Recall the explicit definition of $\mathcal{S}'$ from equation (5). First we claim that for every $n\widehat{\ }g \in \mathcal{C}$ there is $m\widehat{\ }h \in \mathcal{S}'$ with $h \equiv_T g$ such that $\Phi(m\widehat{\ }h)(0) = 0$, that is, something from $\deg_T(g)$ is sent to the $\mathcal{B}$-side. To see this, let m be such that $\Phi_m(f \oplus h') = f$ for all h', and let h be of the form $f \oplus h'$ such that $\Phi(m\widehat{\ }h)(0) = 0$. Such h exists because $\mathcal{C}$ is finite, and for any finite number of elements $\{f_0, \ldots, f_k\}$ strictly T-above f it is always possible to build $h \geqslant_T f$ such that h is T-incomparable to all the f_i's, cf. [17, p491]. Now the computation $\Phi(m\widehat{\ }h)(0) = 0$ will use only a finite part of h, so we can actually make h of the same T-degree as g by copying g after this finite part. This establishes the claim.

To finish the proof we note that from the claim it follows that $\mathcal{B} \leqslant_M \mathcal{S}'$: If something is sent to the $\mathcal{C}$-side by Φ we can send it on to the $\mathcal{B}$-side by the claim. Because $\mathcal{C}$ is finite we can do this uniformly. More precisely, $\mathcal{B} \leqslant_M \mathcal{S}'$ by the following procedure. By the claim fix for every $n\widehat{\ }g \in \mathcal{C}$ a corresponding $m\widehat{\ }h \in \mathcal{S}'$ and a code e such that $\Phi_e(g) = h$. Given an input $n_0\widehat{\ }g_0$, check whether $\Phi(n_0\widehat{\ }g_0)(0)$ is 0 or 1. In the first case, output $\Phi(n_0\widehat{\ }g_0)^-$, i.e. $\Phi(n_0\widehat{\ }g_0)$ minus the first element. This is then an element of $\mathcal{B}$. In the second case $\Phi(n_0\widehat{\ }g_0)^- \in \mathcal{C}$. Since $\mathcal{C}$ is finite we can separate its elements by finite initial segments and determine exactly which element of $\mathcal{C}$ $\Phi(n_0\widehat{\ }g_0)^-$ is by inspecting only a finite part of it. Now using the corresponding code e that was chosen above we output $\Phi\big(m\widehat{\ }\Phi_e\big(\Phi(n_0\widehat{\ }g_0)^-\big)\big)$, which is again an element of $\mathcal{B}$. $\square$

Theorem 8.4. *If $(\mathbf{A}, \mathbf{B}) \neq \emptyset$ then there is a pair of incomparable degrees in $(\mathbf{A}, \mathbf{B})$.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be mass problems of degree $\mathbf{A}$ and $\mathbf{B}$, respectively. If $\mathcal{A}$ and $\mathcal{B}$ satisfy condition (6) then Lemma 8.2 immediately gives the pair $\mathcal{B} \times \mathcal{C}_0$ and $\mathcal{B} \times \mathcal{C}_1$ of incomparable elements between $\mathcal{A}$ and $\mathcal{B}$.

Suppose next that $\mathcal{A}$ and $\mathcal{B}$ do not satisfy condition (6): Let $\mathcal{C} \subseteq \mathcal{A}$ be finite such that $\mathcal{B} \times \mathcal{C} \leqslant_M \mathcal{A}$. Since we also have $\mathcal{A} \leqslant_M \mathcal{B} \times \mathcal{C}$ we then have $\mathcal{A} \equiv_M \mathcal{B} \times \mathcal{C}$.

Suppose that there are T-incomparable $f, g \in \mathcal{C}$ such that $\{f\}, \{g\} \not\geqslant_M \mathcal{B}$. Then one easily checks that $\mathcal{B} \times \{f\}$ and $\mathcal{B} \times \{g\}$ are M-incomparable problems in $(\mathcal{A}, \mathcal{B})$. Otherwise,

$$\forall f, g \in \mathcal{C}\big(f, g \text{ T-comparable } \vee \{f\} \geqslant_M \mathcal{B} \vee \{g\} \geqslant_M \mathcal{B}\big). \tag{9}$$

We deduce:

1. We cannot have $\{f\} \geqslant_M \mathcal{B}$ for all $f \in \mathcal{C}$ that are of minimal T-degree in $\mathcal{C}$, for otherwise $\mathcal{C} \geqslant_M \mathcal{B}$ and hence $\mathcal{A} \equiv_M \mathcal{B}$.
2. From (9) it follows that there cannot be two $f, g \in \mathcal{C}$ of different minimal T-degree both not above $\mathcal{B}$.

From 1. and 2. it follows that there is exactly *one* T-degree $\deg_T(f)$, $f \in \mathcal{C}$, that is minimal in $\mathcal{C}$ such that $\{f\} \not\geqslant_M \mathcal{B}$. But then $\mathcal{B} \times \mathcal{C} \equiv_M \mathcal{B} \times \{f\}$: $\leqslant_M$ is clear, and for $\geqslant_M$, if $g \in \mathcal{C}$ then $\{g\} \geqslant_M \mathcal{B}$ or $g \geqslant_T f$, so $\geqslant_M$ now follows from finiteness of $\mathcal{C}$.

Thus we have $\mathcal{A} \equiv_M \mathcal{B} \times \{f\}$ with $\mathcal{B} \not\leqslant_M \{f\}$. Let $\mathcal{S}' \in \deg_M(\{f\})'$. If $\mathcal{B} \leqslant_M \mathcal{S}'$ then $(\mathcal{A}, \mathcal{B}) = \emptyset$ by Theorem 8.1. If $\mathcal{B} \not\leqslant_M \mathcal{S}'$ then we apply Lemma 8.2 to $\mathcal{S}'$ and $\mathcal{B}$. This is possible because $\mathcal{S}'$ and $\mathcal{B}$ satisfy condition (6) by Lemma 8.3. Lemma 8.2 now produces incomparable $\mathcal{B} \times \mathcal{C}_0$ and $\mathcal{B} \times \mathcal{C}_1$. They are clearly below $\mathcal{B}$, and they are also above $\mathcal{A}$ since $\mathcal{C}_0, \mathcal{C}_1 \geqslant_M \mathcal{S}' \geqslant_M \{f\} \geqslant_M \mathcal{A}$. So we have again a pair of incomparable problems in the interval $(\mathcal{A}, \mathcal{B})$. $\square$

Corollary 8.5. *There are incomparable degrees below every $\mathbf{A} >_M \mathbf{0}'$.*

Proof. Apply Theorem 8.4 to the interval $(\mathbf{0}', \mathbf{A})$. Note that any interval $(\mathbf{0}', \mathbf{A})$ with $\mathbf{A} >_M \mathbf{0}'$ is indeed nonempty. This can be seen using Theorem 8.1: It suffices to show that for any degree of solvability $\mathbf{S}$, $\mathbf{A} \times \mathbf{S} \neq_M \mathbf{0}'$. This follows from Lemma 8.3, but also because $\mathbf{0}'$ is meet-irreducible, for example because $\mathbf{0}'$ is effectively homogeneous (Dyment, cf. [23, Corollary 5.2]). So if $\mathbf{A} \times \mathbf{S} \neq_M \mathbf{0}'$ we must have $\mathbf{A} \leqslant_M \mathbf{0}'$, since clearly $\mathbf{S} \leqslant_M \mathbf{0}'$ is impossible for $\mathbf{S}$ solvable.

Alternatively, one can also use Lemma 8.2 directly for a proof of the corollary. In fact, one can give a simplified proof of Lemma 8.2 for the case of the interval $(\mathbf{0}', \mathbf{A})$. Namely, the conflict arising there does not arise in this special case, so that a more direct proof is possible. $\square$

From Corollary 8.5 it follows in particular that the linearity axiom

$$(p \to q) \vee (q \to p)$$

is not in any of the theories $\mathrm{Th}(\mathfrak{M}/\mathcal{A})$ for $\mathcal{A} >_M 0'$. In particular it is not possible to realize the intermediate Gödel logics G_n and the Gödel-Dummett logic G_∞ (cf. Hájek [5]) by a factor of $\mathfrak{M}$.

We note that one can prove the following variant of Lemma 8.2, with a weaker hypothesis and a weaker conclusion, and with a similar proof.

Proposition 8.6. *Let $\mathcal{A}$ be a mass problem that is not of finite degree, and let $\mathcal{B}$ be any mass problem such that $\mathcal{B} \not\leqslant_M \mathcal{A}$. Then there exists a pair $\mathcal{C}_0$, $\mathcal{C}_1$ of M-incomparable mass problems above $\mathcal{A}$ such that neither of them is above $\mathcal{B}$.*

We note that Theorem 8.4 in general cannot be improved since there are non-empty intervals that contain exactly two intermediate elements. In fact, in Terwijn [24] it is proved that every interval in $\mathfrak{M}$ is either isomorphic to a finite Boolean algebra 2^n or is as large as set-theoretically possible, namely of size $2^{2^{\aleph_0}}$.

9. AN ALGEBRAIC CHARACTERIZATION OF KP

Kreisel and Putnam [10] studied the following formula in order to disprove a conjecture of Łukasiewicz (that IPC would be the only intermediate logic with the disjunction property):

$$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r). \tag{KP}$$

Here we give an algebraic characterization of the logic of KP.

McKinsey and Tarski [13] proved the following classical result, which also follows easily from the results in Jaśkowski [8]. We include a sketch of a proof for later reference.

Theorem 9.1. (Jaśkowski [8], McKinsey and Tarski [13])

$$\mathsf{IPC} = \bigcap \big\{ \mathrm{Th}(B) : B \text{ a finite Brouwer algebra} \big\}.$$

Proof. Let $\mathcal{L}_{\mathsf{IPC}}$ be the Lindenbaum-Tarski algebra of IPC. It is easily verified that $\mathcal{L}_{\mathsf{IPC}}$ is a Heyting algebra. Hence the dual $\overline{\mathcal{L}}_{\mathsf{IPC}}$ of $\mathcal{L}_{\mathsf{IPC}}$ is a Brouwer algebra. Now suppose that $\mathsf{IPC} \nvdash \varphi$ and that $p_0, \ldots, p_k$ are the propositional atoms occurring in φ. We want to produce a finite Brouwer algebra B such that $B \nvDash \varphi$. Note that we cannot take the subalgebra generated by the $p_0, \ldots, p_k$ since this algebra is infinite. (Cf. the infinity of the Rieger-Nishimura lattice.) Take for B the smallest sub-Brouwer-algebra of $\overline{\mathcal{L}}_{\mathsf{IPC}}$ in which all subformulas of φ occur. B can be described as follows: Let B be the finite distributive sublattice of $\overline{\mathcal{L}}_{\mathsf{IPC}}$ generated by all subformulas of φ together with 0 and 1. Since B is finite it is automatically a Brouwer algebra. Note that $\to$ in B need *not* coincide with $\to$ in $\overline{\mathcal{L}}_{\mathsf{IPC}}$. $\qquad \square$

We now imitate the proof just given to obtain the following characterization of KP:

Theorem 9.2. [4]

$$\mathsf{IPC} + \mathsf{KP} \;=\; \bigcap \big\{ \mathrm{Th}(B) : B \text{ a finite Brouwer algebra such that}$$
$$\text{for every } p \in B, \ \neg p \text{ is meet-irreducible} \big\}.$$

Proof. Let $\mathcal{L}_{\mathsf{KP}}$ be the Lindenbaum-Tarski algebra of $\mathsf{IPC} + \mathsf{KP}$. Again, it is easily verified that $\mathcal{L}_{\mathsf{KP}}$ is a Heyting algebra, hence the dual $\overline{\mathcal{L}}_{\mathsf{KP}}$ is a Brouwer algebra. Furthermore, $\overline{\mathcal{L}}_{\mathsf{KP}}$ satisfies the formula KP: If $\neg\varphi \geqslant \psi \vee \chi$ in $\overline{\mathcal{L}}_{\mathsf{KP}}$ this means that $\neg\varphi$ proves $\psi \vee \chi$, hence since KP is a principle of the logic, $\neg\varphi$ proves ψ or $\neg\varphi$ proves χ. Now suppose that $\mathsf{KP} \nvdash \varphi$ and that $\varphi = \varphi(p_0, \ldots, p_k)$. We want to produce a finite Brouwer algebra B such that $B \nvDash \varphi$. We cannot take B to be, as in the proof of Theorem 9.1, the smallest subalgebra generated by all the subformulas of φ, since it may happen that in this algebra some elements are negations (i.e. of the form $\neg p$) that were not negations in $\overline{\mathcal{L}}_{\mathsf{KP}}$. In particular this may happen for meet-reducible elements. So we have to take for B a larger algebra. Take B to be the smallest sub-Brouwer-algebra of $\overline{\mathcal{L}}_{\mathsf{KP}}$ in which all subformulas of φ occur, as well as 0 and 1, and such that if $\psi \in B$ then also $\neg\psi \in B$. Clearly B refutes φ. In B every negation is meet-irreducible, since for every $\varphi \in B$ its negation $\neg\varphi$ from $\overline{\mathcal{L}}_{\mathsf{KP}}$ is also in B, and if this were meet-reducible in B then it would also be meet-reducible in $\overline{\mathcal{L}}_{\mathsf{KP}}$. So we are done if B is finite. But B is indeed finite since in IPC, for every given finite set of formulas one can only generate finitely many nonequivalent formulas from this set using only $\vee$, $\wedge$, and $\neg$, cf. Hendriks [6]. This

---

[4]Note added in proof, May 2013. R. Kuyper has pointed out that Theorem 9.2 is incorrect. Namely, the intersection in the statement of the theorem is equal to IPC, not $\mathsf{IPC} + \mathsf{KP}$. Indeed, what is written in the proof about the Lindenbaum algebra of KP is also true about the Lindenbaum algebra of IPC.

is because first, every formula in the $\{\vee, \wedge, \neg\}$-fragment can be proven equivalent to a disjunction of formulas in the $\{\wedge, \neg\}$-fragment using the distributive law and the equivalence $\neg(\varphi \vee \psi) \leftrightarrow \neg\varphi \wedge \neg\psi$, and second, it is not hard to see that the $\{\wedge, \neg\}$-fragment over a finite number of propositional variables is finite [6]. $\square$

References

[1] S. Artemov, *Explicit provability and constructive semantics*, Bull. Symbolic Logic 7(1) (2001) 1–36.

[2] R. Balbes and P. Dwinger. *Distributive lattices*. University of Missouri Press, Columbia, 1974.

[3] S. Binns, *A splitting theorem for the Medvedev and Muchnik lattices*, Math. Log. Q. 49(4), 327–335.

[4] E. Z. Dyment, *Certain properties of the Medvedev lattice*, Math. USSR Sb. 30 (1976) 321–340. English translation.

[5] P. Hájek, *Metamathematics of fuzzy logic*, Trends in Logic Vol. 4, Kluwer Academic Publishers, Dordrecht, 1998.

[6] A. Hendriks, *Computations in propositional logic*, PhD thesis, ILLC, University of Amsterdam, 1996.

[7] A. V. Jankov, *Calculus of the weak law of the excluded middle*, Izv. Akad. Nauk SSSR Ser. Mat. 32 (1968) 1044–1051. (In Russian.)

[8] S. Jaśkowski, *Recherches sur le système de la logique intuitioniste*, Actes du Congrès International de Philosophie Scientifique VI, Philosophie des mathématiques, Actualités Scientifiques et Industrielles 393, Paris, Hermann (1936) 58–61.

[9] C. G. Jockusch, Jr. and R. I. Soare, *Degrees of members of Π_1^0 classes*, Pacific J. Math. 40(3) (1972) 605–616.

[10] G. Kreisel and H. Putnam, *Eine Unableitbarkeitsbeweismethode für den intuitionistischen Aussagenkalkül*, Arch. Math. Logik 3 (1957) 74–78.

[11] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, New York, Heidelberg, Berlin, 1971.

[12] L. L. Maximova, D. P. Skvortsov, and V. P. Shekhtman, *On the impossibility of finite axiomatization of the logic of finite problems*, Dokl. Akad. Nauk SSSR 245(5) (1979) 1051–1054.

[13] J. C. C. McKinsey and A. Tarski, *Some theorems about the sentential calculi of Lewis and Heyting*, J. Symbolic Logic 13 (1948) 1–15.

[14] Yu. T. Medvedev, *Degrees of difficulty of the mass problems*, Dokl. Akad. Nauk. SSSR 104(4) (1955) 501–504.

[15] Yu. T. Medvedev, *Finite problems*, Dokl. Akad. Nauk. SSSR (NS) 142(5) (1962) 1015–1018.

[16] P. Minari, *Intermediate logics. An historical outline and a guided bibliography*. Technical Report 79-1983, University of Siena, 1983.

[17] P. Odifreddi, *Classical recursion theory*, Vol. 1, Studies in logic and the foundations of mathematics Vol. 125, North-Holland, Amsterdam, 1989.

[18] S. G. Simpson, *Mass problems and randomness*, Bull. Symbolic Logic 11(1) (2005) 1–27.

[19] E. Z. Skvortsova, *A faithful interpretation of the intuitionistic propositional calculus by means of an initial segment of the Medvedev lattice*, Sibirsk. Math. Zh. 29(1) (1988) 171–178. (In Russian.)

[20] A. Sorbi, *Some remarks on the algebraic structure of the Medvedev lattice*, J. Symbolic Logic 55(2) (1990) 831–853.

[21] A. Sorbi, *Embedding Brouwer algebras in the Medvedev lattice*, Notre Dame J. Formal Logic 32(2) (1991) 266–275.

[22] A. Sorbi, *Some quotient lattices of the Medvedev lattice*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik 37 (1991) 167–182.

[23] A. Sorbi, *The Medvedev lattice of degrees of difficulty*, In: S. B. Cooper, T. A. Slaman, and S. S. Wainer (eds.), Computability, Enumerability, Unsolvability: Directions in Recursion Theory, London Mathematical Society Lecture Notes 224, Cambridge University Press, Cambridge, 1996, 289–312.

[24] S. A. Terwijn, *On the structure of the Medvedev lattice*, to appear in J. Symbolic Logic.

(Andrea Sorbi) University of Siena, Dipartimento di Scienze Matematiche ed Informatiche "Roberto Magari", Pian dei Mantellini 44, 53100 Siena, Italy.
  E-mail address: sorbi@unisi.it

(Sebastiaan A. Terwijn) Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Strasse 25, 1040 Vienna, Austria.
  E-mail address: terwijn@logic.at