HABILITATIONSSCHRIFT

CONSTRUCTIVE LOGIC AND COMPUTATIONAL LATTICES

eingereicht an der Technischen Universität Wien

Fakultät für Mathematik

von

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Acknowledgments

Wenn man aber einmal mit Logik beginnt, wo ein
Gedanke von selbst aus dem vorangehenden folgt,
weiß mann zum Schluß nie, wie das endet.

(R. Musil, Der Mann ohne Eigenschaften, p234)

It is almost ten years since I wrote my thesis, and I never intended to write another one. However, my colleagues from Vienna kept insisting that I write a habilitation thesis, and since the work was already done there did not seem much reason to refuse. So here it is.

According to Nietzsche[1] it is possible to deduce the social background of a scientist from his work. He offers the Darwinists as an example: Since these were all struggling for life in poor shabby England, no wonder they became Darwinists! In the case of the work presented in this thesis, one can at least can deduce from it that the author must have been in contact with both constructivists and computability theorists.

My interest in constructive logic and intuitionism dates from the time that Anne Troelstra refused to grade a set of exercises in which I had borrowed a notation from Kreisel. In subsequent years my interest drifted into other directions while studying computability theory in Heidelberg, but the questions surrounding computability and constructivity never left my mind, until finally I started working on some particular incarnations of them during a visit to Rosalie Iemhoff in San Diego during the years 2001–2002. I was prompted by a quotation from Rogers [35, p289], saying that Medvedev showed that the identities of the Medvedev lattice $\mathfrak{M}$ are the theorems of IPC, the intuitionistic propositional calculus. When I saw this I thought it was a great result, and wondered why nobody had ever pointed it out to me. However, when I tried to prove it I quickly found that the result is false. Since in the exercise section Rogers himself already

[1] F. Nietzsche, Die fröhliche Wissenschaft, Alfred Köner Verlag, 1956, p248.

points to the possibility of considering factors of the Medvedev lattice, I decided to try to prove the next best thing, namely that there is a factor $\mathfrak{M}/G$ which has IPC as its logic. At this point I had the excellent idea of contacting Andrea Sorbi, who subsequently informed me that he had had the exact same sequence of events around the time that he wrote his thesis, and that he had found out that Skvortsova had already proven the result we were after (cf. Theorem 2.3.2 below). However, in the meanwhile I was already too much absorbed by the relations between constructive logic and computational lattices to be seriously discouraged in my studies, and hence I continued to pursue further intriguing questions from this area, at the same time continuously learning more and more about intuitionistic and other constructive logics.

Now that I am able to present a substantial body of work on this topic in the form of this habilitation thesis, I would like to thank everybody with whom I had, at some point or other, discussions about topics relating to the following chapters, including Klaus Ambos-Spies, Matthias Baaz, Lev Beklemishev, Nick Bezhanishvili, Dick de Jongh, Martin Goldstern, Rosalie Iemhoff, Jakob Kellner, Georg Kreisel, George Metcalfe, Pierluigi Minari, Yiannis Moschovakis, Jaap van Oosten, Steve Simpson, Ted Slaman, Andrea Sorbi, Jouko Väänänen, Wim Veldman, Yde Venema, Yang Yue, and Domenico Zambella. Your help and comments have been much appreciated.