LIMIT COMPUTABILITY AND CONSTRUCTIVE MEASURE

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Abstract. In this paper we study constructive measure and dimension in the class Δ^0_2 of limit computable sets. We prove that the lower cone of any Turing-incomplete set in Δ^0_2 has Δ^0_2-dimension 0, and in contrast, that although the upper cone of a noncomputable set in Δ^0_2 always has Δ^0_2-*measure* 0, upper cones in Δ^0_2 have nonzero Δ^0_2-*dimension*. In particular the Δ^0_2-dimension of the Turing degree of $\emptyset'$ (the Halting Problem) is 1. Finally, it is proven that the low sets do not have Δ^0_2-measure 0, which means that they do not form a small subset of Δ^0_2. This result has consequences for the existence of bi-immune sets.

1. Introduction

In his study of randomness [27], Schnorr introduced the notion of a Schnorr null set as a more constructive version of Martin-Löf's [21] notion of null set. We briefly review the relevant definitions. For motivation and discussion of these notions we refer the reader to Schnorr's book [27], the monograph by Li and Vitányi [17], and the recent surveys [5, 8, 31].

For $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$, we write $\sigma \sqsubset X$ to mean that σ is an initial segment of X. A class $\mathcal{A} \subseteq 2^\omega$ is a Σ^0_1-*class* if there is a c.e. set $A \subseteq 2^{<\omega}$ such that $\mathcal{A} = \bigcup_{\sigma\in A}[\sigma]$, where $[\sigma] = \{X \in 2^\omega : \sigma \sqsubset X\}$. Whenever we mention a Σ^0_1-class $\mathcal{A}$, we assume we have fixed such a set of generators A, and identify $\mathcal{A}$ with A. Note that we can assume that A is *prefix-free*, that is, if $\sigma \in A$ and $\sigma \prec \tau$ then $\tau \notin A$.

Let μ be the usual Lebesgue measure on 2^ω. A set $\mathcal{A} \subseteq 2^\omega$ is called *Martin-Löf null* (or Σ^0_1-*null*) if there is a uniformly c.e. sequence $\{\mathcal{U}_i\}_{i\in\omega}$ of Σ^0_1-classes (called a *test*) such that $\mu(\mathcal{U}_i) \leqslant 2^{-i}$ and $\mathcal{A} \subseteq \bigcap_i \mathcal{U}_i$. The set $\mathcal{A}$ is *Schnorr null* if in addition the measures $\mu(\mathcal{U}_i)$ are uniformly computable reals. A test with this extra property is called a *total test* or a *Schnorr test*. Equivalently, $\mathcal{A}$ is Schnorr null if there is a test $\{\mathcal{U}_i\}_{i\in\omega}$ such that $\mu(\mathcal{U}_i) = 2^{-i}$ and $\mathcal{A} \subseteq \bigcap_i \mathcal{U}_i$.

The corresponding randomness notions are defined by saying that $A \in 2^\omega$ is Σ^0_1-*random* (or 1-*random* or *Martin-Löf random*) if $\{A\}$ is not Σ^0_1-null, and A is *Schnorr random* if $\{A\}$ is not Schnorr null.

A different treatment of measure is the one of Ville [32] using martingales. A *martingale* is a function $d : 2^{<\omega} \to \mathbb{Q}^+$ that satisfies for every $\sigma \in 2^{<\omega}$ the averaging condition $2d(\sigma) = d(\sigma 0) + d(\sigma 1)$, and d is a *supermartingale* if merely $2d(\sigma) \geqslant d(\sigma 0) + d(\sigma 1)$. A (super)martingale d *succeeds on* a set A if $\limsup_{n\to\infty} d(A \upharpoonright n) = \infty$. We say that d succeeds on, or *covers*, a class $\mathcal{A} \subseteq 2^\omega$ if d succeeds on every $A \in \mathcal{A}$. The *success set* $S[d]$ of d is the class of all sets on which d succeeds. Ville proved that the class of null sets of the form $S[d]$, with d of arbitrary complexity, coincides with the class of classical (Lebesgue) null sets.

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Schnorr gave characterizations of the above notions of effectively null set in terms of martingales. In particular, he introduced null sets of the form

$$S_h[d] = \Big\{ X : \limsup_{n\to\infty} \frac{d(X \upharpoonright n)}{h(n)} = \infty \Big\},$$

where d is a martingale and h is a nondecreasing unbounded function (called an *order*), and proved the following theorem.

Theorem 1.1. (Schnorr [27], Sätze 9.4, 9.5) *A set $\mathcal{A} \subseteq 2^\omega$ is Schnorr null if and only if there are a computable martingale d and a computable order h such that $\mathcal{A} \subseteq S_h[d]$.*

Schnorr also addressed null sets of *exponential* order, that is, of the form $S_h[d]$ with $h(n) = 2^{\varepsilon n}$ and $\varepsilon \in (0, 1]$. Although he did not make an explicit reference to Hausdorff dimension, it turns out that the theory of Hausdorff dimension can be cast precisely in terms of such null sets of exponential order, so that Schnorr's notion of effective measure in a natural way leads us into the theory of dimension.

Lutz [18] used effective martingales to develop his theory of resource bounded measure. He defined $A \in 2^\omega$ to be *computably random* if there is no computable martingale d such that $A \in S[d]$.[1] This framework for studying measure and randomness at the level of complexity classes can be used to constructivize Hausdorff dimension along the same lines:

Definition 1.2. For a complexity class $\mathcal{C}$, a set $\mathcal{A} \subseteq 2^\omega$ has *$\mathcal{C}$-dimension* α if

$$\alpha = \inf \big\{ s \in \mathbb{Q} : \exists d \in \mathcal{C} \ (\, d \text{ is a supermartingale and } \mathcal{A} \subseteq S_{2^{(1-s)n}}[d]\,) \big\}.$$

Lutz [19, 20] used a variant of martingales called *gales* in his presentation. In this paper we stick to martingales and the null sets of the form $S_h[d]$ used by Schnorr in our treatment of Hausdorff dimension. That this makes no difference was pointed out by several authors, including those of [2, 3, 31]. If $\mathcal{C}$ consists of all functions, then the notion of $\mathcal{C}$-dimension is equivalent to classical Hausdorff dimension. We say that a function $2^{<\omega} \to \mathbb{Q}^+$ is in Σ^0_1 if it is approximable from below by a nondecreasing computable function. If $\mathcal{C} = \Sigma^0_1$ then the notion of $\mathcal{C}$-dimension is equivalent to Lutz's definition [20] of constructive Σ^0_1-dimension.

In this paper we are interested in the quantitative structure of Δ^0_2. The appropriate measures to use in this context are those for which Δ^0_2 itself does not have measure 0, but for which every element of Δ^0_2 does have measure 0. Since there are Σ^0_1-random sets in Δ^0_2, Martin-Löf's Σ^0_1-measure is too weak for our purposes. For Σ^0_2-measure, obtained by relativizing Σ^0_1-measure to the halting set $\emptyset'$, the class Δ^0_2 has measure 0, so this measure is too strong. However, relativizing the notions of Schnorr null and computably null to $\emptyset'$ gives measures that meet our requirements:

Definition 1.3. A set $\mathcal{A} \subseteq 2^\omega$ has *Δ^0_2-measure* 0 (or is *Δ^0_2-null*) if there is a $\emptyset'$-computable martingale that succeeds on $\mathcal{A}$.

A set $\mathcal{A} \subseteq 2^\omega$ has *Schnorr Δ^0_2-measure* 0 (or is *Schnorr Δ^0_2-null*) if there is a $\emptyset'$-computable Schnorr test that covers $\mathcal{A}$.

A first study of the quantitative structure of Δ^0_2 using these measures was made in Terwijn [29, 30].

[1] Schnorr also considered this definition in relativized form [27], p. 55.

Relativizing computable randomness yields Δ^0_2-*randomness*, and relativizing Schnorr randomness yields *Schnorr* Δ^0_2-*randomness*. The relations between the various notions are as follows:

$$\begin{array}{ccc}
\Delta^0_2\text{-random} & & \\
\Downarrow & & \\
\text{Schnorr } \Delta^0_2\text{-random} & \Longrightarrow & \Delta^0_2\text{-dimension } 1 \\
\Downarrow & & \Downarrow \\
\Sigma^0_1\text{-random} & \Longrightarrow & \Sigma^0_1\text{-dimension } 1 \\
\Downarrow & & \\
\text{computably random} & & \Big\Downarrow \\
\Downarrow & & \\
\text{Schnorr random} & \Longrightarrow & \text{computable dimension } 1
\end{array}$$

No other implications hold than the ones indicated. That there are Schnorr random sets that are not computably random was proved by Wang [33]. (See Nies, Stephan, and Terwijn [22] for more information on the separation between the various randomness notions.) The strictness of the other implications in the first column follows from elementary observations and results in Schnorr [27], and is discussed in [8, 30]. That there are no more implications between the first and the second column follows from the next proposition. The strictness of the two implications in the second column follows by similar means.[2]

Proposition 1.4. *There are sets A such that A is not Schnorr random and A has Δ^0_2-dimension* 1.

Proof. Let R be Δ^0_2-random, and let $D = \{2^x : x \in \omega\}$ be an exponentially sparse computable domain. Then $A = R \cup D$ is not Schnorr random, since no Schnorr random set contains an infinite computable subset, but no Δ^0_2-martingale can succeed on A exponentially fast. □

Clearly, the "Δ^0_2-dimension 1" in Proposition 1.4 can be improved to "Δ^0_n-dimension 1" by the same proof, if one is considering higher orders of randomness.

The rest of this paper is organized as follows. Ambos-Spies, Merkle, Reimann, and Stephan [2] investigated resource bounded dimension in the exponential time class E. Among other things, they proved that under polynomial time many-one reducibility the complete degree in E has dimension 1, and that the set of possible dimensions of p-m-degrees in E is dense in $[0, 1]$. In Section 2 we show that under Turing reducibility in Δ^0_2 the complete degree has Δ^0_2-dimension 1, and all other degrees have Δ^0_2-dimension 0. In Section 3 we present a proof that the low sets do not have Δ^0_2-measure 0 by showing that for every $\emptyset'$-computable martingale there is a low set that is not covered by it. This means that the low sets do not form a small subset of Δ^0_2.

Our notation generally follows Odifreddi [23, 24] and Soare [28]. We write ${}^{\leqslant_{\mathrm{T}}}A$ for the lower cone $\{B : B \leqslant_{\mathrm{T}} A\}$ and $A^{\leqslant_{\mathrm{T}}}$ for the upper cone $\{B : A \leqslant_{\mathrm{T}} B\}$.

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[2]It is easy to see (cf. [20]) that the class of computable sets has Σ^0_1-dimension 0, but is not computably null, so in particular this class has computable dimension 1. Also, Lutz [20] has shown that there are sets in Δ^0_2 of any given rational Σ^0_1-dimension, but it is obvious that every set in Δ^0_2 has Δ^0_2-dimension 0.

2. Δ^0_2-DIMENSION

The next theorem is a strengthening of Theorem 5.5 in [30], which states that the lower cone of every $A <_{\mathrm{T}} \emptyset'$ has Schnorr Δ^0_2-measure 0. We will make use of the following definition and lemma.

Definition 2.1. For functions f and g and rational $q \in (0,1]$, we say that f is *q-dominated* by g if

$$\liminf_{n\to\infty} \frac{|\{i \leqslant n : g(i) \geqslant f(i)\}|}{n} \geqslant q. \tag{1}$$

Lemma 2.2. *For every $q \in (0,1]$ there is a function $f \leqslant_T \emptyset'$ such that f is not q-dominated by any function $g <_T \emptyset'$.*

Proof. Let $h \leqslant_{\mathrm{T}} \emptyset'$ be a function not dominated by any function $g <_{\mathrm{T}} \emptyset'$, for example, $h(x) = \mu s(\emptyset'_s \restriction x = \emptyset' \restriction x)$ (the smallest s such that all the $y \in \emptyset'$ smaller than x are enumerated into $\emptyset'$ within s steps). Without loss of generality, $q = \frac{1}{c}$ for some $c \in \omega$. Define $f(x) = h(\lfloor \log_c x \rfloor)$ (where $\log_c$ is the logarithm with base c). If g satisfies (1) then for almost every k there is a natural number $x \in [c^k, c^{k+1})$ such that $g(x) \geqslant f(x)$. But then the function $\widehat{g}$ defined by $\widehat{g}(k) = \max\{g(x) : x \in [c^k, c^{k+1})\}$ dominates h, a contradiction. $\square$

Theorem 2.3. *Let $A \in \Delta^0_2$ be any Turing-incomplete set. Then the Δ^0_2-dimension of the lower cone ${}^{\leqslant_T}\!A$ is 0.*

Proof. Let $q \in (0,1]$ be rational and suppose that $A <_{\mathrm{T}} \emptyset'$. We define uniformly in $\emptyset'$ for every $e \in \omega$ a martingale d_e such that

$$R_e: \quad \Phi^A_e \text{ total and } \{0,1\}\text{-valued} \Longrightarrow \Phi^A_e \in S_{2^{(1-q)n}}[d_e].$$

By the usual sum trick this suffices to prove the theorem: The sum $d(\sigma) = \sum_{e\in\omega} 2^{-e} d_e(\sigma)$ is again a $\emptyset'$-computable martingale, and if $X \in S_{2^{(1-q)n}}[d_e]$ then for all $q' > q$ we have $X \in S_{2^{(1-q')n}}[d]$, which shows that $\{B : B \leqslant_{\mathrm{T}} A\}$ has Δ^0_2-dimension $\leqslant q$. Since $q > 0$ was arbitrary the theorem follows.

By Lemma 2.2, let $f \leqslant_{\mathrm{T}} \emptyset'$ be a function that is not q-dominated by any function $g <_{\mathrm{T}} \emptyset'$.

We now define d_e in stages s. At stage s we define d_e on all strings $\sigma \in 2^{<\omega}$ of length s. The value $d_e(\sigma)$ will depend only on $|\sigma|$. (Ambos-Spies, Mayordomo, Wang, and Zheng [1] called martingales with this property 'oblivious'.)

Stage $s = 0$. Define $d_e(\lambda) = 1$, where λ is the empty string.

Stage $s+1$. Given $d_e(\sigma)$ with $|\sigma| = s$, use the oracle $\emptyset'$ to search for a string $\tau \sqsubset A$ with $|\tau| \leqslant f(s)$ such that $\Phi^\tau_{e,|\tau|}(s)\downarrow$. If such τ does not exist, or if $\Phi^\tau_{e,|\tau|}(s)\downarrow \notin \{0,1\}$, do not make a bet; that is, let $d_e(\sigma i) = d_e(\sigma)$ for $i \in \{0,1\}$. If τ exists and $\Phi^\tau_{e,|\tau|}(s)\downarrow = i \in \{0,1\}$, define $d_e(\sigma i) = 2d_e(\sigma)$; that is, bet all our capital on $\Phi^A_e(|\sigma|) = i$. This concludes the definition of d_e.

It is clear that d_e is defined on all strings for every e, uniformly in $\emptyset'$. We check that R_e is satisfied. Suppose that Φ^A_e is total and computes a set. Then the function

$$g_e(n) = \mu t\Big((\exists \tau \sqsubset A)\left[\, |\tau| = t \wedge \Phi^\tau_{e,t}(n)\downarrow \,\right]\Big)$$

is A-computable. By the choice of f, there are infinitely many N such that for more than $(1-q)N$ many $n < N$ we have $f(n) \geqslant g_e(n)$. For these n, in the definition of d_e the string τ is found and a bet is placed successfully. (Note that we never make a wrong bet.) Hence $\Phi_e^A \in S_{2^{(1-q)n}}[d_e]$. $\square$

Theorem 2.4. *For every noncomputable $A \in \Delta_2^0$, the upper cone $A^{\leqslant_T}$ has Schnorr Δ_2^0-measure* 0.

Proof. This theorem is an effectivization of the well-known result of de Leeuw, Moore, Shannon and Shapiro [16] and Sacks [26] that the upper cone of a noncomputable set has Lebesgue measure 0.[3] Lutz and Terwijn [29] showed that there is a $\emptyset'$-computable martingale that succeeds on $A^{\leqslant_\mathrm{T}}$ when $A \in \Delta_2^0$ is noncomputable. We give here a direct proof using total $\emptyset'$-computable tests, which gives the stronger result of the theorem. Fix a noncomputable $A \in \Delta_2^0$, and define for every i and n the open sets

$$\mathcal{E}_{i,n} = \big\{B : A \restriction n = \Phi_i^B \restriction n\big\}.$$

For every i we have $\{B : A = \Phi_i^B\} = \bigcap_n \mathcal{E}_{i,n}$, so $\mu(\bigcap_n \mathcal{E}_{i,n}) = 0$. Furthermore, the $\mathcal{E}_{i,n}$ are uniformly $\emptyset'$-computable because A is $\emptyset'$-computable, and the $\mu(\mathcal{E}_{i,n})$ are uniformly $\emptyset'$-computable reals. So if we let $f(k)$ be the least n such that $\mu(\mathcal{E}_{i,n}) \leqslant 2^{-k}$ and define $\mathcal{F}_{i,k} = \mathcal{E}_{i,f(n)}$, then $\mathcal{F}_{i,0}, \mathcal{F}_{i,1}, \ldots$ is a total $\emptyset'$-computable test, and we still have $\{B : A = \Phi_i^B\} = \bigcap_k \mathcal{F}_{i,k}$.

Because the tests $\bigcap_k \mathcal{F}_{i,k}$ are $\emptyset'$-uniform in i, it follows from an easily proved effective union lemma that $\bigcup_i \bigcap_k \mathcal{F}_{i,k} = A^{\leqslant_\mathrm{T}}$ is also of Schnorr measure 0 relative to $\emptyset'$. $\square$

The next theorem shows that for every $A \in \Delta_2^0$ the Δ_2^0-dimension of the upper cone of A is maximal. In particular, although the Schnorr Δ_2^0-measure of $\deg_\mathrm{T}(\emptyset')$ is 0, there is no Δ_2^0-martingale that succeeds on this Turing degree exponentially fast.

Theorem 2.5. *The Δ_2^0-dimension of* $\deg_T(\emptyset')$ *is* 1.

Proof. [4] Given a martingale $d \in \Delta_2^0$ and a rational $\varepsilon > 0$, we build a $B \equiv_\mathrm{T} \emptyset'$ such that $B \notin S_{2^{\varepsilon n}}[d]$. The idea is simple: We code $\emptyset'$ on an exponentially sparse computable domain D, and define B by $\emptyset'$-effectively diagonalizing against d outside D and taking the coded version of $\emptyset'$ on D. Since D is exponentially sparse, d cannot succeed fast on B, and we have $B \leqslant_\mathrm{T} \emptyset' \leqslant_\mathrm{T} B \oplus D \leqslant_\mathrm{T} B$ since D is computable. Note that this idea works for every computably sparse domain D, so that in fact $\deg_\mathrm{T}(\emptyset')$ is not included in any null set of the form $S_h[d]$ for a Δ_2^0-martingale d and a computable order h. Theorem 2.4 shows that the same is not true for all $\emptyset'$-computable orders h. For the theorem as stated it suffices to take $D = \{2^m - 1 : m \in \omega\}$ and define

$$B(x) = \begin{cases} \emptyset'(n) & \text{if } x = 2^m - 1 \\ 0 & \text{if } x \notin D \text{ and } d((B \restriction x)0) < d((B \restriction x)1) \\ 1 & \text{otherwise.} \end{cases}$$

Then $d(B \restriction n-1) \leqslant d(B \restriction n)$ except possibly when $n = 2^m - 1$ for some n, so

$$\limsup_{n\to\infty} \frac{d(B \restriction n)}{2^{\varepsilon n}} \leqslant \limsup_{m\to\infty} \frac{d(B \restriction 2^m)}{2^{\varepsilon 2^m}} \leqslant \limsup_{m\to\infty} \frac{2^m}{2^{\varepsilon 2^m}} < 1,$$

[3]For another approach to effectivizing this result, see Hirschfeldt, Nies, and Stephan [10].

[4]This proof is a few years old. Coding techniques similar to the one used in it have meanwhile been used in the context of Hausdorff dimension independently by several authors, cf. e.g. Reimann [25].

and hence $B \notin S_{2^{\varepsilon n}}[d]$. □

It follows from Theorems 2.3 and 2.5 that the only possibilities for the Δ^0_2-dimension of a Turing degree are 0 or 1:

Corollary 2.6. *For $A \in \Delta^0_2$, the Δ^0_2-dimension of $\deg_T(A)$ is 1 if A is Turing complete, and 0 otherwise.*

3. The measure of the low sets

It is known that the class of sets that are bounded by a 1-generic set has Σ^0_1-measure 0 (by effectivizing Theorem 4.2 in Kurtz [14], cf. [30], or by Demuth and Kučera [4]). In particular the subclass of the low sets consisting of the Δ^0_2 1-generic sets has Σ^0_1-measure 0. In this section we prove that the low sets do not form a small subset of Δ^0_2, that is, that they do not have Δ^0_2-measure 0. It is easily verified that the computable sets have Δ^0_2-measure 0, and that most sets in Δ^0_2 are bi-immune for the computable sets.[5] Although the low sets do not have Δ^0_2-measure 0, Downey, Hirschfeldt, Lempp, and Solomon [6] were able to construct a Δ^0_2 set A that is bi-immune for the low sets (i.e. there is no infinite low subset of either A or its complement). The set they constructed in fact truth-table reduces to $\emptyset'$. It is not difficult to see that the sets that tt-reduce to $\emptyset'$ have Schnorr Δ^0_2-measure 0, i.e. there is a total $\emptyset'$-computable test covering them. So in this sense the set constructed in [6] does not exhibit the typical behavior of a Δ^0_2-set. Theorem 3.1 shows that indeed it is not the case that almost every set in Δ^0_2 is bi-immune for the low sets.

Theorem 3.1. *The low sets do not have Δ^0_2-measure 0.*

Proof. Let M be a universal Σ^0_1-martingale and let N be an arbitrary Δ^0_2-martingale. We will exhibit a low set B on which N does not succeed. Define a new martingale L by

$$L(\emptyset) = \tfrac{1}{2}\big(M(\emptyset) + N(\emptyset)\big),$$
$$L(\sigma) = \tfrac{1}{2}\big(M(\sigma) + N(\sigma(0)\sigma(2)\dots\sigma(2n))\big) \text{ if } |\sigma| = 2n+1 \text{ or } 2n+2.$$

So L is essentially a sum of the behaviour of M and the behaviour of N restricted to the even bits. We leave it to the reader to check that L is indeed a martingale.

Now let $A \in \Delta^0_2$ be such that L does not succeed on A. (Such an A exists because Δ^0_2 does not have Δ^0_2-measure 0.) Then M is bounded on A, and hence A is Σ^0_1-random. Also, N is bounded on the set B defined by $B(n) = A(2n)$ for every n. We claim that B is low, being half of a Σ^0_1-random set below $\emptyset'$. Thus we have exhibited a low set on which the arbitrary Δ^0_2-martingale N does not succeed.

To prove the claim that B is low, suppose that C is the odd part of A, i.e. the unique set with $A = B \oplus C$. Since A is Σ^0_1-random, by a result of van Lambalgen [15] the set C is Σ^0_1-random relative to B. Nies and Stephan (see Theorem 3.4 in [7]) showed that this implies that B is low. For completeness, we include a proof of this fact.

Let $\mathrm{use}(\Phi^B_e(e))$ be the partial function that for every e measures the number of computation steps of $\Phi^B_e(e)$, if this is defined. Since C is a Δ^0_2 set, it has a computable approximation C_s such that $\lim_s C_s(n) = C(n)$ for every n. Let the convergence modulus of this approximation be the function $m(n) = \mu t(\forall s \geqslant$

[5] Indeed, this is even true for Σ^0_1-measure: The class of sets that are not bi-immune for the computable sets has Σ^0_1-measure 0.

$t)[C_s(n) = C_t(n)]$. Now if there were infinitely many e such that $\text{use}(\Phi_e^B(e)) \geqslant m(e)$ then B could compute infinitely many points of C, contradicting the fact that C is Σ_1^0-random relative to B. Hence, for almost all e, whenever $\Phi_e^B(e)$ is defined we have that $\text{use}(\Phi_e^B(e)) < m(e)$. Since $m \leqslant_T \emptyset'$ we obtain that $B' \leqslant_T \emptyset'$. □

Corollary 3.2. *The Δ_2^0-Hausdorff dimension of the low sets is* 1.

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