CONSTRUCTIVE LOGIC AND THE MEDVEDEV LATTICE

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ABSTRACT

We study the connection between factors of the Medvedev lattice and constructive logic. The algebraic properties of these factors determine logics lying in between intuitionistic propositional logic and the logic of the weak law of the excluded middle (also known as De Morgan, or Jankov logic). We discuss the relation between the weak law of the excluded middle and the algebraic notion of join-reducibility. Finally we discuss autoreducible degrees.

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1 INTRODUCTION

Ever since Heyting wrote down the axioms of intuitionistic logic (in 1930), people have tried to give a semantics for this logic that explains their constructive content. Many people felt that such an explanation should have something to do with the theory of computation, but most approaches based on this idea (such as Kleene's realizability) failed to capture intuitionistic provability. In 1932 Kolmogorov proposed a semantics for intuitionistic propositional logic based on a "calculus of problems" [5]. Kolmogorov's exposition was rather sketchy, but later several more complete formalizations based on Kolmogorov's idea were given by Medvedev. Although initially these contributions also failed to provide a complete semantics for intuitionistic propositional logic IPC, later work by Skvortsova based on this did succeed in capturing IPC. Furthermore, the algebraic structures introduced by Medvedev turned out to be interesting for other reasons as well. In particular, there are many connections to the study of other structures

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from computability theory. For example, the Medvedev lattice contains the Turing degrees (as an upper semilattice).

The approach discussed in this paper by no means exhausts the possibilities for interesting connections between constructive logic and computation using Kolmogorov's idea. For example, in Shen and Vereshchagin [13] connections are made to the theory of Kolmogorov complexity. The paper [19] contains results on Π^0_1 classes bearing on constructive logic.

We briefly review the basic definitions of the Medvedev lattice in section 2. In section 3 we then discuss the connection with logic, including Skvortsova's result. In section 4 we take some steps in exploring the algebraic structure of the Medvedev degrees. In particular, we discuss join-irreducible elements, that are related to factors of the Medvedev lattice where the weak law of the excluded middle $\neg\alpha \vee \neg\neg\alpha$ holds. Finally, in section 5 we discuss autoreducible degrees.

2 The Medvedev lattice

First we briefly recall the definition of the Medvedev lattice $\mathfrak{M}$, originally introduced in Medvedev [8]. Let ω denote the naturals and let ω^ω be the set of all functions from ω to ω (Baire space). A *mass problem* is a subset of ω^ω. We think of such subsets as a "problem", namely the problem of producing an element of it, and so we can think of the elements of the mass problem as its set of solutions. We say that a mass problem $\mathcal{A}$ *Medvedev reduces* to mass problem $\mathcal{B}$ if there is an effective procedure of transforming solutions to $\mathcal{B}$ into solutions to $\mathcal{A}$. Formally: $\mathcal{A} \leq \mathcal{B}$ if there is a partial computable functional $\Psi : \omega^\omega \to \omega^\omega$ such that for all $f \in \mathcal{B}$, $\Psi(f)$ is defined and $\Psi(f) \in \mathcal{A}$. This can be seen as an implementation of Kolmogorov's idea of a calculus of problems. The relation $\leq$ induces an equivalence relation on the mass problems: $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$. The equivalence class of $\mathcal{A}$ is denoted by $[\mathcal{A}]$ and is called the *Medvedev degree*, or the *degree of difficulty* of $\mathcal{A}$. We usually denote Medvedev degrees by boldface symbols. Note that there is a smallest Medvedev degree, denoted by $\mathbf{0}$, namely the degree of any mass problem containing a computable function. There is also a largest degree $\mathbf{1}$, the degree of the empty mass problem, of which it is impossible to produce an element by whatever means. Finally, it is possible to define a meet operator $\times$ and a join operator $+$ on mass problems: For functions f and g, as usual define the function $f \oplus g$ by $f \oplus g(2x) = f(x)$ and $f \oplus g(2x+1) = g(x)$. Let $n\widehat{\ }\mathcal{A} = \{n\widehat{\ }f : f \in \mathcal{A}\}$, where $\widehat{\ }$ denotes concatenation. Define

$$\mathcal{A} + \mathcal{B} = \{f \oplus g : f \in \mathcal{A} \wedge g \in B\}$$

and

$$\mathcal{A} \times \mathcal{B} = 0\widehat{\ }\mathcal{A} \cup 1\widehat{\ }\mathcal{B}.$$

It is not hard to show that $\times$ and $+$ indeed define a greatest lower bound and a least upper bound operator on the Medvedev degrees:[1]

Theorem 2.1 (Medvedev [8]) *The structure $\mathfrak{M}$ of all Medvedev degrees, ordered by $\leq$ and together with $\times$ and $+$ is a distributive lattice.*

Let $\mathcal{F} = \{f : f \text{ noncomputable}\}$. We note the following important fact, namely that for all mass problems $\mathcal{A}$, if $[\mathcal{A}] \not\leq \mathbf{0}$ (i.e. $\mathcal{A}$ does not contain any computable function) then $\mathcal{F} \leq \mathcal{A}$ via the identity. That is, the Medvedev degree of $\mathcal{F}$, which is denoted by $\mathbf{0}'$, is the unique nonzero minimal degree of $\mathfrak{M}$.

A distributive lattice $\mathfrak{L}$ with 0, 1 is called a *Brouwer algebra* if for any elements a and b one can show that the element $a \to b$ defined by

$$a \to b := \text{least}\{c \in \mathfrak{L} : b \leq a + c\}$$

always exists. We even have:

Theorem 2.2 (Medvedev [8]) *$\mathfrak{M}$ is a Brouwer algebra.*

Proof. Define $\mathcal{A} \to \mathcal{B} = \{n\widehat{\ }f : (\forall g \in \mathcal{A})[\Phi_n(g \oplus f) \in \mathcal{B}]\}$, where Φ_n is the n-th partial computable functional. This definition extends to the Medvedev degrees in the obvious way. $\square$

$\mathfrak{L}$ is called a *Heyting algebra* if its dual is a Brouwer algebra. Sorbi [15] has shown that $\mathfrak{M}$ is not a Heyting algebra. Some more discussion and facts about $\mathfrak{M}$ can be found in Rogers [12]. A good survey of what is known

[1] There is an annoying notational conflict between the various papers in this area. Sorbi [18] maintains the usual lattice theoretic notation with $\wedge$ for meet and $\vee$ for join, but e.g. Rogers [12] and Skvortsova [14] use $\wedge$ and $\vee$ exactly the other way round! The advantage of the latter choice will become clear below, namely that $\wedge$ and $\vee$ then nicely correspond with "and" and "or" in the propositional logic corresponding to the lattice (see section 3). To avoid headaches we have introduced separate notation for the lattices ($+$ for join and $\times$ for meet) and the logic (the usual $\wedge$ for "and" and $\vee$ for "or") here. This is in line with notation that is used in some textbooks on lattice theory, cf. [1]. It has as an additional advantage that the join operator $+$ in $\mathfrak{M}$ corresponds to the usual notation $\oplus$ for the join operator in the Turing degrees.

about $\mathfrak{M}$ is Sorbi [18], where also a more complete list of references can be found.

We conclude this section with one more definition that we will use later. Note that $\mathcal{A} \leq \mathcal{B}$ means that there is a *uniform* way to transform solutions for the one problem into solutions to the other. There is also an interesting *nonuniform* variant of this definition [10]: We say that $\mathcal{A}$ *Muchnik reduces* to $\mathcal{B}$, denoted $\mathcal{A} \leq_w \mathcal{B}$, if $(\forall f \in \mathcal{B})(\exists g \in \mathcal{A})[f \leq_T g]$, where $\leq_T$ denotes Turing reducibility. The corresponding degrees are called *Muchnik degrees*. They form a distributive lattice in the same way as the Medvedev degrees.

Define $C(\mathcal{A}) = \big\{f : (\exists e)[\Phi_e(f) \in \mathcal{A}]\big\}$, where Φ_e is the e-th partial computable functional. The Muchnik degrees can be seen as a sublattice of the Medvedev degrees by the embedding $[\mathcal{A}] \mapsto [C(\mathcal{A})]$. The Muchnik degrees are then precisely the Medvedev degrees that contain a mass problem $\mathcal{A}$ such that $C(\mathcal{A}) = \mathcal{A}$. This is equivalent to saying that $\mathcal{A}$ is upward closed under Turing reducibility.

3 Logic and computation

As we have seen, the Medvedev lattice implemented an idea of Kolmogorov that was supposed to give a computational meaning to the logical connectives. Below we make precise what is meant by this and point out that, as already observed by Medvedev himself, unfortunately also this approach does not succeed to capture intuitionistic logic, at least not directly. However, a slight extension of the idea *does* work, and gives us, in an algebraically very natural way, a computational semantics for intuitionistic propositional logic IPC.

In section 2 we have already defined the operations $\times$, $+$, and $\rightarrow$ on $\mathfrak{M}$. We can also define a negation operator $\neg$ by defining $\neg\mathbf{A} = \mathbf{A} \rightarrow \mathbf{1}$ for any Medvedev degree $\mathbf{A}$.

Given any Brouwer algebra $\mathfrak{L}$ (such as $\mathfrak{M}$) with join denoted by $+$ and meet by $\times$, we can evaluate formulas as follows. An $\mathfrak{L}$-*valuation* is a function v : Form $\rightarrow \mathfrak{L}$ from formulas to $\mathfrak{L}$ such that for all formulas α and β, $v(\alpha \vee \beta) = v(\alpha) \times v(\beta)$, $v(\alpha \wedge \beta) = v(\alpha) + v(\beta)$, $v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$, $v(\neg\alpha) = v(\alpha) \rightarrow 1$. (Note the upside-down reading of $\wedge$ and $\vee$ when compared to the usual lattice theoretic interpretation, see also note 1.) Write $\mathfrak{L} \models \alpha$ if $v(\alpha) = 0$ for any $\mathfrak{L}$-valuation v. Finally, define

$$\mathrm{Th}(\mathfrak{L}) = \big\{\alpha : \mathfrak{L} \models \alpha\big\}.$$

On page 289 of Rogers [12] it is stated that Medvedev has shown that the identities of $\mathfrak{M}$ (i.e. $\mathrm{Th}(\mathfrak{M})$) are the theorems of IPC, the intuitionistic propositional calculus. This however seems to be a misquotation. It is certainly *not* true that $\mathrm{Th}(\mathfrak{M}) = \mathrm{IPC}$. (That would have been a great result!) Indeed, Medvedev already noted that for every $\mathbf{A} \in \mathfrak{M}$ we have that either $\neg\mathbf{A} = \mathbf{0}$ or $\neg\mathbf{A} = \mathbf{1}$, hence that always $\neg\mathbf{A} \times \neg\neg\mathbf{A} = \mathbf{0}$. That is, $\mathfrak{M}$ satisfies the *weak law of the excluded middle* $\neg\alpha \vee \neg\neg\alpha$. In fact, we have the following result:

Theorem 3.1 (Medvedev [9], Jankov [3], Sorbi [16])[2] $\mathrm{Th}(\mathfrak{M})$ *is the deductive closure of* IPC *and the weak law of the excluded middle (also known as De Morgan, or Jankov logic).*

This is already very interesting, but in the light of the quest for a computational semantics for IPC it may be a disappointment. Since $\mathfrak{M}$ does not do the trick, we need to look at other Brouwer algebras. A very natural idea, from an algebraic point of view, is to look at *factors* of $\mathfrak{M}$, i.e. to study $\mathfrak{M}$ modulo a filter or an ideal. Given a Brouwer algebra $\mathfrak{L}$ and an ideal I in $\mathfrak{L}$, $\mathfrak{L}/I$ is still a Brouwer algebra. If G is a filter in $\mathfrak{L}$ then $\mathfrak{L}/G$ is not necessarily a Brouwer algebra, but if G is principal then $\mathfrak{L}/G$ is again a Brouwer algebra. In such a factorized lattice G plays the role of 1. E.g. if G is the principal filter in $\mathfrak{M}$ generated by the degree $\mathbf{D}$ then negation in $\mathfrak{M}/G$ can be defined by $\neg\mathbf{A} = \mathbf{A} \to \mathbf{D}$.

Now it is quite easy to find a factor $\mathfrak{M}/G$ of $\mathfrak{M}$ such that $\mathrm{Th}(\mathfrak{M}/G)$ is classical propositional logic. (Take G the principal filter generated by $\mathbf{0}'$, the degree containing the set of all noncomputable functions, see section 2. Note that $\mathbf{0}' = \mathbf{1}$ in $\mathfrak{M}/G$, so that $\mathfrak{M}/G$ has exactly the elements $\mathbf{0}$ and $\mathbf{1}$, corresponding to the classical truth values 1 and 0, respectively.) Of course, what we *really* would like is a factor of $\mathfrak{M}$ that captures IPC. That such a factor indeed exists is the content of the following beautiful theorem.

Theorem 3.2 (Skvortsova [14]) *There exists a principal filter G such that* $\mathrm{Th}(\mathfrak{M}/G)$ *equals* IPC*.*

The proof of Theorem 3.2 consists of a number of clever algebraic coding techniques, combined with some computability theory. Through a series of lattice embedding results (including one by Lachlan for the Turing degrees)

[2]Medvedev [9] actually stated that the *positive* fragments of $\mathrm{Th}(\mathfrak{M})$ and IPC coincide. Jankov [3] proved that $\mathrm{IPC} + \neg\alpha \vee \neg\neg\alpha$ is the largest propositional calculus that is conservative over IPC with respect to positive formulas. The theorem follows from these results, since the weak law of the excluded middle holds in $\mathfrak{M}$. The result also follows directly from the embedding result in Sorbi [18] that characterizes the finite Brouwer algebras that are embeddable in $\mathfrak{M}$.

it is shown that the magic interval can be found. The main problem is the control of the infima, which is taken care of by making use of so-called *canonical* subsets on which the infima are well-behaved. As a canonical subset of $\mathfrak{M}$ those degrees are used that contain a mass problem that is upward closed under Turing reducibility. Note that these are precisely the Muchnik degrees defined at the end of section 2. So, interestingly, both the Turing degrees and the Muchnik degrees play a role in the proof of Theorem 3.2.

4 Irreducible elements

In the previous section we saw how the algebraic structure of $\mathfrak{M}$ and its factors $\mathfrak{M}/G$ relates to the theories $\mathrm{Th}(\mathfrak{M}/G)$. In this section we discuss one special aspect of the algebraic structure of $\mathfrak{M}$, namely its join-irreducible elements. Recall that an element a of a lattice $\mathfrak{L}$ is join-reducible if there are $b, c \in \mathfrak{L}$ such that $a = b + c$ and $a \not\leq b$, $a \not\leq c$. In this case we say that a *splits* into b and c. In this section we discuss the join-irreducible elements of $\mathfrak{M}$. For a discussion of the dual notion of meet-reducibility see e.g. [18]. Join- and meet-irreducible elements also play a crucial role in various results about embeddings of degree structures that are needed in the proof of Theorem 3.2.

In Theorem 3.1 we saw that $\mathfrak{M}$ satisfies the weak law of the excluded middle $\neg\alpha \vee \neg\neg\alpha$. This is due to the fact that $\mathbf{1}$ is join-irreducible, as the following proposition shows.

Proposition 4.1 *Let G be the principle filter generated by Medvedev degree $\mathbf{D}$. Then the weak law of the excluded middle holds in $\mathfrak{M}/G$ if and only if $\mathbf{D}$ is join-irreducible.*

Proof. Suppose that $\mathbf{D}$ is join-reducible, say $\mathbf{A}$ and $\mathbf{B}$ are incomparable such that $\mathbf{A} + \mathbf{B} = \mathbf{D}$. First note that $\neg\mathbf{A} \neq \mathbf{1}$ in $\mathfrak{M}/G$ (where $\mathbf{1}$ is now the top element $\mathbf{D}$ of $\mathfrak{M}/G$) because $\neg\mathbf{A} \leq \mathbf{B} \notin G$. Hence $\neg\neg\mathbf{A} \neq \mathbf{0}$, for otherwise it would hold that $\mathbf{D} \leq \neg\mathbf{A}$. Also, $\neg\mathbf{A} \neq \mathbf{0}$ since $\mathbf{A} \not\geq \mathbf{D}$. Now from $\neg\mathbf{A} \neq \mathbf{0}$ and $\neg\neg\mathbf{A} \neq \mathbf{0}$ it follows that $\neg\mathbf{A} \times \neg\neg\mathbf{A} \neq \mathbf{0}$, since $\mathfrak{M}$ does not have any minimal pairs (because there is exactly one nonzero minimal degree $\mathbf{0}'$ in $\mathfrak{M}$, see section 2). So the weak law of the excluded middle does not hold in $\mathfrak{M}/G$.

Conversely, if $\mathbf{D}$ is join-irreducible it is easy to see that for $\mathbf{A} \neq \mathbf{1}$ we have that $\neg\mathbf{A} = \mathbf{1}$. Since $\neg\mathbf{1} = \mathbf{0}$ we then have that $\neg\mathbf{A} \times \neg\neg\mathbf{A} = \mathbf{0}$ for every $\mathbf{A}$. □

In fact, Sorbi proved the following theorem about the connection between irreducible elements and the theories $\mathrm{Th}(\mathfrak{M}/G)$:

Theorem 4.2 (Sorbi [17, Theorem 4.3]) *For every principal filter G generated by a join-irreducible element greater than $\mathbf{0}'$ it holds that* $\mathrm{Th}(\mathfrak{M}/G) = \mathrm{IPC} + \neg\alpha \vee \neg\neg\alpha$.

We note that it is not the case that if G is a filter generated by a join-reducible element $\mathbf{D}$ then automatically $\mathrm{Th}(\mathfrak{M}/G) = \mathrm{IPC}$. For example, if $\mathbf{D}$ is a join-reducible Muchnik degree then by Skvortsova [14] $\mathrm{Th}(\mathfrak{M}/G)$ satisfies the Kreisel-Putnam formula

$$(\neg p \rightarrow q \vee r) \rightarrow (\neg p \rightarrow q) \vee (\neg p \rightarrow r),$$

which shows that $\mathrm{Th}(\mathfrak{M}/G)$ is strictly larger than IPC.

The Medvedev degrees $\mathbf{0}$ and $\mathbf{1}$ are trivial examples of join-irreducible elements. More interesting examples of irreducible elements are the degrees $[\mathcal{B}_f]$, for any noncomputable f, where $\mathcal{B}_f$ is defined as $\mathcal{B}_f = \{g : g \not\leq_T f\}$, cf. Sorbi [16]. (To see that $[\mathcal{B}_f]$ is join-irreducible suppose that $\mathcal{B}_f \equiv \mathcal{A} + \mathcal{C}$ and that $\mathcal{B}_f \not\leq \mathcal{A}$. Then it cannot be that $\mathcal{A} \subseteq \mathcal{B}_f$ (for otherwise the identity would be a reduction) so there is $h \in \mathcal{A}$ with $h \leq_T f$. Now $\mathcal{B}_f \leq \big\{h \oplus g : g \in \mathcal{C}\big\}$, via Ψ say. But then $\Psi(h \oplus g) \leq_T h \oplus g \not\leq_T f$, hence all $g \in \mathcal{C}$ satisfy $g \not\leq_T f$. So $\mathcal{B}_f \leq \mathcal{C}$ via the identity.) Notice that $[\mathcal{B}_f]$ together with $[\{f\}]$ forms a maximal antichain of size two in $\mathfrak{M}$.

Splittings in the Turing degrees give many examples of join-reducible elements of $\mathfrak{M}$, as the next lemma shows.

Lemma 4.3 (Sorbi [18]) *Suppose $\mathcal{A}$ is a mass problem such that the following condition holds:*

$$\text{\textit{There exist functions} } g, h \notin C(\mathcal{A}) \text{ \textit{such that} } g|_T h \text{ \textit{and} } g \oplus h \in C(\mathcal{A}). \quad (1)$$

Then the Medvedev degree $[\mathcal{A}]$ is join-reducible.

Proof. If condition (1) holds then it is easy to see that

$$[\mathcal{A}] = [\mathcal{A} \times \{g\}] + [\mathcal{A} \times \{h\}].$$

On the other hand, by incomparability of g and h and the fact that they cannot compute anything in $\mathcal{A}$, it follows that the degrees $[\mathcal{A} \times \{g\}]$ and $[\mathcal{A} \times \{h\}]$ are incomparable. □

Problem 5.4 in Sorbi [18] asks for a characterization of the join-irreducible elements of $\mathfrak{M}$. Below we show that condition (1) of Lemma 4.3 characterizes the join-reducible *Muchnik degrees*, and that it does *not* characterize the join-reducible elements of $\mathfrak{M}$.

Recall from section 2 that the Muchnik degrees are precisely the Medvedev degrees containing a mass problem $\mathcal{A}$ such that $\mathcal{A} \equiv C(\mathcal{A})$.

Proposition 4.4 *Condition (1) characterizes the join-reducible Muchnik degrees.*

Proof. Suppose that $[\mathcal{A}]$ is a Muchnik degree. Lemma 4.3 holds for the Muchnik degrees just as well as for the Medvedev degrees, so we only have to show that if condition (1) does not hold for $\mathcal{A}$ then $\mathcal{A}$ is join-irreducible. So suppose (1) does not hold, and suppose that $\mathcal{A} \equiv \mathcal{B} + \mathcal{C}$ and $\mathcal{A} \not\leq \mathcal{B}$. We show that $\mathcal{A} \leq \mathcal{C}$. Since $\mathcal{A} \equiv C(\mathcal{A})$, $\mathcal{A} \not\leq \mathcal{B}$ implies that there is $g \in \mathcal{B} \backslash C(\mathcal{A})$. Then $\mathcal{A} \leq \{g \oplus h : h \in \mathcal{C}\}$, via Ψ say. But then, since $\Psi(g \oplus h) \leq_T g \oplus h$ and by the failure of (1), all $h \in \mathcal{C}$ must be in $C(\mathcal{A})$. Hence $\mathcal{A} \equiv C(\mathcal{A}) \leq \mathcal{C}$ via the identity. $\square$

In Dyment [2] it was shown that *every* Muchnik degree is meet-reducible. (It may be informative to note here that E. Z. Dyment and E. Z. Skvortsova are in fact the same person.)

Proposition 4.4 points out a way in which a Medvedev degree $[\mathcal{A}]$ can be join-reducible without satisfying condition (1): It may happen that $\mathcal{B} \subseteq C(\mathcal{A})$ but that nevertheless $\mathcal{A} \not\leq \mathcal{B}$ because there is no *uniform* procedure that reduces $\mathcal{A}$ to $\mathcal{B}$.

Theorem 4.5 *Condition (1) does not characterize the join-reducible elements of $\mathfrak{M}$: There is a join-reducible Medvedev degree $[\mathcal{A}]$ such that (1) does not hold.*

Proof. We prove this by constructing such an $\mathcal{A}$ by brute force. Let B_e, $e \in \omega$, and X be subsets of ω such that each pair of them forms a minimal pair in the Turing degrees, and such that every B_e does not bound a minimal Turing degree. That such sets exist follows from standard results about lattice embeddings into the Turing degrees. (One can use here the result of Lachlan and Lebeuf [6] that every countable upper semilattice with a least element is isomorphic to an initial segment of the Turing degrees. See e.g. Lerman [7].) Now define $\mathcal{B}' = \{B_e : e \in \omega\}$ (we identify sets with their characteristic functions here) and $\mathcal{C}' = \{f : \emptyset <_T f \leq_T X\}$. Finally, define

$$\mathcal{A} = \{f : f \not\leq_T X\} \setminus \{\Phi_e(B_e) : \Phi_e(B_e) \text{ total}\},$$

and

$$\mathcal{B} = \mathcal{A} \times \mathcal{B}' \qquad \text{and} \qquad \mathcal{C} = \mathcal{A} \times \mathcal{C}'.$$

Then $\mathcal{B}$, $\mathcal{C} \leq \mathcal{A}$ so $\mathcal{B} + \mathcal{C} \leq \mathcal{A}$. We further prove that $\mathcal{A} \leq \mathcal{B} + \mathcal{C}$, $\mathcal{A} \not\leq \mathcal{B}'$, $\mathcal{A} \not\leq \mathcal{C}'$, and that $\mathcal{A}$ satisfies the negation of condition (1).

$\mathcal{A} \leq \mathcal{B} + \mathcal{C}$: It is enough to show that $\mathcal{A} \leq \mathcal{B}' + \mathcal{C}'$. In fact we have $\mathcal{B}' + \mathcal{C}' \subseteq \mathcal{A}$: To see this, let $B_e \oplus f \in \mathcal{B}' + \mathcal{C}'$. Then we cannot have that $B_e \oplus f \leq_T X$ since otherwise $B_e \leq_T X$. Also, there is no i such that $B_e \oplus f = \Phi_i(B_i)$ since otherwise $f \leq_T B_i$, contradicting that X and B_i form a minimal pair. So $B_e \oplus f \in \mathcal{A}$.

$\mathcal{A} \not\leq \mathcal{B}'$: This is by definition of $\mathcal{A}$. Φ_e cannot map $\mathcal{B}'$ into $\mathcal{A}$ since $\mathcal{A}$ excludes $\Phi_e(B_e)$.

$\mathcal{A} \not\leq \mathcal{C}'$: If $f \in \mathcal{C}'$ then $f \leq_T X$ so f cannot compute any $f' \not\leq_T X$.

$\mathcal{A}$ satisfies $\neg$(1): Suppose that $g \notin C(\mathcal{A})$. Then in particular $g \notin \mathcal{A}$. Now if $g = \Phi_e(B_e)$ for some e then, since B_e does not bound a minimal degree, we can find $f \leq_T g$ with $f \neq \Phi_e(B_e)$, which is not Turing-below any other B_i nor is Turing-below X. Then $f \in \mathcal{A}$, and therefore $g \in C(\mathcal{A})$. So we must have $g \leq_T X$. Now take any other $h \notin C(\mathcal{A})$. By the same reason $h \leq_T X$. Hence $g \oplus h \leq_T X$, so $g \oplus h \notin C(\mathcal{A})$. □

Lemma 4.3 gives a special example of a situation where $[\mathcal{A}]$ is join-reducible, namely when an $f \in \mathcal{A}$ can be split into g and h both not in $C(\mathcal{A})$. Note that if we generalize g and h to *sets* of functions we more or less get the definition back: $[\mathcal{A}]$ is join-reducible if and only if there is a set of g's that do not uniformly compute elements in $\mathcal{A}$ (generalizing that $g \notin C(\mathcal{A})$) and a set of h's that also does not uniformly compute elements in $\mathcal{A}$ (generalizing that $h \notin C(\mathcal{A})$), such that the pairs $g \oplus h$ uniformly compute elements of $\mathcal{A}$ (generalizing that $g \oplus h \in C(\mathcal{A})$).

5 Autoreducibility

In computability theory, a set A is called autoreducible if A can compute the answers to membership questions of the form "$x \in A$?" without using the bit $A(x)$, that is, if there is a code e such that for all x, $\{e\}^{A-\{x\}}(x) = A(x)$. E.g. for every set A one can easily see that $A \oplus A$ is autoreducible, since all information of the form $x \in A$ is doubly stored. This shows that every m-degree contains an autoreducible set (Trakhtenbrot). A noncomputable Turing degree is completely autoreducible if it contains *only* autoreducible sets. That there is a completely autoreducible Turing degree was shown by Jockusch and Paterson [4], using the same method with which one can build a minimal Turing degree. Now let us define in an analogous way autoreducibility for Medvedev degrees:

Definition 5.1 A mass problem $\mathcal{A}$ is *autoreducible* if for every $f \in \mathcal{A}$, $\mathcal{A} - \{f\} \leq \mathcal{A}$. A Medvedev degree is autoreducible if it contains an autoreducible mass problem, and *completely autoreducible* if it contains only autoreducible mass problems.

First we note that every Medvedev degree is autoreducible: Given any mass problem $\mathcal{A}$, $\mathcal{A} + \mathcal{A} \equiv \mathcal{A}$ and $\mathcal{A} + \mathcal{A}$ is autoreducible. (Note the similarity to Trakhtenbrots argument for sets quoted above.) Next we turn to completely autoreducible degrees.

Proposition 5.2 *There exists a completely autoreducible Medvedev degree.*

Proof. Let $\mathcal{A} = \big\{X_n : n \in \omega\big\}$ be a uniform sequence of sets of descending Turing degree: $X_{n+1} <_T X_n$ for every n and there exists a computable functional Φ such that $\Phi(X_n) = X_{n+1}$ for every n. Such a sequence can be constructed by standard methods (even in the c.e. degrees), cf. [11]. Now suppose that $\mathcal{B} \equiv \mathcal{A}$. Then $\mathcal{B}$ is also autoreducible: Suppose that $\mathcal{A} \leq \mathcal{B}$ via Ψ_0 and $\mathcal{B} \leq \mathcal{A}$ via Ψ_1. Suppose $Y \in \mathcal{B}$. Suppose that $\Psi_0(Y) = X_n$. Let $\Phi^{(n)}$ denote the n-th iterate of Φ. Then for every $X \in \mathcal{B}$, $\Psi_1 \circ \Phi^{(n)} \circ \Psi_0(X) \in \mathcal{B}$, and moreover

$$\begin{aligned}
\Psi_1 \circ \Phi^{(n+1)} \circ \Psi_0(X) &\leq_T \Phi^{(n+1)} \circ \Psi_0(X) \\
&\leq_T \Phi^{(n+1)}(X_0) \\
&<_T X_n \\
&\leq_T Y
\end{aligned}$$

for every $X \in \mathcal{B}$. In particular $\mathcal{B} - \{Y\} \leq \mathcal{B}$ via $\Psi_1 \circ \Phi^{(n+1)} \circ \Psi_0$. $\square$

We could also have defined a mass problem $\mathcal{A}$ to be autoreducible if $\{f\} \leq \mathcal{A} - \{f\}$ for every $f \in \mathcal{A}$. (The reader may find that this definition more closely resembles the one from computability theory.) Under this alternative definition the autoreducible Medvedev degrees are precisely the degrees of solvability, i.e. the ones containing a mass problem of the form $\{f\}$:

Proposition 5.3 *Under the new definition, $\mathcal{A}$ is autoreducible if and only if $\mathcal{A} \equiv \{f\}$ for some f.*

Proof. Every degree of solvability is autoreducible because $\{f\}+\{f\}$ is autoreducible. Conversely, if $\mathcal{A}$ is autoreducible then we claim that $\{f\}\equiv\mathcal{A}$ for some $f\in\mathcal{A}$: If $\mathcal{A}$ contains an isolated branch f (in the usual tree topology on ω^ω) then this is easy to see: Suppose $\sigma\in\omega^{<\omega}$ is a finite string such that f is the only element of $\mathcal{A}$ extending σ. Then $\{f\}\leq\mathcal{A}$ by using the functional for $\{f\}\leq\mathcal{A}-\{f\}$ for elements that do not extend σ, and by using the identity otherwise.

Now suppose that $\mathcal{A}$ has no isolated branches. Then in particular $\mathcal{A}$ is uncountable. By autoreducibility for every $f\in\mathcal{A}$ there is a computable functional Φ_f such that $\{f\}\leq\mathcal{A}-\{f\}$ via Φ_f. Since $\mathcal{A}$ is uncountable, and since there are only countably many computable functionals, there must be $f, g\in\mathcal{A}$, $f\neq g$, such that $\Phi_f=\Phi_g$. We then have that $\{f\}\leq\mathcal{A}-\{f\}$ via Φ_g and $\{g\}\leq\mathcal{A}-\{g\}$ via Φ_g. Let Ψ be such that $\{f\}\leq\{f,g\}$ via Ψ. (Ψ exists since by autoreducibility $f\equiv_T g$.) Then for all $h\in\mathcal{A}$, $\Psi\circ\Phi_g(h)=f$, hence $\{f\}\leq\mathcal{A}$ via $\Psi\circ\Phi_g$. □

Since for noncomputable f clearly $\{f\}$ is not autoreducible, we see that under this definition noncomputable completely autoreducible degrees do not exist.

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