Computational randomness and lowness*

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ABSTRACT. We prove that there are uncountably many sets that are low for the class of Schnorr random reals. We give a purely recursion theoretic characterization of these sets and show that they all have Turing degree incomparable to $0'$. This contrasts with a result of Kučera and Terwijn [5] on sets that are low for the class of Martin-Löf random reals.

The Cantor space 2^ω is the set of infinite binary sequences; these are called *reals* and are identified with subsets of ω. If $\sigma \in 2^{<\omega}$, that is, σ is a finite binary sequence, we denote by $[\sigma]$ the set of reals that extend σ. These form a basis of clopen sets for the usual discrete topology on 2^ω. Write $|\sigma|$ for the length of $\sigma \in 2^{<\omega}$. The Lebesgue measure μ on 2^ω is defined by stipulating that $\mu[\sigma] = 2^{-|\sigma|}$. With every set $U \subseteq 2^{<\omega}$ we associate the open set $\bigcup_{\sigma\in U}[\sigma]$. When it is convenient, we confuse U with the open set associated to it, in particular we write μU for the measure of the open set corresponding to U. We use the following abbreviation for the measure conditioned to σ:

*Research supported by the Netherlands Foundation for Scientific Research (NWO) Project PGS 22-262.

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$$\mu_\sigma U = \frac{\mu(U \cap [\sigma])}{\mu[\sigma]}$$

A *Martin-Löf test* [7] is a recursive set $U \subseteq \omega \times 2^{<\omega}$ such that $\mu U_n \leq 2^{-n}$, where U_n denotes the n–th section of U. A *Schnorr test* [11] has in addition the property that $\mu U_n = 2^{-n}$ or, alternatively, that there is a recursive enumeration of U such that $\mu(U_n - U_{n,s}) \leq 2^{-s}$ for all n and s, where $U_{n,s}$ are the elements enumerated into U_n before stage s. This latter is a more flexible notion of test but defines the same Schnorr null sets —these are what we are finally interested in. A set of reals is **Martin-Löf null** if it is contained in the G_δ set

$$\bigcap_{n \in \omega} \bigcup_{\sigma \in U_n} [\sigma],$$

for some Martin-Löf test U. We concisely write $\bigcap U$ for the null set above. Similarly, a set of reals is **Schnorr null** if it is contained in $\bigcap U$ for some Schnorr test U.

Martin-Löf tests were introduced to give a consistent definition of the notion of "random sequence". A real $R \in 2^\omega$ is *Martin-Löf random* if it does not belong to any Martin-Löf null set. R is *Schnorr random* if it does not belong to any Schnorr null set. We denote the set of Martin-Löf random reals by $\mathcal{R}$ and the set of Schnorr random reals by $\mathcal{S}$. Martin-Löf tests give essentially only positive information and Schnorr tests give both positive and negative information. For this reason Martin-Löf randomness is sometimes called Σ_1-randomness and Schnorr randomness is sometimes called recursive randomness. Martin-Löf randomness has received the most attention in the past. Recently more restricted notions of null set like Schnorr's have gained popularity because of applications to structural complexity theory (see [2] for a survey).

The definitions above naturally relativize to an arbitrary parameter. When U is recursive in A, we shall speak of tests and null sets *relative to A*. Relativized randomness has been first considered by Demuth and Kučera [3] and Kučera [4]. Some more details on effective null sets are given below as they are needed. For a more complete account we refer the reader to the literature (e.g. [6] and the recent survey [1]).

When a class of sets $\mathcal{C} \subseteq 2^\omega$ has a natural relativized version $\mathcal{C}^A$, it is often fruitful to ask for nontrivial examples of sets such that $\mathcal{C} = \mathcal{C}^A$. These sets are called *low for* $\mathcal{C}$. For example, the ordinary low sets from recursion theory are just the sets that are low for the class of Turing–complete sets. A set A is **low for** $\mathcal{R}$ if every Martin-Löf null set relative to A is contained in a Martin-Löf null set. A set A is **low for** $\mathcal{S}$ if every Schnorr null set relative to A is contained in a Schnorr null set. Clearly every recursive set is low for $\mathcal{R}$ and $\mathcal{S}$. The existence of nonrecursive sets that are low for $\mathcal{R}$ was proved in [5]. Here we want to prove the existence of nonrecursive sets that are low for $\mathcal{S}$ and study their complexity.

We introduce some recursion–theoretic notions that we use to characterize sets that are low for $\mathcal{S}$. A set $T \subseteq \omega \times \omega$ is called a *trace* if all its sections $T^{[k]}$ are finite. If the function mapping k to the canonical code of $T^{[k]}$ is a recursive function, we call T a *recursive trace.* Let $g : \omega \to \omega$ be any function. We say that T *traces* or *captures* g if $g(k) \in T^{[k]}$ for every k. A *bound* is a function $h : \omega \to \omega$ that is nondecreasing and has infinite range. We say that a trace T *has bound* h if $\|T^{[k]}\|$ (the cardinality of $T^{[k]}$) is less than $h(k)$ for all k. Trivially, every recursive trace has a recursive bound, but there is no uniform recursive bound for all recursive traces.

A set A is **(recursively) traceable** if there is a recursive bound h such that all (total) functions $g \leq_T A$ have a recursive trace bounded by h. The following easy fact says that all recursive bounds are essentially equivalent. This has interesting consequences for traceable degrees.

FACT 1. Let h be a recursive bound. Suppose that every function recursive in A has a recursive trace T with bound h. Then every function recursive in A has a recursive trace T with arbitrary small recursive bound.

Proof sketch. Identify in a canonical way each sequence $\sigma \in \omega^{<\omega}$ with a natural number. When $k < |\sigma|$ we write $\sigma(k)$ for the k-th digit of the sequence σ. Let $g \leq_T A$. Let f be an increasing recursive function to be specified below (roughly: an inverse of h). Let T be a recursive trace with bound h that captures the function $i \mapsto g \upharpoonright f(i)$ (the string that codes the first $f(i)$ values of g). Let S be the set defined by

$$S^{[k]} \quad = \quad \{\sigma(k) : \sigma \in T^{[i_k]}\},$$

where i_k is least i such that $|\sigma| = f(i) > k$. Clearly, S is a recursive trace. The cardinality of $S^{[k]}$ is easily computed and is bounded by $h(i_k)$. So, the faster f grows, the slower the cardinality of $S^{[k]}$ grows. It is easy to design an f that makes S attain a given recursive bound. Q.E.D.

From the fact above it follows immediately that recursively traceable degrees are hyperimmune–free. Recall (see e.g. [9, Definition V.5.2]) that a set A has hyperimmune–free degree if every function recursive in A is dominated by some recursive function. Indeed, this is actually the case for traceable degrees: every function g is bounded by the recursive function $\max T^{[k]}$ where T is a recursive trace of g. (Incidentally, observe that being dominated by a recursive function is trivially equivalent to being captured by a recursive trace, but this does not make every hyperimmune–free degree recursively traceable because a uniform bound may not exist.) Miller and Martin's construction of hyperimmune–free degrees [8] is easily adapted to yield traceable degrees —actually, a continuum of such degrees. The proof is postponed to the end of this paper. Finally, recall that nonrecursive hyperimmune–free degrees are incomparable to $0'$, so the same holds for nonrecursive traceable degrees. The next theorem gives a "measure–theoretic" characterization of traceable degrees that is inspired by the combinatorics used in a proof of

Raisonnier [10].

THEOREM 2. A set is recursively traceable iff it is low for $\mathcal{S}$.

Proof. For the "only if" direction, let A be a recursively traceable set. Let U be a given Schnorr test recursive in A. We want to construct a Schnorr test V such that $\bigcap V \supseteq \bigcap U$. We can approximate the set U with an A-computable function that yields the finite sets $U_{n,s}$. By hypothesis, $U_{n,s}$ (that is, the function mapping $\langle n, s\rangle$ to the canonical code of the finite set $U_{n,s}$) has a recursive trace T. This T we use to enumerate V. In order to enumerate not too much measure into V_n, we have to make sure that the bulk of U_n is approximated by $U_{n,s}$ fast, that is, while T is still informative. After all, the longer we wait, the worse T gets. The following will suffice: we require $\mu U_{n,s} > 2^{-n} - 2^{-s}$. We also have to fix a bound h for T that is sharp enough.

Recapitulate. We fix a recursive trace T with bound h (for convenience this h will be specified below) and such that $U_{n,s} \in T^{[\langle n,s\rangle]}$ for all n, s. (Finite sets are identified with their code.) Now we prune T to eliminate elements that are not a candidate for $U_{n,s}$: define $\hat{T}$ as follows. Let $\hat{T}^{[\langle n,s\rangle]}$ be the set of those $D \in T^{[\langle n,s\rangle]}$ such that D is a finite subset of $2^{<\omega}$ and

$$2^{-n} - 2^{-s} \leq \mu D \leq 2^{-n} \quad \text{and} \quad C \subseteq D \text{ for some } C \in \hat{T}^{[\langle n,s-1\rangle]}$$

(for $s = 0$, $\hat{T}^{[\langle n,s-1\rangle]}$ is defined to be empty). Observe that $\hat{T}$ is still a recursive trace that captures $U_{n,s}$. Finally, define

$$V_{n,r} = \bigcup_{s<r} \hat{T}^{[\langle n,s\rangle]} \qquad \text{and} \qquad V_n = \bigcup_{r\in\omega} V_{n,r}.$$

Observe that

$$\mu V_n \leq 2^{-n} \cdot \|\hat{T}^{[\langle n,0\rangle]}\| + \sum_{s\in\omega} 2^{-s} \cdot \|\hat{T}^{[\langle n,s\rangle]}\|.$$

So, we can make μV_n recursively converge to 0, by choosing the bound h

of $\|\hat{T}^{[\langle n,s\rangle]}\|$ small enough. To see that $\mu V_{n,r}$ recursively converges to μV_n observe that

$$\mu \bigcup_{s>r} \hat{T}^{[\langle n,s\rangle]} \;<\; \sum_{s>r} 2^{-s} \cdot \|\hat{T}^{[\langle n,s\rangle]}\|.$$

Again it is just a question of choosing a bound h that is sufficiently small.

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Now we prove the "if" direction. For $k, l \in \omega$ define the clopen set

$$B_{k,l} = \left\{ \tau * 1^k \;:\; \tau \in 2^{<\omega} \;\;\&\;\; |\tau| = l \right\},$$

where 1^k is a string of 1's of length k and $*$ denotes concatenation. Note that $\mu B_{k,l} = 2^{-k}$ for all l. Now, given a function $g \leq_T A$, we define the test U^g by stipulating that

$$U_n^g \;:=\; \bigcup_{k>n} B_{k,g(k)}.$$

It is easy to see that μU_n^g can be approximated recursively in A, so U^g is a Schnorr test relative to A. By lowness of A, we can find a Schnorr test that contains $\bigcap U^g$. A fortiori we can find a recursive set $V \subseteq 2^{<\omega}$ with $V \supseteq \bigcap U^g$ and a recursive enumeration of V such that μV_s converges recursively to $\mu V < 1/4$, where V_s is the set of the elements of V enumerated before stage s. To simplify the proof below we also require that V satisfies a technical assumption. Namely that for every k and l,

$$(*) \qquad \mu(B_{k,l} - V) \;\neq\; 2^{-(l+3)}.$$

We leave to the reader to check that, if necessary, one can always enlarge V to some V' satisfying $(*)$. (Hint. If at stage s for some $\langle k, l\rangle < s$ the difference between the two numbers above appears to be "small", add to V' a fraction of $B_{k,l}$ that ensures that equality will never obtain. One can ensure that in the end $\mu(V' - V) < \varepsilon$ for an arbitrarily small ε and that $\mu V'$ can still be approximated recursively.)

The construction below is simpler if we assume that $\mu(U_n^g - V) = 0$ for some n. So, we make this provisional assumption, and we shall eliminate it later. We define a trace T for g (to be precise, a set T such that $g(k) \in T^{[k]}$ for $k > n$, so a trace of g is obtained immediately from T). First we define T and show that it is recursive. Then we show that there is a recursive upper bound on the largest element of $T^{[k]}$. This is enough to conclude that $\|T^{[k]}\|$ is a recursive function of k. Define

$$(**) \qquad T^{[k]} \ := \ \left\{ l \ : \ \mu(B_{k,l} - V) < 2^{-(l+3)} \right\}$$

By assumption $\mu(U_n^g - V) = 0$, so T traces g, with a possible exception of the first n values. It is evident that T is recursively enumerable. We show how to enumerate the complement of T. Let $s_0 = 0$ and define s_{i+1} and ε_i such that

$$\varepsilon_i \ := \ \mu(B_{k,l} - V_{s_i}) - 2^{-(l+3)} \qquad \text{and} \qquad \mu V_{s_{i+1}} > \mu V - \frac{\varepsilon_i}{2}.$$

Suppose $l \notin T^{[k]}$. Then $\varepsilon_i > 0$ for all i. It is clear that ε_i converges to a limit ε and, by the assumption $(*)$ above we have that $\varepsilon > 0$. So, $\varepsilon_i/2 < \varepsilon$ for some i. Therefore $\varepsilon_i/2 < \varepsilon_{i+1}$ for some i. So, enumerating V up to stage s_{i+1} we know for sure that $l \notin T^{[k]}$. To show that T is a recursive trace it remains to show that we can compute $\|T^{[k]}\|$. It suffices to show that we can effectively find an l_k such that $l \notin T^{[k]}$ for all $l > l_k$. Find a stage s such that $\mu V_s > \mu V - 2^{-(k+2)}$. Let l_k be larger than k and larger than the length of all strings in V_s. From the definition of $B_{k,l}$ it is clear that V_s and $B_{k,l}$ are independent for every $l > l_k$. This implies immediately that $\mu(B_{k,l} - V_s) = 2^{-k}(1 - \mu V_s) > 2^{-k}(3/4)$. Consequently, we cannot have that $\mu(B_{k,l} - V) < 2^{-(k+2)}$ and a fortiori that $\mu(B_{k,l} - V) < 2^{-(l+3)}$.

Now, note that l_k depends on the recursive enumeration of V and, indirectly, on g, so we still have to show that there is an uniform bound on $\|T^{[k]}\|$. We claim that $\|T^{[k]}\| < 2^k k$ for every k. Observe that $(**)$ above guarantees that

$$\sum_{l \in T^{[k]}} \mu(B_{k,l} - V) \quad < \quad \frac{1}{4}$$

so,

$$\mu \bigcup_{l \in T^{[k]}} B_{k,l} - \mu V \quad \leq \quad \mu \bigcup_{l \in T^{[k]}} (B_{k,l} - V) \quad \leq \quad \frac{1}{4}.$$

We obtain that

$$\mu \bigcup_{l \in T^{[k]}} B_{k,l} \leq \frac{1}{2}.$$

As observed above $\mu B_{k,l} = 2^{-k}$ and, for k fixed, the $B_{k,l}$'s are mutually independent as soon as the l's are taken sufficiently far apart. So,

$$1 - \left(1 - 2^{-k}\right)^{\frac{\|T^{[k]}\|}{k}} \quad \leq \quad \mu \left(2^\omega - \bigcap_{l \in T^{[k]}} (2^\omega - B_{k,l}) \right) \quad \leq \quad \frac{1}{2}.$$

From the inequality above we obtain $\|T^{[k]}\| \leq 2^k k$. So, as required, we have a recursive bound independent of g.

To complete the proof we show that the hypothesis that $\mu(U_n^g - V) = 0$ for some n can be weakened to: $\mu_\sigma(U_n^g - V) = 0$ for some σ and some n such that $\mu_\sigma V < 1/4$. (Recall that μ_σ is the measure conditioned to $[\sigma]$.) Then we show that this latter hypothesis is indeed true. So, suppose first that $\mu_\sigma(U_n^g - V) = 0$ and $\mu_\sigma V < 1/4$. For a set of strings W we use the notation

$$W|\sigma = \{\tau \in 2^{<\omega} : [\sigma * \tau] \subseteq W\}.$$

We may assume that $g(k) > k$ for every k because a trace for $g(k) + k$ immediately gives a trace for g. Clearly we can also assume that $n > |\sigma|$. We claim that $\mu(U_n^{\tilde{g}} - \tilde{V}) = 0$ where $\tilde{V} = V|\sigma$ and $\tilde{g}$ is the translation of g defined by $k \mapsto g(k) \dot{-} |\sigma|$. Namely, if $l > |\sigma|$ then $B_{k,l}|\sigma = B_{k,l-|\sigma|}$. Since $g(k) > k$ and $n > |\sigma|$ we have that $U_n^g|\sigma = U_n^{\tilde{g}}$, so $\mu(U_n^{\tilde{g}} - \tilde{V}) = \mu_\sigma(U_n^g - V) = 0$. This proves the claim. Now, it is clear that $\mu\tilde{V} < 1/4$ has also a recursively approximable measure. So the proof given above is valid when $\tilde{V}$ and $\tilde{g}$ are substituted for V and g and ensures the existence of a recursive trace for $\tilde{g}$. But from a trace of $\tilde{g}$ we immediately obtain a trace for g.

Now, suppose that no σ and n exist such that $\mu_\sigma(U^g_n - V) = 0$ and $\mu_\sigma V < 1/4$. We shall obtain a contradiction by constructing a real in $\bigcap U^g - V$. Let σ_0 be the empty string and assume we have defined σ_n such that $\mu_{\sigma_n} V < 1/4$. By hypothesis $\mu_{\sigma_n}(U^g_n - V) > 0$, so there is a $\tau \in U^g_n$ such that $\mu_{\sigma_n}([\tau] - V) > 0$. In particular $\tau \supseteq \sigma_n$ and $\mu_\tau V < 1$. Apply the Lebesgue density theorem to find $\sigma_{n+1} \supseteq \tau$ such that $\mu_{\sigma_{n+1}} V < 1/4$. Let R be the real that extends all σ_n's constructed in this way. Since $[\sigma_{n+1}] \subseteq U^g_n$ for all n we have that $R \in \bigcap U^g$. But $[\sigma_n] \not\subseteq V$ for every n, so, since V is open, $R \notin V$. This contradiction completes the proof of Theorem 2. Q.E.D.

The following corollary contrasts with the main result of [5] which asserts the existence of a nonrecursive recursively enumerable set that is low for $\mathcal{R}$. It is worthwhile to note that it is is unknown whether there are low sets for $\mathcal{R}$ that are not below $\emptyset'$.

COROLLARY 3. There are $2^{\aleph_0}$ many sets that are low for $\mathcal{S}$. Nonrecursive degrees that are low for $\mathcal{S}$ are incomparable with $0'$.

Proof. This follows immediately from the discussion below Fact 1, Theorem 2 above and Theorem 4 below. Q.E.D.

As promised above, we conclude this paper with a sketch of the proof of the existence of nonrecursive traceable degrees. We merely check that the construction of Miller and Martin [8] produces such degrees.

THEOREM 4. There are $2^{\aleph_0}$ recursively traceable degrees.

Proof sketch. A binary tree T is a subset of $2^{<\omega}$ in which every string has exactly two minimal extensions. We write T^τ for the full subtree of T above

τ. The set of infinite branches of T is denoted by $[T]$. We construct a chain of trees $T_0 \supseteq \ldots \supseteq T_s \supseteq \ldots$ such that the set $\bigcap_s [T_s]$ is perfect and contains only traceable sets with bound $h(k) = 2^k k$.

Let $T_0 = 2^{<\omega}$. For every minimal string ν of T_{2e}, if for some $\tau \supseteq \nu$ all branches B of T_{2e}^{τ} are such that $\{e\}^B$ is not total, replace T_{2e}^{ν} with T_{2e}^{τ} in T_{2e}. Otherwise, replace T_{2e}^{ν} with the tree S defined by the following recursive procedure: let $S_0 = \{\nu\}$, then, for each maximal string in S_k enumerate in S_{k+1} two incomparable extensions $\sigma_0, \sigma_1 \in T_{2e}$ such that $\{e\}^{\sigma_i}(k)\downarrow$ for both $i = 0, 1$. Finally, to make $\bigcap_s [T_s]$ perfect, construct T_{2e+2} from T_{2e+1} by erasing from T_{2e+1} all minimal elements.

We check that all branches in $\bigcap_s [T_s]$ are recursively traceable. Let $\nu_1, \ldots, \nu_i$ be those minimal elements of T_{2e+1} for which $\{e\}^B$ is total for all branches B that extend them. By construction, $\{e\}^B(k)$ attains at most 2^{k+e} different values as B is one of the branches above (note there are 2^e minimal elements in T_{2e+1}). Branches that do not extend one of $\nu_1, \ldots, \nu_i$, make $\{e\}^B$ nontotal and may be considered "irrelevant". Each T_{2e+1} is a recursive set, so all relevant values of $\{e\}^B(k)$ are computable. Q.E.D.

We note that the traceable degrees do not coincide with the hyperimmune–free degrees. This can be shown using a chain of finitely branching trees in a fashion as in the proof above.

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