Computability and Measure

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Voor Laetitia

Contents

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Most of this thesis reports on joint work, some of which is published and some of which is not. In particular, Chapter 2 is joint work with Klaus Ambos-Spies, Hans-Christian Neis, and Xizhong Zheng and comprises the papers [8] and [9]. Chapter 3 is joint work with Ambos-Spies done in the spring of 1997. Chapter 4 is joint work [75] with Leen Torenvliet from 1996. The first three sections of Chapter 5 consist of joint work [39] with Antonín Kučera done in April 1997 in Prague. The sections 5.4 and 5.5 of Chapter 5 report on joint work [76] with Domenico Zambella from the spring and summer of 1997. Most of the work in Chapter 6 was done jointly with Jack Lutz between August and December 1995. Proposition 6.1.6 was proved with Leen Torenvliet in January 1997.

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Chapter 1

Introduction

In this thesis we discuss some questions from recursion theory relating to measure and randomness. In this chapter we introduce and discuss some material that is needed in the subsequent chapters. Section 1.6 contains a brief description of the contents of the thesis.

1.1 Computability

The origins of recursion theory, or computability theory, can be traced to the eighteenth century, or even further back, but the subject really took off during the 1930's through the work of Gödel, Church, Kleene, Post, and Turing. They gave various precise definitions of the intuitive notion of *algorithmically computable function*. They proved that all of these definitions were in fact equivalent, i.e., that they defined the same class of functions, thus providing a firm basis for a mathematical theory of computability: recursive function theory. The most important formalizations include the class of *recursive* functions (Kleene [35]) and that of functions computable by a Turing machine (Turing [77]). The notion of recursive function is now one of the cornerstones of the area of mathematical logic. The next era of recursion theory started with the seminal paper by Post [68]. Here numerous notions which play an important role in modern recursion theory were introduced. In particular it stated the problem which became known as Posts problem: are there any r.e. Turing-degrees that are different from 0 (the degree of the recursive sets) and $0'$ (the degree of the halting problem)? The positive answer, independently found by Friedberg and A. A. Muchnik entailed the discovery of a diagonalization technique, the finite injury priority method, that would prove to be of great importance in the subsequent development of the subject.

With the growth of computer science came an interest in developing a mathematical theory of functions computable not only in theory but also in practice. Thus from the 1960s onward the theory of *feasible* computation developed: complexity theory. Here the classes P of sets computable deterministically in polynomial time and NP of sets nondeterministically computable in polynomial time were defined. Today it is still unknown whether these classes are equal or not. The setting for Chapter 2 will be complexity-theoretic.

Of the many applications of recursion theory we mention here only one: the application to the old problem of formalizing the notion of randomness. Although measure theory may serve as a foundation for probability theory, it does not answer the question of what a random *individual* object is. Indeed, in the context of Lebesgue measure, no individual object is different from another since every single real has measure zero. Still, we feel that the outcome of one hundred tails when we flip a coin one hundred times is very special, even if the probability for this outcome is the same as that of any other outcome. The viewpoint that every attempt of formalizing the notion of randomness must fail has been expressed by several mathematicians in this century. In general, of course, no absolute notion of randomness exists. For example, R. von Mises [63] started a line of research in which an infinite binary sequence A is called 'random' (a 'Kollektiv') with respect to a class of functions if for every subsequence B of A specified by one of the functions from the class it holds that in the limit B has the same number of zeros and ones. If the class of selection functions is countable this notion is nontrivial. Although von Mises' notion is flawed in a sense (Ville [78]), it points the way by having in its definition a class of functions as a parameter that specifies how random random objects should be. What class of functions should we choose? Recursion theory provides us with a canonical answer: The class of *recursive* functions is a natural choice (Church). After all, if a mathematical object has an internal regularity of a kind that can not be described in an effective way then it will seem random when any kind of computable analysis is applied to it. The Church-Turing thesis thus provides an argument for this choice. In 1966, Martin-Löf [59] proposed a definition of randomness, measure-theoretic in nature, that did not suffer from the same defects as von Mises' concept. This concept would prove to be important in various ways. We will say more about it in Chapters 4 and 5. In Section 1.5 we introduce random sets in the context of effective measure theory.

We now review some of the basic definitions from recursion theory. For the larger part our notation follows Soare [72]. The set of natural numbers is denoted by ω, the rational numbers by $\mathbb{Q}$, and the real numbers by $\mathbb{R}$. We use the notation $\mathbb{Q}^+$ and $\mathbb{R}^+$ for the sets of positive rationals and reals, respectively. $\exists^\infty x$ and $\forall^\infty x$ denote "there are infinitely many x" and "for all but finitely many x", respectively. We will work mostly in Cantor space, the class of subsets of ω, denoted by 2^ω. An element of 2^ω is called a *set* or a *real*. As this notation suggests, we identify a set A with its characteristic function $\chi_A : \omega \to \{0,1\}$. That is, $A(x) = 1$ if $x \in A$ and $A(x) = 0$ if $x \notin A$. We fix a recursive pairing function $\langle .,. \rangle : \omega \times \omega \to \omega$ and define the n-th section of the set A by

$$A^{[n]} = \{\langle x, n\rangle : \langle x, n\rangle \in A\}.$$

For $A \in 2^\omega$, $\overline{A}$ denotes the complement, $\omega - A$, of A; $A \triangle B$ is the symmetric difference of the sets A and B. The *join* of A and B is defined by

$$A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}.$$

The empty set is denoted by $\emptyset$, and the cardinality of A is denoted by $\|A\|$.

The set of finite strings of zeros and ones is denoted by $2^{<\omega}$. The empty string in $2^{<\omega}$ is denoted by λ. For elements $v, w \in 2^{<\omega}$, $|w|$ is the length of w, vw or $v\hat{\ }w$ denotes the concatenation of v and w, and w^n is the n-th iteration of w. The $(i+1)$-th bit of the string v is denoted by $v(i)$, so $v = v(0)v(1)\dots v(|v|-1)$. The notations $v \sqsubset w$, $v \sqsubseteq w$ denote that v is a (proper) prefix of w. Also, $w \sqsubset A$ denotes that w is an initial segment of (the characteristic sequence of) the set A. wA (or $w\hat{\ }A$) is the concatenation of the finite string w with the infinite binary sequence A. $A\restriction n = A(0)A(1)\dots A(n-1)$ is the initial segment of length n of A. For a function $f : \omega \to \omega$, $f\restriction n$ denotes the string $f(0)f(1)\dots f(n-1)$, and if f is partial then $(f\restriction n)\downarrow$ denotes that this string is defined. The range of f is denoted by $\mathrm{rng}(f)$. For a string $w \in 2^{<\omega}$ we have the basic open set

$$C_w = \{A \in 2^\omega : w \sqsubset A\}.$$

1.1.1. Definition. A set $X \subset 2^{<\omega}$ is called *prefix-free* if any two elements of X are incomparable, i.e. if for every σ and τ in X, $\tau \not\sqsubset \sigma$.

Subsets of 2^ω are called *classes*, and for a class $\mathcal{A}$ we have the dual class co-$\mathcal{A} = \{\overline{A} : A \in \mathcal{A}\}$ and the complement $\mathcal{A}^c = \{A : A \notin \mathcal{A}\}$ of $\mathcal{A}$.

We fix a standard numbering $\{\varphi_e\}_{e\in\omega}$ of the partial recursive functions. As an alternative notation for φ_e we sometimes use $\{e\}$. By $\varphi_{e,s}(x) = y$ we denote that the e-th Turing program computing on x halts within s computation steps and outputs y. We write $\varphi_{e,s}(x)\downarrow$ if such a y exists and $\varphi_e(x)\downarrow$ if such s and y exist. Similarly, we write $\varphi_{e,s}(x)\uparrow$ if the computation $\varphi_e(x)$ does not halt within s steps, and we write $\varphi_e(x)\uparrow$ if $\varphi_{e,s}(x)\uparrow$ for every s. A *recursive* function is a partial recursive function that halts on every input. The e-th recursively enumerable (r.e.) set is defined by $W_e = \mathrm{dom}(\varphi_e)$. We fix a canonical coding of the finite sets and denote the n-th finite set by D_n; the number n is called the *canonical code* of the finite set D_n. The notions of (partial) recursiveness and recursive enumerability *relativize* to an arbitrary set A by giving the Turing programs access to the information coded in A. If the bit $A(x)$ is used in the computation $\{e\}^A(y)$ we say that x is *queried*. The *use* $u(A; e, x)$ of a computation $\{e\}^A(x)$ is defined to be 1 + the maximal number used in the computation (including all the queries) if it halts, and ∞ otherwise. Similarly for $\{e\}_s^A(x)$ and the corresponding use function $u(A;e,x,s)$. The e-th function partial recursive in A is denoted by φ_e^A or $\{e\}^A$. The *halting problem* is the set

$$K = \{e : \{e\}(e)\downarrow\}.$$

The *jump* A' of A is K^A, the halting problem relative to A. We sometimes use the alternative notation $\emptyset'$ for K. The the n-th iterated jump of A is denoted by $A^{(n)}$.

We will make use of the following recursive reducibilities (A and B are arbitrary sets of natural numbers):

- One-one reducibility: $A \leq_1 B$ if there is a recursive one-one function f such that $x \in A \Leftrightarrow f(x) \in B$.
- Many-one reducibility: $A \leq_m B$ if there is a recursive function f such that $x \in A \Leftrightarrow f(x) \in B$.
- Truth-table reducibility: Let X be a set variable. A *truth-table condition* is a propositional formula built from atomic formulas of the form $n \in X$, for $n \in \omega$. Clearly such formulas can be coded by natural numbers in an effective way. Now $A \leq_{tt} B$ if there is a recursive function f such that $x \in A \Leftrightarrow B$ satisfies truth-table condition $f(x)$.
- Bounded truth-table reducibility: A truth-table reduction f is called a *bounded* truth-table reduction if there is a number m, called the *norm* of the reduction, such that for every x the truth-table condition $f(x)$ contains at most m numbers. $A \leq_{btt} B$ denotes that A reduces to B via a bounded truth-table reduction.
- Weak truth-table reducibility: $A \leq_{wtt} B$ if there is code e and a recursive function f such that $A = \{e\}^B$ and the use function $u(B;e,x)$ is bounded by $f(x)$.
- Turing reducibility: $A \leq_T B$ if there is a code e such that $A = \{e\}^B$.

For a reducibility $r \in \{1, m, btt, tt, wtt, T\}$, $A|_r B$ denotes that the sets A and B are r-incomparable, i.e. that $A \not\leq_r B$ and $B \not\leq_r A$. The *r-upper cone* and *r-lower cone* of A are defined by

$$\begin{aligned} A^{\leq_r} &= \{B : A \leq_r B\} \\ {}^{\leq_r}A &= \{B : B \leq_r A\}. \end{aligned}$$

The *r-degree* of A is defined $\deg_r(A) = {}^{\leq_r}A \cap A^{\leq_r}$.

As usual, the classes of the arithmetical hierarchy are denoted by Σ_n, Π_n, and Δ_n. In particular, Δ_2 is the class of sets computable in the halting problem K, Σ_1 is the class of r.e. sets, also denoted by RE, and Δ_1 is the class of recursive sets, also denoted by REC.

In Chapter 2 we use the length-lexicographic ordering $<$ of the set of binary words Σ^*. The n-th word in this ordering is denoted by z_n. In this ordering, $x < y$ if y is longer than x, or if $|y| = |x|$ and x comes lexicographically before y. A *time bound* t is a recursive, time-constructible function satisfying $t(n) \geq n$ for almost every n. For a time bound t we have the class DTIME(t) of sets computable in time $O(t)$, i.e. sets computable by a Turing machine for which the running time on input z_n is bounded by $f(|z_n|)$ for a function $f \in O(t)$. We will use the complexity classes

$$
\begin{aligned}
\mathrm{P} &= \bigcup_{c \geq 1} \mathrm{DTIME}(n^c) \\
\mathrm{E} &= \bigcup_{c \geq 1} \mathrm{DTIME}(2^{cn}) \\
\mathrm{E}_2 &= \bigcup_{c \geq 1} \mathrm{DTIME}(2^{n^c})
\end{aligned}
$$

We will use the following polynomial reducibilities (A and B are arbitrary sets of words)

- Polynomial many-one reducibility: $A \leq_m^p B$ if there is a function $f \in \mathrm{P}$ such that $x \in A \Leftrightarrow f(x) \in B$.

- Polynomial bounded truth-table reducibility: $A \leq_{btt}^p B$ is defined as btt-reducibility above, with f polynomial time computable.

- Polynomial bounded truth-table reducibility of norm k: $A \leq_{k\text{-}tt}^p B$ if $A \leq_{btt}^p B$ by a reduction of norm k. (For the definition of norm see the definition of btt-reducibility above.)

- Polynomial truth-table reducibility: $A \leq_{tt}^p B$ defined as tt-reducibility above, with f polynomial time computable.

- Polynomial Turing reducibility: $A \leq_T^p B$ if there is a code e such that $A = M_e^B$, where M_e is the e-th polynomial time machine.

1.2 Measure

We give the definition of Lebesgue measure on 2^ω. Although 2^ω and the real unit interval $\mathbb{I} = [0,1]$ are not homeomorphic, 2^ω is isomorphic to $\mathbb{I}$ in a measure-theoretic sense. Cantor space has as basic opens the sets $C_w = \{A : w \sqsubset A\}$, $w \in 2^{<\omega}$. Every C_w is given a measure $\mu(C_w) = 2^{-|w|}$. The *Borel* sets of 2^ω are obtained by closing the class of basic open sets under the operations of complementation and countable union. The definition of μ uniquely extends to a *measure* on the Borel sets, that is, a function on the Borel sets taking values in $[0,\infty]$ that is countably additive (or σ*-additive*): for every disjoint sequence of Borel sets $\{\mathcal{B}_n\}$ we have $\mu(\bigcup \mathcal{B}_n) = \sum \mu(\mathcal{B}_n)$. This measure μ on the Borel sets in turn extends to a measure defined on the class of sets of the form $\mathcal{B} \triangle \mathcal{N}$, where $\mathcal{B}$ is Borel and $\mathcal{N}$ is a subset of a Borel set of measure zero. This is the class of *Lebesgue measurable* sets (often shortened to 'measurable sets'), and the measure μ defined on it by $\mu(\mathcal{B} \triangle \mathcal{N}) = \mu(\mathcal{B})$ is called the *Lebesgue measure.* The Lebesgue measure is *complete*, i.e. if $\mathcal{X}$ is a measurable set, $\mu(\mathcal{X}) = 0$, and $\mathcal{Y} \subseteq \mathcal{X}$ then $\mathcal{Y}$ is also measurable and $\mu(\mathcal{Y}) = 0$. It has the following property: For every measurable class $\mathcal{X}$,

$$\mu(\mathcal{X}) = \inf\{\mu(\mathcal{U}) : \mathcal{X} \subseteq \mathcal{U} \wedge \mathcal{U} \text{ open}\}. \tag{1.1}$$

In the next section a different definition of the notion of measure zero will be given that will be useful for our further studies.

For a measurable class $\mathcal{A}$, instead of $\mu(\mathcal{A})$ we sometimes write $\Pr(\mathcal{A})$. For two measurable classes $\mathcal{A}$ and $\mathcal{B}$ with $\Pr(\mathcal{B}) \neq 0$, we define the *conditional probability of $\mathcal{A}$ given $\mathcal{B}$* by

$$\Pr(\mathcal{A}|\mathcal{B}) = \frac{\Pr(\mathcal{A} \cap \mathcal{B})}{\Pr(\mathcal{B})}.$$

For a set $X \subseteq 2^{<\omega}$ of initial segments we simply write $\mu(X)$ for $\mu(\bigcup_{x\in X} C_x)$. Note that if X is prefix-free then $\mu(X) = \sum_{x\in X} 2^{-|x|}$.

1.3 Martingales

Martingales were invented by P. Levy, and first applied to the study of random sequences by J. Ville [78]. Later J. L. Doob used martingales in the study of stochastic processes, making the concept of martingale well-known. In the following, rather than giving the most general definition of martingales, we immediately give the special case of the definition that will be used throughout this thesis.

1.3.1. Definition. A function $d : 2^{<\omega} \to \mathbb{R}^+$ is a *martingale* if for every $w \in 2^{<\omega}$, d satisfies the averaging condition

$$2d(w) = d(w0) + d(w1) \tag{1.2}$$

Similarly, d is a *supermartingale* if d satisfies

$$2d(w) \geq d(w0) + d(w1) \tag{1.3}$$

and d is a *submartingale* if d satisfies

$$2d(w) \leq d(w0) + d(w1) \tag{1.4}$$

1.3.2. Definition. A (super)martingale d *succeeds on* a set A if

$$\limsup_{n\to\infty} d(A{\upharpoonright}n) = \infty.$$

We say that d succeeds on a class $\mathcal{A} \subseteq 2^\omega$ if d succeeds on every $A \in \mathcal{A}$. The class of all sets on which d succeeds is denoted by $S[d]$.

From (1.3) one immediately sees that for a (super)martingale d and every $v \sqsubseteq w$ it holds that $d(w) \leq 2^{|w|-|v|}d(v)$. Similarly, we have

1.3.3. Lemma. *Let d be a (super)martingale. For any string v and any prefix-free set $X \subseteq \{x : v \sqsubseteq x\}$ it holds that $2^{-|v|}d(v) \geq \sum_{x\in X} 2^{-|x|}d(x)$.*

Proof. It suffices to prove this for finite X (Bolzano-Weierstrass). Use induction on the cardinality of X. The base step $\|X\| = 1$ is immediate from (1.3). Suppose the lemma holds for all X of cardinality n. Let X be prefix-free and of cardinality $n+1$. Choose w of maximal length such that $X \subseteq \{x : w \sqsubseteq x\}$. Then both $X_0 = \{x \in X : w0 \sqsubseteq x\}$ and $X_1 = \{x \in X : w1 \sqsubseteq x\}$ have cardinality less than or equal to n. It follows by induction hypothesis that

$$\begin{aligned}\sum_{x\in X} 2^{|w|-|x|}d(x) &= \frac{1}{2}\sum_{x\in X_0} 2^{|w0|-|x|}d(x) + \frac{1}{2}\sum_{x\in X_1} 2^{|w1|-|x|}d(x)\\ &\leq \frac{1}{2}(d(w0)+d(w1))\\ &\leq d(w).\end{aligned}$$

Since any v with $X \subseteq \{x : v \sqsubseteq x\}$ satisfies $v \sqsubseteq w$ and by (1.3) it holds that $d(w) \leq 2^{|w|-|v|}d(v)$ the lemma follows by multiplying the above equations with $2^{-|w|}$. $\square$

The following result is sometimes called "Kolmogorov's inequality for martingales".

1.3.4. LEMMA. (Ville [78]) *Let d be a (super)martingale and define $S^k[d] = \{x \in 2^{<\omega} : d(x) \geq k\}$. Then $\mu(S^k[d]) \leq d(\lambda)k^{-1}$.*

PROOF. Let $X \subseteq S^k[d]$ be prefix-free such that $\mu(X) = \mu(S^k[d])$. By Lemma 1.3.3 we have

$$k \cdot \mu(X) = k \sum_{x \in X} 2^{-|x|} \leq \sum_{x \in X} 2^{-|x|} d(x) \leq d(\lambda).$$

□

1.3.5. DEFINITION. (Lutz [50]) A *density system* is a function $d : \omega \times 2^{<\omega} \to \mathbb{R}^+$ such that for every $k \in \omega$ the function $d_k(w) = d(k, w)$ is a martingale and $d_k(\lambda) \leq 2^{-k}$. The set *covered* by d_k is

$$S^1[d_k] = \{A \in 2^\omega : (\exists w \sqsubset A)[d_k(w) \geq 1]\}.$$

We say that a class $\mathcal{A}$ is *covered by d* if $\mathcal{A} \subseteq \bigcap_{k \in \omega} S^1[d_k]$.

1.3.6. THEOREM. (Ville [78]) *For any class $\mathcal{A} \subseteq 2^\omega$ the following statements are equivalent:*

(i) *$\mathcal{A}$ has Lebesgue measure zero,*

(ii) *There exists a density system that covers $\mathcal{A}$,*

(iii) *There exists a martingale that succeeds on $\mathcal{A}$,*

(iv) *There exists a supermartingale that succeeds on $\mathcal{A}$.*

PROOF. (i)⇒(ii). Suppose $\mu(\mathcal{A}) = 0$. By (1.1) there are open sets $\mathcal{U}_k \subseteq 2^\omega$ such that $\mathcal{A} \subseteq \bigcap_k \mathcal{U}_k$ and $\mu(\mathcal{U}_k) \leq 2^{-k}$. Define the martingales d_k by

$$d_k(w) = \Pr(\mathcal{U}_k | C_w).$$

Then $d_k(w) = 1$ if $C_w \subseteq \mathcal{U}_k$, so $\mathcal{U}_k \subseteq S^1[d_k]$. Also, $d_k(\lambda) = \Pr(\mathcal{U}_k) \leq 2^{-k}$, and $\mathcal{A} \subseteq \bigcap_k \mathcal{U}_k \subseteq \bigcap_k S^1[d_k]$, so the d_k form a density system that covers $\mathcal{A}$.
(ii)⇒(iii). Let $\{d_k\}_{k \in \omega}$ be a density system. Define

$$d(w) = \sum_{k=0}^{\infty} 2^k d_{2^k}(w).$$

Then $d(w) \leq \sum_k 2^{k+|w|-2^k} < \infty$ and it is easy to check that d is a martingale. Furthermore, if $d_{2^k}(w) \geq 1$ then $d(w) \geq 2^k d_{2^k}(w) \geq 2^k$. So if $\mathcal{A} \subseteq \bigcap_{k \in \omega} S^1[d_k]$ then d succeeds on $\mathcal{A}$.
(iii)⇒(iv). Immediate.
(iv)⇒(i). Suppose that the supermartingale d succeeds on $\mathcal{A}$. Then the sets

$S^k[d] = \{x \in 2^{<\omega} : d(x) \geq k\}$ determine open sets that have measure smaller than $d(\lambda)k^{-1}$ by Lemma 1.3.4, so $\mathcal{A}$ has measure zero by (1.1). $\square$

The next result shows that if we linearly transform the betting percentages $d(wi)/d(w)$ of a martingale d then the resulting martingale covers the same sets as d.

1.3.7. PROPOSITION. *Let d be any martingale and let $k \in \omega$. Let d' be the martingale defined by $d'(\lambda) = d(\lambda)$ and*

$$d'(wi) = \frac{\frac{d(wi)}{d(w)} + k - 1}{k} \cdot d'(w).$$

Then $S[d'] = S[d]$.

PROOF. From the inequality $1 + x \leq e^x$ follows

LEMMA. *Let $\{\alpha_n\}_{n\in\omega}$ be a sequence of reals. Then*

$$\limsup_{m\in\omega} \prod_{n=0}^{m} \alpha_n = \infty \Leftrightarrow \limsup_{m\in\omega} \sum_{n=0}^{m} (\alpha_n - 1) = \infty.$$

Now suppose that $A \in S[d]$. Let $\alpha_i = d(A{\restriction}(i+1))/d(A{\restriction}i)$. Then $\limsup_m \prod_{i=0}^m \alpha_i = \infty$, so by the lemma $\limsup_m \sum_{i=0}^m (\alpha_i - 1) = \infty$. But then also $\limsup_m \sum_{i=0}^m \frac{\alpha_i - 1}{k} = \infty$, hence, again by the lemma, $\limsup_m \prod_{i=0}^m \frac{\alpha_i + k - 1}{k}$ $= \infty$, which means that A is in $S[d']$. Since the reverse implications in this argument hold too it follows that $S[d'] = S[d]$. $\square$

The previous result immediately yields the following

1.3.8. COROLLARY. (cf. Schnorr [73, Satz 13.4]) *Let $\varepsilon > 0$ be any real number. For every martingale d there exists a martingale d' of the same complexity as d (modulo a linear factor) with $S[d'] = S[d]$ such that for every $w \in 2^{<\omega}$ and $i \in \{0, 1\}$ it holds that $d'(wi)/d'(w) \in [1 - \varepsilon, 1 + \varepsilon]$.*

1.4 SOME ELEMENTARY THEOREMS

A measurable set $\mathcal{A} \subseteq 2^\omega$ has density d at X if $\lim_n \mu(\mathcal{A} \cap C_{X{\restriction}n})2^n = d$. Define $\phi(\mathcal{A}) = \{X \in 2^\omega : \mathcal{A}$ has density 1 at $X\}$. Note that $\mathcal{A}$ has density 0 at each point of $\phi(\mathcal{A}^c)$.

We now prove a classical theorem of Lebesgue. The proof is essentially the proof given in Oxtoby [67, p17]. The proof below is somewhat simpler because the basic open sets C_x have a more specific form than an arbitrary real interval.

1.4.1. THEOREM. (Lebesgue Density Theorem) *If $\mathcal{A}$ is measurable then so is $\phi(\mathcal{A})$, and $\mu(\mathcal{A} \triangle \phi(\mathcal{A})) = 0$.*

PROOF. It suffices to show that $\mathcal{A}-\phi(\mathcal{A})$ is a null set since $\phi(\mathcal{A})-\mathcal{A} \subseteq \mathcal{A}^c-\phi(\mathcal{A}^c)$ and $\mathcal{A}^c$ is measurable. Define for every positive rational ε

$$\mathcal{B}_\varepsilon = \{X \in \mathcal{A} : \liminf_{n\to\infty} \mu(\mathcal{A} \cap C_{X\restriction n})2^n < 1-\varepsilon\}.$$

Then $\mathcal{A}-\phi(\mathcal{A}) = \bigcup_\varepsilon \mathcal{B}_\varepsilon$, hence it suffices to prove that every $\mathcal{B}_\varepsilon$ is a null set. Suppose for a contradiction that for $\mathcal{B} = \mathcal{B}_\varepsilon$ we have that the outer measure $\mu^*(\mathcal{B}) := \inf\{\mu(\mathcal{U}) : \mathcal{B} \subseteq \mathcal{U} \wedge \mathcal{U} \text{ open}\} > 0$. Then there exists $\mathcal{G} \supseteq \mathcal{B}$ open with $\mu(\mathcal{G})(1-\varepsilon) < \mu^*(\mathcal{B})$. Define

$$I = \{x \in 2^{<\omega} : C_x \subseteq \mathcal{G} \wedge \mu(\mathcal{A} \cap C_x) < (1-\varepsilon)2^{-|x|}\}.$$

Then

(i) for any $X \in \mathcal{B}$, I contains $X\restriction n$ for infinitely many n, and

(ii) if $\{x_i\}_{i\in\omega}$ is a sequence of elements of I such that $C_{x_i} \cap C_{x_j} = \emptyset$ for $i \neq j$, then $\mu^*(\mathcal{B} - \bigcup_i C_{x_i}) > 0$.

The first statement holds since $\mathcal{G}$ is open and the second statement holds because $\mu^*(\mathcal{B} \cap \bigcup_i C_{x_i}) \leq \sum_i \mu(\mathcal{A} \cap C_{x_i}) < \sum_i (1-\varepsilon)2^{-|x_i|} \leq (1-\varepsilon)\mu(\mathcal{G}) < \mu^*(\mathcal{B})$. Now construct a sequence $\{x_i\}_{i\in\omega}$ as follows. Let x_1 in I be arbitrary, and if x_i, $i \leq n$ are defined let $I_n = \{x \in I : C_x \text{ disjoint with } C_{x_i}, i \leq n\}$. I_n is infinite by (i) and (ii). Define $x_{n+1} \in I_n$ such that $2^{-|x_{n+1}|} > d_n/2$, where $d_n = \sup\{2^{-|x|} : x \in I_n\}$. Let $X \in \mathcal{B} - \bigcup_i C_{x_i}$. X exists by (ii). By (i), let $x \in I$ be such that $X \in C_x$. Let k be the smallest number with $C_x \cap C_{x_k} \neq \emptyset$. Note that k exists since otherwise C_x is disjoint from every C_{x_i} and hence $2^{-|x|} \leq d_i < 2 \cdot 2^{-|x_i|}$ for every i, contradicting that $\mu(\bigcup_i C_{x_i}) \leq 1$. For k it holds that $2^{-|x|} \leq d_{k-1} < 2 \cdot 2^{-|x_k|}$, whence that $|x| \geq |x_k|$, and thus $C_x \subseteq C_{x_k}$ because $C_x \cap C_{x_k} \neq \emptyset$. But this contradicts $X \notin \bigcup_i C_{x_i}$. □

Next we prove a classical theorem from computability theory: Sacks' theorem on the measure of upper cones.

1.4.2. THEOREM. (Sacks [71]) *For every nonrecursive set $A \in 2^\omega$ the upper cone*

$$A^{\leq_T} = \{B : A \leq_T B\}$$

has measure zero.

PROOF. Let A be an element of 2^ω such that $\mu(\{B : A \leq_T B\}) \neq 0$. Note that this set is measurable hence must have positive measure. We will show that A is recursive. For two classes $\mathcal{C}$ and $\mathcal{D}$ define $\mu(\mathcal{C}|\mathcal{D}) = \mu(\mathcal{C} \cap \mathcal{D})/\mu(\mathcal{D})$ and for a formula P write $\mu(P(B))$ for $\mu(\{B \in 2^\omega : P(B)\})$. Since $\{B : A \leq_T B\} = \bigcup_e \{B : A = \{e\}^B\}$ has positive measure there exists $e \in \omega$ such that $\mu(\{B : A = \{e\}^B\}) > 0$. It follows from Theorem 1.4.1 that there is a point $X \in 2^\omega$ such that $\mathcal{A}$ has density 1 at X. From this follows the existence of a $\sigma \in 2^{<\omega}$ such that $\mu(A = \{e\}^B | C_\sigma) \geq 3/4$. Using σ we can compute A as follows. For any $x \in \omega$ the sets $T_n = \{\tau \sqsupseteq \sigma : \{e\}^\tau(x) \downarrow = n\}$ are uniformly r.e., so we can enumerate $T_n = \bigcup_s T_{n,s}$ until we find n with $\sum_{\tau \in T_{n,s}} \mu(\tau) \geq 3/4$. Then $A(x) = n$. □

Next we prove Kolmogorov's 0-1 law for measurable sets. As Sacks' theorem above, it can be proved directly, but it also follows very quickly from Lebesgue's density theorem.

1.4.3. Definition. $\mathrm{E} \subseteq 2^\omega$ is a *tailset* if $\mathcal{A}$ is closed under finite variances, i.e., if $v \in 2^{<\omega}$ and $X \in 2^\omega$ are such that $vX \in \mathrm{E}$ then $wX \in \mathrm{E}$ for every string w of length $|v|$.

1.4.4. Theorem. (Kolmogorov's 0-1 law) *If $\mathcal{A} \subseteq 2^\omega$ is a measurable tailset then either $\mu(\mathcal{A}) = 0$ or $\mu(\mathcal{A}) = 1$.*

Proof. Suppose $\mu(\mathcal{A}) > 0$. By Theorem 1.4.1, choose $X \in \mathcal{A}$ such that $\mathcal{A}$ has density 1 at X. Let $\varepsilon \in (0,1)$ be arbitrary. Choose n large enough such that $\mu(\mathcal{A} \cap C_{X\restriction n})2^n > 1 - \varepsilon$. Because $\mathcal{A}$ is a tailset we then have that $\mu(\mathcal{A} \cap C_w)2^n > 1 - \varepsilon$ for *any* w of length n. So $\mu(\mathcal{A}) > 1 - \varepsilon$. Since ε was arbitrary it follows that $\mu(\mathcal{A}) = 1$. □

1.5 Resource bounded measure theory

Classical measure theory, as introduced above, allows one to make mathematical assertions of a quantitative nature, like 'almost every infinite binary sequence has the same limiting frequency of zeros and ones'. However, Lebesgue measure is too coarse to make such assertions about *countable* classes, since by σ-additivity every countable class has measure zero. Still, it is possible to use the concepts from measure theory to give meaning to statements like 'almost every *recursive* binary sequence has the same limiting frequency of zeros and ones'. The key to this is to use more constructive versions of the classical measure, and require for example only that *constructive* infinite unions of small classes are small again, rather than arbitrary countable unions. It may then happen that a countable class is not small in a constructive sense, and that the use of constructive measure *inside* this class gives the desired definitions, provided that the class is natural enough. From the 1960's onward, this approach has been taken by several authors (including Schnorr [73], Freidzon [23], Mehlhorn [61], Lisagor [48], Lutz [49, 50]), making use of various constructive measures. Constructive measure theory had been developed before in the broader context of constructive mathematics within the various schools of constructivism. Both Martin-Löf [60] and Schnorr [73] used ideas going back to Brouwer. In recent years the approach of Schnorr and Lutz in particular has become popular in complexity theory. Below we present Lutz's framework, usually referred to as *resource bounded measure theory*, in which the above ideas are incorporated. (Recent survey articles of this area are [6] and [53].) Theorem 1.3.6 gives a definition of the notion of 'measure zero' that is particularly easy to make constructive. Also, it has the advantage that the level of constructiveness, or the 'resources' used in the theory, is built in as a parameter in the definition in an elegant way and can be varied over a large number of classes.

1.5.1. DEFINITION. (Schnorr [73], Lutz [49, 50]) Let Δ be a class of martingales. For a class $\mathcal{A} \subseteq 2^\omega$, we say that $\mathcal{A}$ has Δ-*measure zero* ($\mu_\Delta(\mathcal{A}) = 0$) if there is a martingale in Δ that succeeds on $\mathcal{A}$. $\mathcal{A}$ has Δ-*measure one* ($\mu_\Delta(\mathcal{A}) = 1$) if $\mu_\Delta(\mathcal{A}^c) = 0$.

Since martingales are real-valued functions, and we want to use them in a constructive context, we need a notion of computability for them. This is provided by the next definition.

1.5.2. DEFINITION. Let $d : 2^{<\omega} \to \mathbb{R}^+$ be a martingale, and let Δ be a class of functions. A *computation* of d is a function $\hat{d} : \omega \times 2^{<\omega} \to \mathbb{Q}^+$ satisfying $|d(w) - \hat{d}(k, w)| \leq 2^{-k}$ for every k and w. If $\hat{d}$ is in Δ, we say that d is a Δ-*martingale* and that the function $\hat{d}$ is a Δ-*computation* of d.

Define the function classes

$$
\begin{array}{rcl}
\mathrm{all} & = & \{f : 2^{<\omega} \to \mathbb{Q}^+ : f \text{ is a martingale}\} \\
\mathrm{rec} & = & \{f \in \mathrm{all} : f \text{ has a recursive computation}\} \\
\Delta_2 & = & \{f \in \mathrm{all} : f \text{ has a computation recursive in } \emptyset'\} \\
\mathrm{p} & = & \{f \in \mathrm{all} : f \text{ has a polynomial time-computation}\} \\
\mathrm{p}_2 & = & \{f \in \mathrm{all} : (\exists k)[f \text{ has a computation computable in time } O(2^{(\log n)^k})]\}
\end{array}
$$

Note that μ_{all} is just μ, the Lebesgue measure. For the function classes above we will use the corresponding measures μ_{rec}, μ_{p}, μ_{p_2}, μ_{Δ_2}. In addition, in Chapter 4 we will use the measures μ_{Σ_n}, μ_{Π_n}, μ_{Δ_n} corresponding to the levels of the arithmetical hierarchy. However, not everything that is proved below for the former holds for the latter, so we will treat these separately.

1.5.3. DEFINITION. A *constructor* is a function $\delta : 2^{<\omega} \to 2^{<\omega}$ with the property that $\delta(x) \sqsupset x$ for every $x \in 2^{<\omega}$. The *set constructed by* δ is the unique set $R(\delta)$ with $\delta^n(\lambda) \sqsubset R(\delta)$, where δ^n denotes the n-th iterate of δ. For a class of functions Δ, $R(\Delta)$ denotes the class

$$\{R(\delta) : \ \delta \text{ is a constructor and } \delta \in \Delta\}.$$

The following observations by Lutz are easily made:

$$\begin{aligned} R(\text{all}) &= 2^\omega \\ R(\text{rec}) &= \text{REC} \\ R(\Delta_2) &= \Delta_2 \\ R(\text{p}) &= \text{E} \\ R(\text{p}_2) &= \text{E}_2 \end{aligned}$$

We now come to the definition that embodies the central idea of resource bounded measure theory, namely that one can use bounded measure theories to investigate the internal structure of classes.

1.5.4. DEFINITION. (Lutz [49, 50]) Let Δ be one of the classes all, rec, Δ_2, p, or p_2. A class $\mathcal{A} \subseteq R(\Delta)$ has *measure zero in* $R(\Delta)$, denoted $\mu(\mathcal{A}|R(\Delta)) = 0$, if $\mu_\Delta(\mathcal{A} \cap R(\Delta)) = 0$. $\mathcal{A}$ has *measure one in* $R(\Delta)$, denoted $\mu(\mathcal{A}|R(\Delta)) = 1$, if $\mu(\mathcal{A}^c|R(\Delta)) = 0$.

In Theorem 1.5.6 we show that this definition is consistent.

We have already seen in Theorem 1.3.6 that for $\Delta =$ all it does not matter whether we use martingales or supermartingales in our measure theory. The following proposition shows that this holds also for the other measure theories defined above, as well as some other robustness properties. In Chapter 4 we will encounter a measure for which the distinction between martingales and supermartingales *is* crucial.

1.5.5. PROPOSITION. *Let Δ be one of the classes* all, rec, Δ_2, p, *or* p_2.

(i) *For a class $\mathcal{A}$, if there is a supermartingale in Δ that succeeds on $\mathcal{A}$ then there is also a martingale in Δ that succeeds on $\mathcal{A}$. Hence, for the definition of Δ-measure, it does not matter whether we use martingales or supermartingales.*

(ii) *In Definition 1.3.2, it does not matter whether we write* 'lim sup' *or* 'lim'. *More precisely, if for every (super)martingale that succeeds on a class $\mathcal{A}$ in the* lim sup*-sense there is a (super)martingale that succeeds on $\mathcal{A}$ in the* lim*-sense.*

(iii) *For Δ-measure theory, we may assume that all martingales have values in $\mathbb{Q}^+$ and that they are exactly Δ-computable, rather than using Δ-computations for them.*

PROOF. We prove (i), (ii), and (iii) all at once. Suppose d is a supermartingale that succeeds on the class $\mathcal{A}$ in the lim sup-sense.

First we transform d into a supermartingale d_0 that has values in $\mathbb{Q}$, that is exactly computable in Δ, and that succeeds on $\mathcal{A}$. Let $\hat{d} : \omega \times \Sigma^* \to \mathbb{Q}^+$ be a Δ-computation of d:

$$\forall k \in \omega \, \forall w \in \Sigma^* (|d(w) - \hat{d}_k(w)| \leq 2^{-k}).$$

Define a martingale d_0 which succeeds on every $A \in \mathcal{C}$ as follows: $d_0(w) = \hat{d}_{|w|}(w) + 4 \cdot 2^{-|w|}$. Then $d_0(w) \geq d(w) + 3 \cdot 2^{-|w|}$ and $d_0(w) \leq d(w) + 5 \cdot 2^{-|w|}$. Furthermore,

$$\begin{aligned} d_0(w0) + d_0(w1) &\leq d(w0) + 5 \cdot 2^{-|w|-1} + d(w1) + 5 \cdot 2^{-|w|-1} \\ &\leq 2(d(w) + 5/2 \cdot 2^{-|w|}) \\ &\leq 2(d(w) + 3 \cdot 2^{-|w|}) \\ &\leq 2d_0(w), \end{aligned}$$

so d_0 is a martingale, and d_0 succeeds on every $A \in \mathcal{A}$ because $d_0(w) \geq d(w)$ and d succeeds on every $A \in \mathcal{A}$. Clearly, d_0 is computable in time Δ.

Secondly we transform the supermartingale d_0 into a martingale d_1 that succeeds on at least the same sets as d_0. For this we simply (inductively) define $d_1(w0) = d_0(w0)$ and $d_1(w1) = 2d_1(w) - d_1(w0)$. Clearly d_1 is a martingale, and $S[d_1] \supseteq S[d_0]$ because for all w, $d_1(w) \geq d_0(w)$ (this follows easily by induction on $|w|$ from supermartingale property of d_0), so d_1 also succeeds on $\mathcal{A}$.

Thirdly we transform the martingale d_1 into a martingale d_2 that succeeds on $\mathcal{A}$ in the lim-sense. Define $d_2(\lambda) = d_1(\lambda)$ and

$$d_2(wi) = \frac{d_1(wi)}{d_1(w)} \Big(d_2(w) - \lfloor d_2(w) - 1 \rfloor \Big) + \lfloor d_2(w) - 1 \rfloor,$$

where $\lfloor x \rfloor$ is the greatest natural number smaller than or equal to x, and $\lfloor x \rfloor = 0$ when x is negative. It is easy to check that d_2 is a martingale and that if $\limsup_n d_1(A{\restriction}n) = \infty$ then also $\limsup_n d_2(A{\restriction}n) = \infty$. So we see that d_2 succeeds on at least the same sets as d_1. Note that for every v and w with $v \sqsubseteq w$, if $d_2(v) \geq x$ then $d_2(w) \geq \lfloor x - 1 \rfloor$. So if $\limsup_n d_2(A{\restriction}n) = \infty$ then $\lim_n d_2(A{\restriction}n) = \infty$, hence, if d_2 succeeds on a set, it does so in the lim-sense.

In conclusion; we have transformed Δ-supermartingale d that succeeds on $\mathcal{A}$ in the lim sup-sense into the exactly Δ-computable martingale d_2 with rational values that succeeds on (at least) $\mathcal{A}$ in the lim-sense. This proves the proposition. □

It is now easily proved that Definition 1.5.4 is consistent:

1.5.6. THEOREM. (Lutz [49, 50]) $\mu_\Delta(R(\Delta)) \neq 0$.

PROOF. As an example we prove that $\mu_{\mathrm{p}}(\mathrm{E}) \neq 0$. Let d be any p-martingale. By Proposition 1.5.5 we may assume that d has values in $\mathbb{Q}$ and that d is exactly p-computable, say that $d \in \mathrm{DTIME}(n^k)$. Inductively define a set A as follows: given $A{\restriction}z_n$ define $A(z_n) = 1$ iff $d((A{\restriction}z_n)1) \leq d((A{\restriction}z_n)0)$. Then clearly d does not succeed on A, and to compute $A(z_n)$ we need $n \cdot |A{\restriction}n|^k = O(2^{(k+1)|z_n|})$ steps, so $A \in \mathrm{DTIME}(2^{(k+1)n})$. $\square$

We are now ready to state the definition of randomness in the context of resource bounded measure theory.

1.5.7. DEFINITION. A set A is Δ-*random* if no Δ-martingale succeeds on A. (Equivalently: if $\mu_{\Delta}(\{A\}) \neq 0$.)

Note that rather than answering what a random sequence is, this definition gives us a parameter that we may set ourselves, depending on the context in which we want to use the random sets. This makes it very flexible. In Chapter 4 we will see how this definition relates to the concept of Martin-Löf mentioned in Section 1.1.

We may note here that Δ-randomness is quite strong compared to the notion that von Mises had in mind. As an example let us consider p-randomness. It is easy to see that if a set A is p-random, then for every infinite set $B \in \mathrm{P}$ the set of strings on which A and B agree has density 1/2, so A is 'von Mises-random' with respect to P. However, p-random sets do not exist in E, whereas Wilber [80] has shown that sets with the above property do exist in E.

1.6 OVERVIEW

The effective measures introduced in the previous section are the central theme of this thesis. In Chapter 2 we look at the polynomial time bounded measure μ_{p} and use it to investigate the quantitative structure of the exponential time class E. In that chapter we also use generic sets, and we compare the two approaches.

In Chapter 3 we investigate a measure $\mu_{\mathfrak{a}}$ with a certain asymmetry property which makes it suitable for the study of the class of recursively enumerable sets. The study of the various completeness notions for this measure also sheds light on a question not mentioning measure, namely to what extent an incomplete set can resemble a complete set.

In Chapter 4 we study classes of martingales corresponding to the classes of the arithmetical hierarchy. In particular we study r.e.-martingales and the corresponding random sets that were introduced by Martin-Löf. We describe the distribution of these sets in terms of the reduction relations commonly used in recursion theory. We also locate the class $R(\text{r.e.})$ of sets constructed by r.e.-functions in terms of the same relations. This class is the analogue of the classes $R(\Delta)$ occurring in the framework introduced above. Finally, we treat similar questions for the measures corresponding to Δ_n, and we prove that these measures coincide with the measures corresponding to Π_n.

In Chapter 5 we study sets that are *low* for the the class of Martin-Löf random sets and the class of Schnorr random sets, respectively. Low sets are sets that are weak in the sense that they "do not help as an oracle". Although they are weak in this sense, low sets can still code substantial information. In particular they can be nonrecursive. Lowness for $\mathcal{C}$ indicates that this information is inaccessible to elements from $\mathcal{C}$.

Finally, in Chapter 6 we consider various questions about recursive measure theory, relating to the distribution of recursively random sets, Kolmogorov complexity, the measure of Schnorr studied in Chapter 5, partial recursive martingales, and martingales that are recursive in the halting set K.

Chapter 2

Genericity and Randomness in Exponential Time

In this chapter we use the polynomial time bounded measure μ_{p} introduced in Section 1.5 to investigate the structure of the exponential time class E. Ambos-Spies, Fleischhack and Huwig [4, 5] introduced polynomial time bounded genericity concepts and used them for the investigation of structural properties of NP (under appropriate assumptions) and E. In Section 2.2 we relate these concepts to each other. In Section 2.3 we prove that the amount of genericity in a successor (under bounded truth-table reducibility) of a generic set is bounded, and we deduce from this a generalization of the Small Span Theorem of Juedes and Lutz [31]. In Section 2.4 we consider polynomially random sets and derive some basic properties, and in Section 2.5 we prove that n^{c+1}-random sets are n^c-generic, whereas the converse fails. It follows from the results from Section 2.3 that the amount of randomness in a successor of a random set is bounded. This contrasts with the result from Section 2.6 stating that every n^c-random set in E has n^k-random predecessors in E for any $k \geq 1$. We apply this result to answer a question raised by Lutz [52]: We show that the class of weakly complete sets has measure 1 in E and that there are weakly complete problems which are not p-btt-complete for E.

We now introduce some notation and terminology that we will use only in this chapter. We use the notation Σ^* instead of $2^{<\omega}$. Σ denotes the set $\{0,1\}$. For a set of strings A, $A^{=n}$ denotes the set of strings in A of length n. Similarly we have the set $A^{\leq n}$. We use the words *'problem'* and *'language'* as synonyms for 'set', i.e. for subsets of Σ^*. $<$ is the length-lexicographical ordering on Σ^*; z_n is the n^{th} string under this ordering. For a string $x \in \Sigma^*$, $x+1$ denotes the $<$-successor of x. For $A \subseteq \Sigma^*$ and $x \in \Sigma^*$ we let $A{\restriction}x$ denote the finite initial segment of A below x, i.e. $A{\restriction}x = \{y : y < x \wedge y \in A\}$, and we identify this initial segment with its characteristic string, i.e. $A\restriction z_n = A(z_0)\dots A(z_{n-1}) \in \Sigma^*$. For the calculations below it is crucial to note that

$$2^{|x|} - 1 \leq |A\restriction x| < 2^{|x|+1} - 1, \tag{2.1}$$

whence $O(|A\restriction x|^c) = O(2^{c|x|})$ for any $c \geq 1$. The lower case letters c, k, n always denote elements of ω.

2.1 Generic sets

Ambos-Spies, Fleischhack, and Huwig [4, 5] introduced different types of resource bounded genericity. Here we shortly review one of their concepts which is closely related to resource bounded measure (see [8]).

2.1.1. Definition. (Ambos-Spies et al. [5]) A *condition* is a set $C \subseteq \Sigma^*$. A language A *meets* the condition C if, for some string x, $A\restriction x \in C$. C is *dense along* A if

$$(\exists^\infty x \in \Sigma^*)(\exists i \in \Sigma)[(A\restriction x)i \in C];$$

and C is *dense* if C is dense along all languages. A language A is $\mathcal{C}$*-generic* if A meets every condition $C \in \mathcal{C}$ which is dense along A. We say that A is $t(n)$*-generic* if it is DTIME$(t(n))$-generic.

This genericity concept was introduced by Ambos-Spies, Fleischhack and Huwig in [5]. Of the three types of genericity concepts introduced there, here we consider only the second type. In [5], $\mathcal{C}$-generic sets were called $\mathcal{C}$-2-generic sets. For deterministic time classes we abbreviate DTIME$(t(n))$-generic by $t(n)$-*generic* and we call a condition $C \in$ DTIME$(t(n))$ a $t(n)$-*condition.*

A condition C should be viewed as a finitary property P of languages, where C contains all finite initial parts $X\restriction x$ of languages such that all languages Y extending $X\restriction x$ have the property P. So a language A has the property P if and only if A meets $\mathcal{C}$. The class $\mathcal{C}$ is dense along A if and only if in a construction of A along the ordering $<$, where at stage s of the construction we decide whether or not the string z_s belongs to A, there are infinitely many stages s such that by appropriately defining $A(z_s)$ we can ensure that A has the property P (i.e. $A\restriction(z_s + 1) \in C$). Finally, in case of a $t(n)$-condition, the complexity for the correct choice for $A(z_s)$ is $t(n)$-time bounded in $|A\restriction z_s|$, i.e., by (2.1), $t(2^n)$-time bounded in the length of z_s. So a $t(n)$-generic set will have all finitary properties P of time complexity $t(n)$ (relative to the length n of the initial segment) which can be ensured in a construction of the above type infinitely often.

In the following we will mainly consider n^c-generic sets ($c \geq 1$) which are adequate for analyzing the structure of E. We start, however, with some more general results.

2.1.2. PROPOSITION.

(i) *Let $\mathcal{C}$ and $\mathcal{D}$ be classes such that $\mathcal{C} \subseteq \mathcal{D}$. Then any $\mathcal{D}$-generic set is $\mathcal{C}$-generic. In particular, if t and t' are recursive functions such that $t(n) \leq t'(n)$ almost everywhere then any $t'(n)$-generic set is $t(n)$-generic.*

(ii) *For any recursive function t, the complement of a $t(n)$-generic set A is also $t(n)$-generic.*

PROOF. The first part is immediate by definition. The second part follows from closure of $\mathrm{DTIME}(t(n))$ under flipping strings. (That is, if $A \in \mathrm{DTIME}(t(n))$ then so is the set $\{\overline{w} : w \in A\}$, where $\overline{w}(i) = 1 - w(i)$ for every $i < |w|$.) $\square$

In [5] Ambos-Spies, Fleischhack and Huwig have shown that there are sparse P-generic sets in $\mathrm{DTIME}(2^{n^2})$. By a simple modification of this proof we obtain a strong general existence theorem for $t(n)$-generic sets.

2.1.3. THEOREM. *Let $t(n)$, $t'(n)$ and $f(n)$ be nondecreasing functions on ω such that $t(n)$ and $t'(n)$ are time-constructible, $t(n), t'(n) \geq n$, $f(n)$ is polynomial time computable with respect to the unary representation, and the range of f is unbounded. Let B be a set in $\mathrm{DTIME}(t'(n))$. Then there is a $t(n)$-generic set A such that*

$$A \in \mathrm{DTIME}(2^{n+1}(t'(n) + n^2 t(2^{n+1}) \log t(2^{n+1})))$$

and for any $n \geq 0$

$$\|(A \triangle B) \cap \Sigma^{=n}\| \leq f(n).$$

PROOF. We construct a $t(n)$generic set A with the required properties in stages, where at stage s we decide whether or not $z_s \in A$. By means of a standard universal machine we may fix a recursive enumeration $\{C_e : e \in \omega\}$ of $\mathrm{DTIME}(t(n))$ such that

$$C = \{0^e 1x : x \in C_e\} \in \mathrm{DTIME}(e \cdot t(|x|) \log(t(|x|)) + e). \tag{2.2}$$

Then to ensure that A is $t(n)$-generic it suffices to meet the requirements

$$R_e \quad : \quad C_e \text{ dense along } A \Rightarrow A \text{ meets } C_e$$

for all numbers $e \in \omega$. Simultaneously with A we enumerate a list Sat of the indices of the requirements which are satisfied by diagonalization and we let Sat_s be the part of Sat enumerated by the end of stage s ($\mathrm{Sat}_{-1} = \emptyset$). So, by the end of stage $s-1$, $A{\upharpoonright}z_s$ and Sat_{s-1} are given.

Stage s. We say that the requirement R_e *requires attention* (at stage s) if $e < f(|z_s|)$, $e \notin \mathrm{Sat}_{s-1}$ and

$$\exists i \in \Sigma\ [(A{\upharpoonright}z_s)i \in C_e]. \tag{2.3}$$

Distinguish the following two cases.

Case 1: Some requirement requires attention. Fix the least e such that R_e requires attention and fix $i \leq 1$ minimal with $(A \upharpoonright z_s)i \in C_e$. Let $A(z_s) = i$ and $\mathrm{Sat}_s = \mathrm{Sat}_{s-1} \cup \{e\}$ and say that R_e *receives attention.*

Case 2: Otherwise. Let $A(z_s) = B(z_s)$ and let $\mathrm{Sat}_s = \mathrm{Sat}_{s-1}$.
This completes the construction. To show that A is $t(n)$-generic, first note that every requirement receives attention at most once and that Sat_s contains the indices of the requirements which received attention by the end of stage s. So, by a straightforward induction, every requirement requires attention only finitely often. Hence if C_e is dense along A, (2.3) will hold at infinitely many stages s, whence R_e will eventually receive attention, thereby ensuring that A meets C_e. So every requirement R_e is met whence A is $t(n)$-generic.

Moreover, at a stage s with $|z_s| = n$, only a requirement R_e with $e < f(n)$ may receive attention. So Case 1 can apply to at most $f(n)$ such stages, whence, by definition of A in Case 2, $\|(A \triangle B) \cap \{x \in \Sigma^* : |x| = n\}\| \leq f(n)$ will hold.

It remains to show that $A \in \mathrm{DTIME}(2^{n+1}(t'(n) + t''(n)))$, where $t''(n) = n^2 t(2^{n+1}) \log t(2^{n+1})$. Fix any string z_s of length n. Then, by (2.1) it holds that $s < 2^{n+1}$ whence it suffices to show that, given $A \upharpoonright z_s$ and Sat_{s-1},$A(z_s)$ and Sat_s can be computed in $t'(n) + t''(n)$ steps. To do so, without loss of generality assume that $f(n) \leq n$. Moreover, since $f(n)$ can be computed in $\mathrm{poly}(n)$ steps, we may assume that $f(n)$ is given. Then $t''(n)$ steps suffice to decide whether Case 1 applies to stage s and if so to perform the corresponding action: Since, by assumption, $A \upharpoonright z_s$ and Sat_{s-1} are given, it suffices to check for each of the n numbers $e < |z_s|$ and for $i \leq 1$ whether $(A \upharpoonright z_s)i \in C_e$ which, by (2.2), can be done in $O(n \cdot t(2^{n+1}) \log t(2^{n+1}))$ steps for each such e. Finally, since Case 2 can be performed in $t'(n)$ steps this implies the claim. □

2.1.4. COROLLARY. *There is a sparse n^c-generic set in* $\mathrm{DTIME}(2^{(c+2)n})$.

PROOF. Apply Theorem 2.1.3 to $t(n) = n^c$, $t'(n) = f(n) = n$ and $B = \emptyset$. Since

$$2^{n+1}(n + n^2(2^{n+1})^c \log(2^{n+1})^c) < 2^{(c+2)n}$$

almost everywhere, this yields an n^c-generic set $A \in \mathrm{DTIME}(2^{(c+2)n})$ with

$$\|A \cap \Sigma^{=n}\| \leq n.$$

□

As the following theorem shows, Theorem 2.1.3 provides an almost optimal lower bound on the time complexity of $t(n)$-generic sets.

2.1.5. THEOREM. *Let A be $t(n)$-generic. Then $A \notin$* $\mathrm{DTIME}(t(2^n))$. *In particular, there is no n^c-generic set in* $\mathrm{DTIME}(2^{cn})$.

PROOF. For a contradiction assume that $A \in \mathrm{DTIME}(t(2^n))$. Then, by (2.1),

$$C = \{X\restriction(x+1) : A(x) \neq X(x)\}$$

is a $t(n)$-condition which is obviously dense. So, by $t(n)$-genericity, A meets $\mathcal{C}$. By definition of $\mathcal{C}$ this implies that $A(x) \neq A(x)$ for some x, a contradiction. □

The argument in the proof of Theorem 2.1.5 is typical for showing that a generic set has a certain property. In the following we give two further examples: we prove that generic sets are incompressible under many-one reductions and bi-immune. Here as in the following we will restrict ourselves to n^c-genericity.

A function $f : \Sigma^* \to \Sigma^*$ is almost 1-1 if the collision set of f,

$$\mathrm{COLL}_f = \{x \in \Sigma^* : \exists y < x\,(f(x) = f(y))\},$$

is finite. f is *consistent* with a set A if, for all $x, y \in \Sigma^*$, $A(x) \neq A(y)$ implies that $f(x) \neq f(y)$. Then A is $\mathcal{C}$*-incompressible* if, for any $f \in \mathrm{F}\mathcal{C}$ which is consistent with A, f is almost 1-1. Again we abbreviate $\mathrm{DTIME}(t(n))$-incompressible by $t(n)$-incompressible and we write p-incompressible for P-incompressible. Note that $A \leq_m B$ via $f \in \mathcal{C}$ implies that f is consistent with A. So, for $\mathcal{C}$-incompressible A, any $\mathcal{C}$-m-reduction from A is almost 1-1.

2.1.6. THEOREM. *Let A be n^c-generic ($c \geq 2$). Then A is $2^{(c-1)n}$-incompressible.*

PROOF. Fix $f \in \mathrm{FDTIME}(2^{(c-1)n})$ such that f is consistent with A. To show that f is almost 1-1, define

$$C = \{X\restriction(x+1) : \exists y < x[f(x) = f(y) \wedge X(x) \neq X(y)]\}.$$

Then C is an n^c-condition. Moreover, by consistency of f with A, A does not meet C. So, by n^c-genericity of A, C is not dense along A. By definition of C, it follows that the collision set of f is finite. □

It is easy to show that any 2^{cn}-incompressible set A is 2^{cn}-*bi-immune*, i.e., $A \cap B \neq \emptyset$ and $\overline{A} \cap B \neq \emptyset$ for any infinite $B \in \mathrm{DTIME}(2^{cn})$ (see [12]). So Theorem 2.1.6 implies that any n^c-generic set is $2^{(c-1)n}$-bi-immune ($c \geq 2$). By a direct argument we can slightly improve this result:

2.1.7. THEOREM. *Let A be n^cgeneric ($c \geq 2$). Then A is 2^{cn}-bi-immune.*

PROOF. By Proposition 2.1.2 it suffices to show that $\overline{A}$ is 2^{cn}-immune, i.e., that $A \cap B \neq \emptyset$ for any infinite $B \in \mathrm{DTIME}(2^{cn})$. So fix such a set B. Define

$$C = \{X\restriction(x+1) : X(x) = B(x) = 1\}.$$

Then, by (2.1), C is an n^c-condition which, by infinity of B, is dense. Hence A meets C which, by definition of C, implies that $A \cap B \neq \emptyset$. $\square$

One can also apply n^c-genericity to separate the standard polynomial time reducibilities between p-one-one and p-bounded-truth-table (see [41]). As a corollary we obtain that n^c-generic sets cannot be p-btt-complete for E.

2.1.8. THEOREM. *Let A be n^c-generic ($c \geq 2$).*

(i) $A \oplus A \not\leq_1^p A$

(ii) $\overline{A} \not\leq_m^p A$

(iii) $A_k \not\leq_{k\text{-}tt}^p A$, *where* $A_k = \{x : \{x, x+1, \ldots, x+k\} \cap A \neq \emptyset\}$ ($k \geq 1$)

(iv) $A_\omega \not\leq_{btt}^p A$, *where* $A_\omega = \{0^k 1x : x \in A_k\}$

PROOF. The proof is very similar to the proof of the corresponding facts for the tally p-generic sets (see Ambos-Spies, Fleischhack, and Huwig [4, Theorem 5.9]). As an example we prove item (ii). Suppose that $\overline{A} \leq_m^p A$ via $f \in \mathrm{P}$. Let $C = \{X\restriction(x+1) : \exists y, z \leq x\,(f(y) = z \wedge X(y) = X(z))\}$. Then $C \in \mathrm{P}$ and C is dense, so A meets C. It follows that $\overline{A}$ does not p-m-reduce to A via f. $\square$

Note that, for any set A, $A \oplus A \leq_m^p A$, $\overline{A} \leq_{1\text{-}tt}^p A$, $A_k \leq_{(k+1)\text{-}tt}^p A$, and $A_\omega \leq_{tt}^p A$. So we can mutually distinguish p-1, p-m, p-1-tt, p-$(k+1)$-tt ($k \geq 1$), p-btt and p-tt reductions to n^c-generic sets:

2.1.9. COROLLARY. *Let A be n^c-generic ($c \geq 2$). There are sets B_1, B_2, $B_{3,k}$ ($k \geq 1$) and B_4 such that*

(i) $B_1 \leq_m^p A$ *but* $B_1 \not\leq_1^p A$

(ii) $B_2 \leq_{1\text{-}tt}^p A$ *but* $B_2 \not\leq_m^p A$

(iii) $B_{3,k} \leq_{(k+1)\text{-}tt}^p A$ *but* $B_{3,k} \not\leq_{k\text{-}tt}^p A$ ($k \geq 1$)

(iv) $B_4 \leq_{tt}^p A$ *but* $B_4 \not\leq_{btt}^p A$

2.1.10. COROLLARY. *Let A be n^c-generic ($c \geq 2$). Then A is not p-btt-complete for* E.

PROOF. Assume that $A \in \mathrm{E}$. Then, for A_ω as in Theorem 2.1.8, $A_\omega \in \mathrm{E}$ but $A_\omega \not\leq_{btt}^p A$. So A is not p-btt-complete for E. $\square$

We conclude this section with the observation that Corollary 2.1.10 is optimal. Given $f : \omega \to \omega$ we say that A is p-$f(n)$-tt-reducible to B if there is a p-tt-reduction from A to B for which the number of oracle queries on inputs of length n is bounded by $f(n)$.

2.1.11. THEOREM. *Let $f : \omega \to \omega$ be nondecreasing, unbounded and polynomial time computable with respect to the unary representation. There is an n^c-generic set A which is complete for* E *under p-$f(n)$-tt-reductions ($c \geq 1$). In particular, there is an n^c-generic set A which is p-tt-complete for* E.

PROOF. Fix a p-m-complete set C for E. We will use the following transitivity law for p-m- and p-$f(n)$-tt-reductions: Since a p-m-reduction increases the size of the input only by a polynomial factor and since f is nondecreasing, for any sets X, Y, Z,

$$X \leq_m^p Y \wedge Y \leq_{f(\log n)\text{-}tt}^p Z \Rightarrow X \leq_{f(n)\text{-}tt}^p Z.$$

Hence it suffices to show that there is an n^c-generic set $A \in$ E with $C \leq_{f(\log n)\text{-}tt}^p A$. Let $B = \{xy : |x| = |y| \wedge x \in C\}$. Then, by Theorem 2.1.3, there is an n^c-generic set $A \in$ E such that

$$\|(A \triangle B) \cap \{x \in \Sigma^* : |x| = 2n\}\| \leq 1/3 \cdot f(\log n).$$

Hence $x \in C$ if and only if $\|A \cap F_x\| \geq 2/3 \cdot f(\log(|x|))$, where F_x consists of the lexicographically first $f(\log(|x|))$ strings xy with $|x| = |y|$. So C is p-$f(\log n)$-tt-reducible to A. □

2.1.12. REMARK. As the results of this section show, the n^c-generic sets pertain to diagonalizations over the levels of the linear exponential time hierarchy $\mathrm{E} = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{cn})$. So, since a set is P-generic if and only if it is n^c-generic for all $c \geq 1$, P-generic sets relate to diagonalizations over E. In particular, by Theorem 2.1.5, no set in E is P-generic. In fact, by Theorems 2.1.6 and 2.1.7, P-generic sets are E-incompressible and E-bi-immune. On the other hand, by Theorem 2.1.3, P-generic sets can be found in all sufficiently closed, smooth deterministic time classes properly containing E. E.g., as shown already in [5], there are P-generic sets in the class $\mathrm{DTIME}(2^{n^2})$. In an analogous way, the $2^{(\log n)^c}$-generic sets pertain to the levels $\mathrm{DTIME}(2^{n^c})$ of the polynomial exponential time hierarchy E_2. E.g., by Theorem 2.1.5, there is no $2^{(\log n)^c}$-generic set in $\mathrm{DTIME}(2^{n^c})$, and the proofs of Theorems 2.1.6 and 2.1.7 can be easily modified to show that $2^{(\log n)^c}$-generic sets are $\mathrm{DTIME}(2^{n^{c-1}})$-incompressible and $\mathrm{DTIME}(2^{n^c})$-bi-immune ($c \geq 2$). On the other hand, by Theorem 2.1.3, there are $2^{(\log n)^c}$-generic sets in $\mathrm{DTIME}(2^{n^{c+2}})$. So, for $\mathrm{P}_2 = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{(\log n)^c})$, the P_2-generic sets relate to diagonalizations over E_2 just as the P-generic sets relate to diagonalizations over E. In particular, observe that the proof of Theorem 2.1.11 can be easily modified to show that, for any function f as there and for any $c \geq 1$, there is a $2^{(\log n)^c}$-generic set (hence a P-generic set) which is complete for E_2 under p-$f(n)$-tt-reductions.

2.2 Genericity and measure

In this chapter, we say that a class $\mathcal{A}$ has *$t(n)$-measure zero* if there is a martingale $d : \Sigma^* \to \mathbb{Q}^+$ that is exactly computable in time $O(t(n))$ such that d succeeds on $\mathcal{A}$. (This is justified by Proposition 1.5.5.) Throughout, $t(n) : \omega \to \omega$ will be a recursive, time constructible function satisfying $t(n) \geq n$ for almost every n.

To show that the class of n^c-random sets has p-measure 1 we need a weak version of σ-additivity for the $t(n)$-measure.

2.2.1. Definition. (Lutz) A class $\mathcal{X}$ is a *$t(n)$-union* of the $t(n)$-measure 0 classes $\mathcal{X}_i$, $i \in \omega$, if $\mathcal{X} = \bigcup_{i\in\omega} \mathcal{X}_i$ and there exists a $t(n)$-computable function $d : \omega \times \Sigma^* \to \mathbb{Q}^+$ such that for every i, $d_i(x) = d(i, x)$ is a martingale and d_i succeeds on every problem in $\mathcal{X}_i$.

By Proposition 1.5.5 (iii) this definition is equivalent to Lutz's definition (see e.g. [50, p 231]). The next lemma is a generalization of Lutz's Δ-Ideal Lemma for arbitrary time bounds $\Delta = O(t(n))$ ([50, Lemma 3.10]).

2.2.2. Lemma. *If $\mathcal{X}$ is a $t(n)$-union of the $t(n)$-measure* 0 *classes $\mathcal{X}_i$, $i \in \omega$, then $\mathcal{X}$ has $n\,t(2n)$-measure* 0.

Proof. By assumption there exists a $t(n)$-computable function $d : \omega \times \Sigma^* \to \mathbb{Q}^+$ such that for every i, d_i is a martingale and d_i succeeds on every problem in $\mathcal{X}_i$. Without loss of generality we may assume that $d_i(\lambda) = 1$ for every i. Define $d' : \Sigma^* \to \mathbb{R}^+$ by

$$d'(w) = \sum_{i=0}^{\infty} 2^{-i} d_i(w).$$

Note that by the martingale property of the d_i and the assumption that $d_i(\lambda) = 1$, $d_i(w) \leq 2^{|w|}$ for every i, so this sum is convergent. Now d' is a martingale because all the d_i are, and $d'(w) \geq 2^{-i} d_i(w)$, so d' succeeds on $\mathcal{X}_i$ for every i, hence d' succeeds on $\mathcal{X}$. We show that d' is $n\,t(2n)$-computable. Define

$$\hat{d}_k(w) = \sum_{i=0}^{k+|w|} 2^{-i} d_i(w).$$

Then

$$\begin{aligned} d'(w) - \hat{d}_k(w) &= \sum_{i=k+|w|+1}^{\infty} 2^{-i} d_i(w) \\ &\leq \sum_{i=k+|w|+1}^{\infty} 2^{-i+|w|} = 2^{-k} \end{aligned}$$

(The inequality holds since $d_i(w) \leq 2^{|w|} \cdot d_i(\lambda) = 2^{|w|}$). Since clearly $\hat{d}_k(w) \in$ FDTIME$(n\,t(2n))$, it follows that the sequence $\{\hat{d}_k(w) : k \in \omega\}$ is an $n\,t(2n)$-computation of d'. □

2.2.3. THEOREM. *For any $c \geq 1$, the class of n^c-generic sets has n^{c+3}-measure 1, hence p-measure 1.*

PROOF. Fix c and let $\{C_e : e \in \omega\}$ be a recursive enumeration of DTIME(n^c) such that

$$C = \{0^e 1x : x \in C_e\} \in \mathrm{DTIME}(e \cdot |x|^c \cdot \log(|x|^c) + e) \tag{2.4}$$

holds. Let $\mathcal{C}_e = \{X : C_e \text{ is dense along } X \text{ and } X \text{ does not meet } C_e\}$ and let $\mathcal{C} = \bigcup_{e \geq 0} \mathcal{C}_e$. Then $\mathcal{C}$ is the class of languages which are not n^c-generic. So, by Lemma 2.2.2, it suffices to define an n^{c+2}-computable function d such that, for $e \in \omega$, d_e is a martingale which succeeds on every language in $\mathcal{C}_e$. For x with $|x| \leq 2^e$ let $d(e,x) = 1$ and, for x with $|x| \geq 2^e$ and for $i \leq 1$, let

$$d(e, xi) = \begin{cases} 0 & \text{if } xi \in C_e \wedge x(1-i) \notin C_e \\ 2d(e,x) & \text{if } x(1-i) \in C_e \wedge xi \notin C_e \\ d(e,x) & \text{otherwise.} \end{cases}$$

Then each d_e is a martingale. Moreover, it easily follows from (2.4) that d is n^{c+2}-computable. So it only remains to prove that each d_e succeeds on the languages in $\mathcal{C}_e$. Fix e and $X \in \mathcal{C}_e$. Then C_e is dense along X but X does not meet C_e. By the latter, $X{\upharpoonright}x \notin C_e$ for all x, whence $d(e, X{\upharpoonright}x) \neq 0$ for all x. It follows that $d(e, X{\upharpoonright}z_n)$ is nondecreasing in n. So it suffices to show that there are infinitely many x such that $d(e, X{\upharpoonright}(x+1)) = 2d(e, X{\upharpoonright}x)$, i.e., by definition of d, such that, for some $i \leq 1$, $X{\upharpoonright}(x+1) = (X{\upharpoonright}x)i$, $(X{\upharpoonright}x)(1-i) \in C_e$ and $(X{\upharpoonright}x)i \notin C_e$. But this is immediate by definition of $\mathcal{C}_e$. □

By Theorem 2.2.3, any property shared by all n^c-generic sets (for some $c \geq 1$) occurs with p-measure 1. E.g., from Theorem 2.1.6 and Theorem 2.1.7 we may conclude that the classes of 2^{cn}-incompressible sets and 2^{cn}-bi-immune sets have p-measure 1. This was first shown by Juedes and Lutz [31] and Mayordomo [57] respectively, using direct arguments. Though, in general, the direct proof that a property P has p-measure 1 uses the same ideas as showing that any n^c-generic set (for some c) has this property, the latter may turn out to be less complex, since it suffices to consider single requirements. In particular in more involved arguments this simplified machinery can help to keep down the combinatorial complexity of proofs. In the next section we will give an example for this.

2.2.4. REMARK. By duplicating the above argument we can show that, for any c, the class of the $2^{(\log n)^c}$-generic sets has p_2-measure 1, hence measure 1 in E_2. In particular, the class of P-generic sets has p_2-measure 1 and measure 1 in E_2.

2.3 Genericity above a generic set

For a polynomial time bounded reducibility $\leq_r^p$ the lower and upper span of a set A are defined by $P_r(A) = \{B : B \leq_r^p A\}$ and $P_r^{-1}(A) = \{B : A \leq_r^p B\}$, respectively. The intersection of the upper and lower span of A is the p-r-degree of A: $\deg_r^p(A) = \{B : B \equiv_r^p A\}$. Juedes and Lutz [31] have shown that, for any set $A \in \mathrm{E}$, the upper span of A or the lower span of A under p-m-reducibility has measure 0 in E. Hence $\deg_m^p(A)$ has measure 0 in E for any set $A \in \mathrm{E}$. So, in particular, the class of p-m-complete problems for E has measure 0 in E.

Here we first deduce the Small Span Theorem for p-m-reducibility from Theorem 2.2.3 and a theorem on the distribution of the n^c-generic sets under p-m-reducibility which we will prove next. Then, by extending this theorem to bounded truth-table reductions, we generalize the Small Span Theorem to these reductions.

2.3.1. Theorem. *Let A and B be sets such that $A \leq_m^p B$, A is n^c-generic and $A \in \mathrm{DTIME}(2^{dn})$ where $c, d \geq 2$. Then B is not n^{d+1}-generic.*

Proof. Fix $f \in \mathrm{FP}$ such that $A \leq_m^p B$ via f and let $D = \{x : |f(x)| \geq |x|\}$. Note that, by Theorem 2.1.6, A is p-incompressible whence f is almost 1-1. This easily implies that D is infinite. So the condition

$$C = \{X\restriction(y+1) : \exists x[|x| \leq |y| \wedge f(x) = y \wedge A(x) \neq X(y)]\}$$

is dense. Moreover, as one can easily check, $C \in \mathrm{DTIME}(n^{d+1})$ and, since $A \leq_m^p B$ via f, B does not meet C. So B is not n^{d+1}-generic. □

Note that the main step in the above proof shows that, f or any p-incompressible $A \in \mathrm{DTIME}(2^{dn})$ and for any B with $A \leq_m^p B$, B is not $2^{(d+1)n}$-bi-immune. The first proof of this fact is due to Lindner [47].

2.3.2. Corollary. (Small Span Theorem of Juedes and Lutz [31]) *Let $A \in \mathrm{E}$. Then $\mu(P_m(A)|\mathrm{E}) = 0$ or $\mu_{\mathrm{p}}(P_m^{-1}(A)) = \mu(P_m^{-1}(A)|\mathrm{E}) = 0$.*

Proof. If there is no n^2-generic set in $P_m(A) \cap \mathrm{E}$ then we have that $\mu(P_m(A)|\mathrm{E}) = \mu_{\mathrm{p}}(P_m(A) \cap \mathrm{E}) = 0$ by Theorem 2.2.3. Otherwise, fix A' and $d \geq 2$ such that A' is n^2-generic, $A' \leq_m^p A$, and $A' \in \mathrm{DTIME}(2^{dn})$. Since $P_m^{-1}(A)$ is contained in $P_m^{-1}(A')$ it follows from Theorem 2.3.1, that $P_m^{-1}(A)$ does not contain any n^{d+1}-generic set. So, again by Theorem 2.2.3, $\mu_{\mathrm{p}}(P_m^{-1}(A)) = \mu(P_m^{-1}(A)|E) = 0$. □

Lutz [51] raised the question whether the Small Span Theorem generalizes to the weaker polynomial reducibilities. Lindner [47] proved the Small Span Theorem for p-1-tt-reducibility. So the positive character of p-m-reducibility is not necessary for the theorem. This still left the question what happens for reducibilities which may use more than one query. Here we prove the Small Span Theorem for p-k-tt-reductions for any $k \geq 1$, i.e. for reductions where the number of queries does not depend on the input. Below we use the notation $h(g_1, \ldots, g_k)$ for p-k-tt-reductions. Here $h : \Sigma^* \times \Sigma^k \to \Sigma$ and $g_i : \Sigma^* \to \Sigma^*$ are polynomial time computable functions, and $A \leq^p_{k\text{-}tt} B$ via $h(g_1, \ldots, g_k)$ if $A(x) = h(x, B(g_1(x)), \ldots, B(g_k(x)))$ for every x. Note that this is just a more explicit version of the definition given on page 5. Given h we define $h_x(a_1, \ldots, a_k) = h(x, a_1, \ldots, a_k)$. The main step in the proof below is an analogue of Theorem 2.3.1 for p-k-tt-reducibility.

2.3.3. THEOREM. *Let A and B be sets such that $A \leq^p_{k\text{-}tt} B$ for some $k \geq 1$, A is n^c-generic for some $c \geq 2$, and $A \in \mathrm{DTIME}(2^{dn})$ for some $d \geq 2$. Then B is not $n^{(k+1)(d+1)}$-generic.*

For the proof of this theorem we need an incompressibility concept for p-k-tt-reductions and some more technical tools.

2.3.4. DEFINITION. The collision set of a p-k-tt-reduction $h(g_1, \ldots, g_k)$ is defined by

$$\mathrm{COLL}_{h(g_1,\ldots,g_k)} = \{x \in \Sigma^* : \exists y < x[g_1(x) = g_1(y) \wedge \ldots \wedge g_k(x) = g_k(y) \wedge h_x = h_y]\}.$$

The reduction $h(g_1, \ldots, g_k)$ is *almost 1-1* if $\mathrm{COLL}_{h(g_1,\ldots,g_k)}$ is finite. We say that $h(g_1, \ldots, g_k)$ is *consistent* with a language A if

$$\forall x, y \in \Sigma^*[(g_1(x) = g_1(y) \wedge \ldots \wedge g_k(x) = g_k(y) \wedge h_x = h_y) \to A(x) = A(y)].$$

A language A is p-k-tt-*incompressible* if any p-k-tt-reduction $h(g_1, \ldots, g_k)$ which is consistent with A is almost 1-1.

Note that $A \leq^p_{k\text{-}tt} B$ via $h(g_1, \ldots, g_k)$ implies that $h(g_1, \ldots, g_k)$ is consistent with A, whence for p-k-tt-incompressible A, $h(g_1, \ldots, g_k)$ is almost 1-1. As we show next, p-k-tt-incompressibility coincides with p-incompressibility. So, by Theorem 2.1.6, n^c-generic sets are incompressible under p-k-tt-reductions.

2.3.5. LEMMA. *For any $k \geq 1$, A is p-k-tt-incompressible if and only if A is p-incompressible.*

PROOF. Since any p-m-reduction may be viewed as a p-k-tt-reduction, obviously any p-k-tt-incompressible set is p-incompressible. For a proof of the nontrivial direction, let A be p-incompressible and fix any p-k-tt-reduction $h(g_1, \ldots, g_k)$ which is consistent with A ($k \geq 1$). We have to show that $h(g_1, \ldots, g_k)$ is almost 1-1. Let

$$A' = \{\langle h_x, g_1(x), \ldots, g_k(x)\rangle : x \in A\}.$$

Since $h(g_1, \dots, g_k)$ is consistent with A it holds that $A \leq_m^p A'$ via $f(x) = \langle h_x, g_1(x), \dots, g_k(x)\rangle$. So, by p-incompressibility of A, f is almost 1-1, whence $h(g_1, \dots, g_k)$ is almost 1-1 too. □

For technical convenience, in the following we assume that all p-k-tt-reductions are in a normal form, where the queries are listed in decreasing order and redundant queries are replaced by $\emptyset$: A p-k-tt-reduction $h(g_1, \dots, g_k)$ is *normal* if, for any $x \in \Sigma^*$, there is some $i \leq k$ such that, for $1 \leq j < i$, $g_j(x) > g_{j+1}(x)$ and, for $j \geq i$, $g_j(x) = \emptyset$. It is easy to show that for any p-k-tt-reduction there is an equivalent normal p-k-tt-reduction. For a normal p-k-tt-reduction $h(g_1, \dots, g_k)$, the rank of $h(g_1, \dots, g_k)$ is defined to be the greatest number $r \in \{1, \dots, k\}$ such that

$$\exists^\infty x \in \Sigma^*(|x| \leq (k+1)|g_r(x)|).$$

(If no such r exists then the rank of $h(g_1, \dots, g_k)$ is 0.)

2.3.6. LEMMA. *Let $h(g_1, \dots, g_k)$ be a normal p-k-tt-reduction which is almost 1-1. Then the rank of $h(g_1, \dots, g_k)$ is greater than 0.*

PROOF. Fix n such that $2^n > 2^{2^k}$ and no x with $|x| \geq n$ is in the collision set of $h(g_1, \dots, g_k)$. It suffices to show that, for some x with $|x| = (k+1)n$, $|g_1(x)| \geq n$. Let

$$BC_n = \{(\alpha, y_1, \dots, y_k) : \alpha \text{ is a } k\text{-ary Boolean function and, } \\ \text{for } 1 \leq i \leq k,\ y_i \in \Sigma^* \text{ and } |y_i| < n\}.$$

Since $h(g_1, \dots, g_k)$ is normal, for any x with $|g_1(x)| < n$ it holds that $(h_x, g_1(x), \dots, g_k(x)) \in BC_n$. So, since, by choice of n, $h(g_1, \dots, g_k)$ is 1-1 on $\{x \in \Sigma^* : |x| = (k+1)n\}$, the existence of an x with the desired properties will follow from

$$\|BC_n\| < \|\{x \in \Sigma^* : |x| = (k+1)n\}\| = 2^{(k+1)n}.$$

This holds since there are $2n-1$ strings of length less than n and 2^{2^k}-ary Boolean functions, whence by choice of n, $\|BC_n\| < 2^{2^k} \cdot (2^n)^k < 2^n \cdot (2^n)^k = 2^{(k+1)n}$. □

We are now ready to prove Theorem 2.3.3.

PROOF OF THEOREM 2.3.3. Fix a normal p-k-tt-reduction $h(g_1, \dots, g_k)$ from A to B of minimal rank, say r. Note that, by Lemma 2.3.5, $h(g_1, \dots, g_k)$ is almost 1-1 whence, by Lemma 2.3.6, $r > 0$. We first show that there are infinitely many strings x satisfying

$$\begin{aligned} &|x| \leq (k+1)|g_1(x)| \wedge \\ h_x(0, B(g_2(x)), \dots, B(g_k(x))) \;&\neq\; h_x(1, B(g_2(x)), \dots, B(g_k(x))). \end{aligned} \tag{2.5}$$

For a contradiction assume that (2.5) fails for almost all strings x, and fix n such that no string x with $|x| \geq n$ has this property. Define a p-k-tt-reduction $h'(g_1', \dots, g_k')$ as follows. For x with $n \leq |x| \leq (k+1)|g_1(x)|$ let

$$(g_1'(x), \dots, g_k'(x)) = (g_2(x), \dots, g_k(x), \emptyset)$$

and

$$h_x'(j_1, \dots, j_k) = h_x(0, j_1, \dots, j_{k-1})$$

(for any $j_1, \dots, j_k \in S$), and let $(g_1'(x), \dots, g_k'(x)) = (g_1(x), \dots, g_k(x))$ and $h_{x'} = h_x$ otherwise. Note that in the first case,

$$\begin{aligned} h_{x'}(B(g_1'(x)), B(g_2'(x)), \dots, B(g_k'(x))) & \\ &= h_x(0, B(g_2(x)), \dots, B(g_k(x))) \\ &= h_x(B(g_1(x)), B(g_2(x)), \dots, B(g_k(x))), \end{aligned}$$

where the second equality follows from failure of (2.5). So $A \leq_{k\text{-}tt}^p B$ via $h'(g_1', \dots, g_k')$. Moreover, this reduction is normal, and, for almost all x with $|x| \leq (k+1)|g_1(x)|$, $h'(g_1', \dots, g_k')$ is obtained from $h(g_1, \dots, g_k)$ by eliminating the greatest query $g_1(x)$. So the rank of $h'(g_1', \dots, g_k')$ is $r-1$, contrary to minimality of r. So (2.5) holds infinitely often. Hence the condition

$$\begin{aligned} C = \{X{\restriction}y+1 : \exists x[|x| \leq (k+1)|y| \wedge g_1(x) = y \wedge & \\ & h_x(X(g_1(x)), \dots, X(g_k(x))) \neq A(x)]\} \end{aligned}$$

is dense. Moreover, $C \in \mathrm{DTIME}(n(k+1)(d+1))$ and, since $A \leq_{k\text{-}tt}^p B$ via $h(g_1, \dots, g_k)$, B does not meet C. So B is not $n^{(k+1)(d+1)}$-generic. □

2.3.7. COROLLARY. (Small Span Theorem for $\leq_{k\text{-}tt}^p$) *Let $A \in \mathrm{E}$ and $k \geq 1$. Then $\mu(P_{k\text{-}tt}(A)|\mathrm{E}) = 0$ or $\mu_{\mathrm{p}}((P_{k\text{-}tt}^{-1}(A)) = \mu((P_{k\text{-}tt}^{-1}(A)|\mathrm{E}) = 0$.*

PROOF. This is shown as Corollary 2.3.2 using Theorem 2.3.3 in place of Theorem 2.3.1. □

2.3.8. COROLLARY. *Every p-k-tt-degree in* E *and* NP *has p-measure zero.*

PROOF. Immediate by Corollary 2.3.7, since $\deg_{k\text{-}tt}^p(A) = P_{k\text{-}tt}(A) \cap P_{k\text{-}tt}^{-1}(A)$. □

We do not know whether Corollary 2.3.7 can be extended to p-btt-reducibility. Note that in Theorem 2.3.3 the polynomial bound on the genericity for the successors (under p-k-tt-reducibility) of the n^c-generic set $A \in \mathrm{E}$ grows with k so that we do not get a polynomial bound for the successors under all btt-reductions. Recently, Burhrman and van Melkebeek [16] have proved that for any $\alpha < 1$,

$$\mu_{\mathrm{p}}(\{A \in \mathrm{E}_2 | \mu_{\mathrm{p}_2}(P_{n^\alpha\text{-}tt}^{-1}(A)) \neq 0\}) = 0.$$

From this follows the following Small Span Theorem:

2.3.9. THEOREM. *Let $\leq^p_{s\text{-}tt}$ denote a polynomial time truth table reducibility that asks only $s(n)$ queries, where $s(n)$ is asymptotically smaller than n^ε for every $\varepsilon > 0$. Then for any set A it holds that $\mu_{\mathrm{p}}(P_{s\text{-}tt}(A) \cap \mathrm{E}_2) = 0$ or $\mu_{\mathrm{p}_2}(P^{-1}_{s\text{-}tt}(A)) = 0$.*

An interesting consequence of Corollary 2.3.7 is that, for any $k \geq 1$, the class of p-k-tt-hard languages for E has p-measure 0. (Likewise, from Theorem 2.3.9 it follows that the class of $\leq^p_{s\text{-}tt}$-hard sets for E_2 has p_2-measure 0.) A corresponding result for generic sets follows from Theorem 2.3.3.

2.3.10. COROLLARY. *Let A be $n^{5(k+1)}$-generic ($k \geq 1$). Then A is not* E*-hard under p-k-tt-reductions. Hence, the class of* E*-hard languages under p-k-tt-reducibility has p-measure 0.*

PROOF. By Corollary 2.1.4 there is an n^2-generic set in $\mathrm{DTIME}(2^{4n})$, whence, by Theorem 2.3.3, no p-k-tt-hard set for E can be $n^{5(k+1)}$-generic. □

Though Corollary 2.3.10 does not settle the question whether the class of the p-btt-hard languages for E has p-measure 0, we obtain two partial results: First, by Corollary 2.3.10, no P-generic set is p-btt hard for E, whence, by Remark 2.2.4, the class of p-btt-hard problems for E has p_2-measure 0. The second partial result concerns the complete sets. Here, p-measure-0 result follows immediately from Corollary 2.1.10 and Theorem 2.2.3:

2.3.11. THEOREM. $\mu_{\mathrm{p}}(A : A$ *p-btt-complete for* $\mathrm{E}) = 0$.

By using a different method, Buhrman and Mayordomo [15] independently proved a weaker version of the latter two results, namely that the class of the p-btt-complete languages for E has p_2-measure 0.

The question whether there are Small Span Theorems for the weak p-reducibilities, namely polynomial truth-table (p-tt) and polynomial Turing (p-T) reducibility, and the more specific question whether the classes of E-hard problems under these reducibilities have p-measure 0 seem to be much more fundamental. By Theorem 2.1.11 our approach by generic sets fails for the weak reducibilities. Moreover, as observed already by Lutz, these questions may depend on the relation between E and BPP: For the classical measure μ, Bennett and Gill [14] have shown that $\mu(P_T^{-1}(A)) = 1$ if and only if $A \in \mathrm{BPP}$ while Ambos-Spies [2] has shown that $\mu(P_m^{-1}(A)) = 1$ if and only if $A \in P$. Moreover, Ambos-Spies [2] and, independently, Tang and Book [74] extended these results to the intermediate reducibilities by showing that $\mu(P_{tt}^{-1}(A)) = 1$ if and only if $A \in \mathrm{BPP}$ while $\mu(P_{btt}^{-1}(A)) = 1$ if and only if $A \in P$. Since $\mu(\mathcal{C}) = 1$ implies that $\mathcal{C}$ does not have p-measure 0, these results imply that, assuming $\mathrm{E} \subseteq \mathrm{BPP}$, the Small Span Theorem fails for p-tt-reducibility and p-Turing-reducibility and the classes of the E-hard sets under these reducibilities do not have p-measure 0. Moreover, Heller [26] has constructed an oracle relative to which $\mathrm{E}_2 = \mathrm{BPP}$. So a proof of the Small Span Theorem for the weak p-reducibilities would require nonrelativizable techniques.

2.4 Resource bounded measure and randomness

As in Definition 1.5.7 we define

2.4.1. Definition. A set A is $t(n)$-random if no $t(n)$-martingale succeeds on A.

Note that a set A is $t(n)$-random if and only if A does not belong to any class of $t(n)$-measure 0, that is, if and only if the singleton $\{A\}$ does not have $t(n)$-measure 0. By Proposition 1.5.5 (iii) it suffices to consider martingales with rational values, which are not just approximable but *exactly* computable within the given time bound. This observation simplifies the construction of random sets.

The existence of recursive $t(n)$-random sets can be shown by diagonalization: Let $\{d_e : e \in \omega\}$ be a recursive enumeration of the $t(n)$-martingales $d : \Sigma^* \to \mathbb{Q}^+$ with $d(\lambda) = 1$ (for a martingale d which succeeds on a problem we may assume that d is normed: $d(\lambda) = 1$). Define $A(\lambda) = 0$ and, for $w \neq \lambda$, $A(w) = 1 \Leftrightarrow f((A{\restriction}w)0) \geq f((A{\restriction}w)1)$, where

$$f(w) = \sum_{i=0}^{|w|} 2^{-2i} d_i(w).$$

Then, as one can easily check, f is bounded on A whence, by definition of f, any d_i is bounded on A, so that by Proposition 1.5.5 (iii) A is $t(n)$-random.

2.4.2. Theorem. *The class of $t(n)$-random sets has $n^3\, t(2n) \log t(2n)$-measure one.*

Proof. Let $f : \omega \times \Sigma^* \to \mathbb{Q}^+$ be a universal function of the class of the unary $t(n)$-computable functions $g : \Sigma^* \to \mathbb{Q}^+$. We may assume that $f \in \mathrm{FDTIME}(n\, t(n) \log t(n))$. For any e, define a martingale d_e as follows.

$$\begin{aligned} d_e(\lambda) &= f(e, \lambda) \\ d_e(wi) &= \begin{cases} f(e, wi) & \text{if } f(e, w0) + f(e, w1) \leq 2d_e(w) \\ d_e(w) & \text{otherwise} \end{cases} \end{aligned}$$

Obviously, if f_e, where $f_e(x) = f(e, x)$, is a martingale then $d_e = f_e$. So $\{d_e : e \in \omega\}$ is an enumeration of all $t(n)$-martingales, i.e. the function d

with $d(e,x) = d_e(x)$ is a universal function of the $t(n)$-martingales and, by definition, $d \in \mathrm{FDTIME}(n^2 t(n) \log t(n))$. Let $\mathcal{X}_e = \{A \subseteq \Sigma^* : d_e \text{ succeeds on } A\}$ and $\mathcal{X} = \bigcup_{e \in \omega} \mathcal{X}_e$. Then $\mathcal{X}$ is an $(n^2 t(n) \log t(n))$-union of the $(n^2 t(n) \log t(n))$-measure 0 classes $\mathcal{X}_e$, whence, by Lemma 2.2.2, $\mu_{n^3 t(2n) \log t(2n)}(\mathcal{X}) = 0$. Since, by Proposition 1.5.5 (iii), the class of $t(n)$-random sets is the complement of $\mathcal{X}$, it has $(n^3 t(2n) \log t(2n))$-measure 1. □

2.4.3. Corollary. *The class of n^c-random sets ($c \geq 1$) has n^{c+4}-measure 1, hence p-measure 1.*

Lutz and others also studied the class $\mathrm{E}_2 = \mathrm{DTIME}(2^{\mathrm{polynomial}})$. In [50] it is shown that the natural measure on this class is the p_2-measure, where p_2 is the class consisting of all the functions $2^{p(\log n)}$, p a polynomial. By the same proof as above we see that

2.4.4. Corollary. *The class of p-random sets has $n^{\log n}$-measure 1, hence p_2-measure 1.*

Note that by Theorem 1.5.6, if $\mu(\mathcal{C}|\mathrm{E}) = 1$ then $\mathcal{C} \cap \mathrm{E} \neq \emptyset$, and if $\mu(\mathcal{C}|\mathrm{E}_2) = 1$ then $\mathcal{C} \cap \mathrm{E}_2 \neq \emptyset$. So Corollaries 2.4.3 and 2.4.4 imply

2.4.5. Corollary. (i) *For any $c \geq 1$, the class of n^c-random sets has measure 1 in* E. *In particular there is an n^c-random set in* E.

(ii) (Lutz [50]) *The class of p-random sets has measure 1 in E_2. In particular there is a p-random set in E_2.*

Note that, for time bounds t and t' such that $t'(n) \leq t(n)$ almost everywhere, any $t(n)$-random set is $t'(n)$-random. So any p-random set is n^c-random, and any n^c-random set is $n^{c'}$-random, for any $c' \leq c$. Conversely, by diagonalization we can show that there are n^c-random sets which are not n^{c+1}-random (for any $c \geq 1$). So these concepts of randomness give rise to a proper hierarchy.

Also note that the existence results for n^c-random and p-random sets in Corollary 2.4.5 can be easily extended to the general case: If in the construction of a $t(n)$-random set A described above (after Definition 2.4.1) we use an enumeration of the $t(n)$-martingales as in the proof of Theorem 2.4.2, then $A \in \mathrm{DTIME}(t'(2^{n+1}))$ for $t'(n) = n^4 t(n) \log t(n)$.

Some further basic properties of random sets are stated in the following lemma.

2.4.6. Lemma. *Let A be a $t(n)$-random set. Then the following hold:*

(i) *The complement $\overline{A}$ of A is $t(n)$-random.*

(ii) *A is* dense, *i.e., there exists an $\epsilon > 0$ such that $|A^{\leq n}| > 2^{n^\epsilon}$ for almost every n.*

PROOF. To prove (i), suppose that the $t(n)$-martingale d succeeds on $\overline{A}$. Then d' defined by $d'(w) = d(\overline{w})$ succeeds on A, where $\overline{w}$ is the unique string of length $|w|$ such that $\overline{w}(i) = 1 - w(i)$ for $i < |w|$.
For a proof of (ii), it suffices to show that the class of nondense sets has n-measure 0, since $t(n) \geq n$ a.e. n. Define the n-martingale $d : \Sigma^* \to \mathbb{Q}^+$ by $d(\lambda) = 1$, $d(w0) = 3/2 \cdot d(w)$, and $d(w1) = 1/2 \cdot d(w)$. If B is a nondense set then $|B_{\leq n}| \leq 2^{\sqrt{n}}$ for infinitely many n. However, $|\Sigma^{\leq n}| = 2^{n+1} - 1$, so

$$\limsup_{n\to\infty} d(B{\restriction}z_n) \geq \lim_{n\to\infty} ((3/2)^{2^{n+1}-1-2^{\sqrt{n}}} \cdot (1/2)^{2^{\sqrt{n}}}) = \infty. \qquad \square$$

Note that many more much stronger properties than the above can be proven (such as the various stochastic properties from probability theory, or such as the Weak Stochasticity Theorem from [54]), but we will not need these in the sequel.

2.5 Resource bounded genericity and randomness

Theorem 2.2.3 shows that many properties which occur with p-measure 1 are shared by all n^c-generic sets ($c \geq 2$). Generic sets are designed to be universal for standard resource bounded diagonalization arguments. In such a diagonalization argument, a single diagonalization step corresponding to one of the subrequirements has to be performed only once and only under the proviso that there are infinitely many chances to do so. Though, in general, this easily implies that the action for a single requirement will be performed infinitely often (provided there are infinitely many chances to do so), we cannot say anything about the frequency with which the opportunities are taken. The latter contrasts with a typical measure one construction where we have to take the majority of the opportunities. To illustrate this difference we consider the density of a set. We have shown already that a generic set can be sparse (Corollary 2.1.4). However, as first observed by Lutz and Mayordomo [54], the class of sparse sets has p-measure 0. To see this consider the n^2-martingale $d : \Sigma^* \to \mathbb{Q}^+$ defined by $d(\lambda) = 1$, $d(x0) = 3/2 \cdot d(x)$, and $d(x1) = 1/2 \cdot d(x)$. Then it is easy to see that d succeeds on any sparse set, in fact on any set which is not exponentially dense (cf. Lemma 2.4.6 (ii)).

Though this example points out limitations of the generic set approach to p-measure 1-results, we would like to emphasize that the generic sets help us to distinguish between those properties that can be forced by standard diagonalizations and those which require a measure diagonalization argument. Moreover, this example also shows that the assumption that a class $\mathcal{C}$ contains an n^c-generic set is weaker than the assumption that $\mathcal{C}$ has nonzero p-measure. This observation might be of particular interest when studying the structure of NP assuming that NP is sufficiently large. Lutz defines that NP is not small if $\mu_{\mathrm{p}}(\mathrm{NP}) \neq 0$, and in [55] he and Mayordomo proved that under this non-smallness hypothesis p-T-completeness and p-m-completeness for NP do not coincide. By Corollary 2.1.9 this result already follows from the (apparently weaker) assumption that NP contains an n^2-generic set.

Moreover, the relations between resource bounded genericity and measure which we explored here for the polynomial case hold for arbitrary time (and space) bounds. In particular, as shortly indicated in Remarks 2.1.12 and 2.2.4 already, we obtain corresponding results for the p_2-measure analysis of E_2 by Lutz.

The next theorem shows that the approach using random sets is a refinement of the method of using generic sets. The proof is essentially the same as the proof of Theorem 2.2.3 showing that the n^c-generic sets have p-measure 1.

2.5.1. THEOREM. *Let A be n^{c+1}-random. Then A is n^c-generic. Hence any p-random set is p-generic.*

PROOF. Let $C \in \mathrm{DTIME}(n^c)$ be a condition which is dense along A. To show that A meets C, define $d : \Sigma^* \to \mathbb{Q}^+$ by $d(\lambda) = 1$ and, for w in Σ^* and $i \leq 1$,

$$d(wi) = \begin{cases} 0 & \text{if } wi \in C \wedge w(1-i) \notin C \\ 2d(w) & \text{if } w(1-i) \in C \wedge wi \notin C \\ d(w) & \text{otherwise} \end{cases}$$

Then $d \in \mathrm{FDTIME}(n^{c+1})$ is a martingale whence, by n^{c+1}-randomness of A, $\limsup_n d(A{\upharpoonright}z_n) < \infty$. By density of C along A and by definition of d this implies that A meets C. □

The converse of Theorem 2.5.1 fails: by Lemma 2.4.6, any n^c-random set is dense whereas by Corollary 2.1.4 there exist sparse n^c-generic sets. Intuitively, the difference between $t(n)$-genericity and $t(n)$-randomness can be described as follows: Both concepts are universal for $t(n)$-bounded diagonalizations. In case of genericity, however, we only require that, for any single condition, if there are infinitely many chances to meet the condition then the condition has to be met at least once, or (as one can easily check) equivalently, *infinitely* often. In case of randomness this does not suffice; here a *majority* of the chances has to be taken.

In Sections 2.1 and 2.2 numerous properties of the n^c-generic sets were proven. By Theorem 2.5.1 these properties are shared by all n^{c+1}-random sets. For instance, in Section 2.1 we showed that n^c-generic sets are not p-btt-complete for E and that p-generic sets are not p-btt-hard for E, and in Section 2.3 that the genericity of successors of n^c-generic sets in E is limited ($c \geq 2$). So we obtain the corresponding results for n^c-random sets:

2.5.2. COROLLARY. (i) *If A is n^c-random ($c \geq 3$), then A is not p-btt-complete for* E.

(ii) *If A is p-random, then A is not p-btt-hard for* E.

2.5.3. COROLLARY. *Let A and B be sets such that $A \leq_m^p B$, A is n^c-random and $A \in \mathrm{DTIME}(2^{dn})$, where $c,d \geq 3$. Then B is not n^{d+1}-random.*

Corollary 2.5.3 shows that, for any n^c-random set $A \in \mathrm{E}$ there is a bound on the polynomial randomness of the successors of A (under p-m-reducibility). The reason for this is the following: If $A \leq_m^p B$ via f, then, by n^2-randomness, f cannot compress A, so that $f(A)$ contains an infinite $2^{(d+1)n}$-computable subset of B. An n^{d+1}-random set, however, does not have such *easy* infinite parts.

2.6 RANDOMNESS BELOW A RANDOM SET

Here we will contrast the preceding result on the limitations on randomness of the successors of an exponential time computable n^c-random set by showing that any such set has *predecessors* of arbitrarily high polynomial randomness.

2.6.1. THEOREM. *Let A be an n^2-random set. For any $k \geq 1$ there is an n^k-random set A_k with $A_k \leq_m^p A$. In fact, there is a p-random set A_∞ with $A_\infty \leq_m^p A$. Also, for any $k \geq 1$, there is a $2^{(\log n)^k}$-random set $B_k \leq_m^p A$. If, moreover, $A \in \mathrm{E}$ then A_k and A_∞ can be chosen so that $A_k \in \mathrm{E}$ and $A_\infty \in \mathrm{DTIME}(2^{n^2})$, and if $A \in \mathrm{E}_2$ then B_k can be chosen to be in E_2.*

The idea underlying the proof of Theorem 2.6.1 is the following. If we restrict the domain D of a random set A then, relative to this domain, $A \cap D$ remains random. So if we take the restriction of A to some polynomially scattered domain D and polynomially compress $A \cap D$ by mapping D onto Σ^* then, for the compressed version A_D of $A \cap D$, time complexity and randomness increase by a polynomial factor but still A_D can be reduced in polynomial time to $A \cap D$ and hence to A. The formal proof of Theorem 2.6.1 requires the following lemma, which uses the idea above in a slightly more general form.

2.6.2. LEMMA. *Let A be $n\,t(n)$-random for a nondecreasing function t with $t(n) \geq n$ a.e. , and let $f : \omega \to \omega$ be a nondecreasing time constructible function. Then*

$$A_f = \{x : 0^{f(|x|)}1x \in A\} \textit{ is } t(2^{f(\log(n)-1)})\textit{-random.}$$

PROOF. Define $t'(n) = t(2^{f(\log(n)-1)})$, and let $d : \Sigma^* \to \mathbb{Q}^+$ be a $t'(n)$-computable martingale. We will show that d does not succeed on A_f. To prove this it suffices, by $n\,t(n)$-randomness of A, to define an $n\,t(n)$-martingale $\hat{d}$ such that

$$d \text{ succeeds on } A_f \Rightarrow \hat{d} \text{ succeeds on } A.$$

For the definition of $\hat{d}$ we will use the following notation: For a string $X\restriction(0^{f(|x|)}1x)$, let $\widetilde{X}\restriction x$ be defined by $\widetilde{X}(y) = X(0^{f(|y|)}1y)$ for $y < x$. Now $\hat{d}$ is defined by induction as follows:

1. $\hat{d}(\lambda) = d(\lambda)$,
2. For $y = 0^{f(|x|)}1x$ and $i \le 1$, $\hat{d}((X\restriction y)i) = d((\widetilde{X}\restriction x)i)$,
3. For y not of the form $0^{f(|x|)}1x$ and $i \le 1$, $\hat{d}((X\restriction y)i) = \hat{d}(X\restriction y)$.

Since d is a martingale, a straightforward induction on $|x|$ shows that

$$\hat{d}((X\restriction x)0) + \hat{d}((X\restriction x)1) \le 2\hat{d}(X\restriction x)$$

and, by definition of A_f and $\hat{d}$, for $i \le 1$

$$\hat{d}(A\restriction(0^{f(|x|)}1x)i) = d((A_f\restriction x)i).$$

So $\hat{d}$ is a martingale which succeeds on A if d succeeds on A_f. It remains to show that $\hat{d}$ is computable in time $n\,t(n)$. By induction, it suffices to show that, given $\hat{d}(\lambda), \dots, \hat{d}(X\restriction y)$, the value of $\hat{d}(X\restriction(y+1))$ can be computed in $O(t(|X\restriction(y+1)|))$ steps. Now fix any y and let $m = |X\restriction(y+1)|$. Since f is time constructible the time required for the decision whether or not y is of the form $0^{f(|x|)}1x$ is polynomial in the length of y, hence, by equation (2.1) on page 18 and $t(n) \ge n$, linear in $t(m)$. So it suffices to analyze the cases 2. and 3. in the definition of $\hat{d}$ individually. The case 3. is trivial by induction hypothesis. For a proof of the case 2. fix $x \in \Sigma^*$ and $i \le 1$ such that $y = 0^{f(|x|)}1x$ and $(X\restriction(y+1))(y) = i$. Then, by definition of $\hat{d}$, $\hat{d}(X\restriction(y+1)) = d((\widetilde{X}\restriction x)i)$, whence it suffices to show that $d((\widetilde{X}\restriction x)i)$ can be computed in $O(t(m))$ steps. Now it follows from $|y| = f(|x|) + |x| + 1$ and the monotonicity of t that

$$\begin{aligned}
t(m) = t(|X\restriction(y+1)|) &\ge t(2^{|y|}) && \text{(by equation (2.1))}\\
&= t(2^{f(|x|)+|x|+1})\\
&\ge t(2^{f(|x|)})\\
&= t'(2^{|x|+1})\\
&\ge t'(|(\widetilde{X}\restriction x)i|) && \text{(by equation (2.1))}.
\end{aligned}$$

Since d is $t'(n)$-computable this implies that $d((\widetilde{X}\restriction x)i)$ can be computed in $O(t(m))$ steps. This completes the proof of Lemma 2.6.2. $\square$

PROOF OF THEOREM 2.6.1. Let $t(n) = n$ and let A be n^2-random. Fix $k \in \omega$ and define $f_0(n) = k \cdot n$, $f_1(n) = (n+1)\log(n+1)$, and $f_2(n) = n^{k+1}$. It is easy to see that for $i \leq 2$,

$$A_{f_i} = \{x : 0^{f_i(|x|)}1x \in A\} \leq_m^p A.$$

Now define $A_k = A_{f_0}$, $A_\infty = A_{f_1}$, and $B_k = A_{f_2}$. Then by Lemma 2.6.2, A_k is $2^{-k} \cdot n^k$-random, hence n^k-random, A_∞ is $n^{\log\log n}$-random, hence p-random, and B_k is $2^{(\log(n)-1)^{k+1}}$-random, hence $2^{(\log n)^k}$-random. For a proof of the second part fix c such that $A \in \mathrm{DTIME}(2^{cn})$. Then, as one can easily check, $A_k \in \mathrm{DTIME}(2^{(k+1)cn}) \subset \mathrm{E}$ and $A_\infty \in \mathrm{DTIME}(2^{c(n+1)(\log(n+1)+1)}) \subset \mathrm{DTIME}(2^{n^2})$. If A is in $\mathrm{DTIME}(2^{n^c})$ then $B_k \in \mathrm{DTIME}(2^{n^{c(k+2)}}) \subset \mathrm{E}_2$. □

It follows from Theorem 2.6.1 that classes that are closed under $\leq_m^p$-reductions, like NP, UP, PP, or PSPACE, contain an n^2-random set if and only if they contain a p-random set if and only if they contain a $2^{(\log n)^k}$-random set.

2.7 Random sets are weakly complete

In this final section we apply our results on random sets to study the weakly complete problems in E and E_2. We first review this concept of Lutz [51, 52]. For any set A, let $P_m(A) = \{B : B \leq_m^p A\}$. Then A is *weakly hard* for E if $\mu(P_m(A)|\mathrm{E}) \neq 0$; if moreover $A \in \mathrm{E}$ then we say that A is *weakly complete* for E. Weak completeness for E_2 is defined in the same way, using p_2 and E_2 instead of p and E. Lutz [52] showed that there is a weakly complete set in E which is not p-m-complete for E. To show this Lutz introduced a quite involved new diagonalization technique which he calls *martingale diagonalization*. Our results on random sets provide an elementary proof of this fact and yield stronger results.

2.7.1. THEOREM. (i) *A is weakly hard for* E *if and only if $P_m(A) \cap \mathrm{E}$ contains an n^2-random set.*

(ii) *A is weakly hard for E_2 if and only if $P_m(A) \cap \mathrm{E}_2$ contains an n^2-random set.*

PROOF. (i) If A is weakly hard for E then $P_m(A) \cap \mathrm{E}$ contains an n^2-random set by Corollary 2.4.5 (i). Now suppose that $P_m(A) \cap \mathrm{E}$ contains an n^2-random set. Then, by Theorem 2.6.1, $P_m(A) \cap \mathrm{E}$ contains an n^k-random set for every $k \in \omega$. But this means that there is no n^k-martingale which succeeds on every set in $P_m(A) \cap \mathrm{E}$, whence $\mu_{\mathrm{p}}(\{B : B \leq_m^p A\} \cap \mathrm{E}) \neq 0$. Assertion (ii) follows from Corollary 2.4.5 (ii) and Theorem 2.6.1 with a similar argument. □

2.7.2. COROLLARY. *Let $A \in \mathrm{E}\,(\mathrm{E}_2)$ be n^2-random. Then A is weakly complete for* $\mathrm{E}\,(\mathrm{E}_2)$.

PROOF. Immediate from Theorem 2.7.1. □

Corollary 2.7.2 can be strengthened a little bit as follows.

2.7.3. THEOREM. *Let $A \in \mathrm{E}_2$ be n^2-random. Then $P_m(A)$ is not* p_2*-measurable in* E_2.

PROOF. By Corollary 2.7.2, $\mu_{\mathrm{p}_2}(P_m(A)) \neq 0$. By Theorem 2.4.6 (i), $\overline{A}$ is also n^2-random, hence by Lemma 2.6.2, the set

$$\widetilde{A} = \{x10^{|x|^{c+1}} : x \in \overline{A}\}$$

is $2^{\log^c n}$-random. Since $\widetilde{A} \equiv^p_m \overline{A}$, and because by Theorem 2.1.8 (ii) it holds that $\overline{A} \not\leq^p_m A$, we have that $\widetilde{A} \not\leq^p_m A$. Hence there is no $2^{\log^c n}$-martingale that succeeds on $\mathrm{E}_2 - P_m(A)$. Because c was arbitrary it follows that $\mu_{\mathrm{p}_2}(\mathrm{E}_2 - P_m(A)) \neq 0$, so $\mu_{\mathrm{p}_2}(P_m(A)|\mathrm{E}_2) \neq 1$. So $P_m(A)$ has neither measure zero nor measure one in E_2, and it follows from an effective version of Theorem 1.4.4 (Lutz [49, Thm 5.15]) that $P_m(A)$ is not measurable in E_2 □

In contrast to Corollary 2.7.2, note that for any $k \geq 1$ there are n^k-generic sets in E which are not weakly complete for E. This follows from the fact (Theorem 2.1.4) that for every k there are sparse n^k-generic sets in E, and the result of Lutz and Mayordomo [54] that for sparse sets A, $\mu_{\mathrm{p}}(P_m(A)) = 0$. (This last result holds because a set that p-m-reduces to a sparse set is either compressible or sparse, and Juedes and Lutz [31] proved that the class of compressible sets has p-measure zero, and we already noted that the class of sparse sets has p-measure zero.

Juedes and Lutz [32] proved the following relation between completeness for E and E_2. We now show how Corollary 2.5.3 and Theorem 2.7.1 can be used to give a very direct proof of their result.

2.7.4. COROLLARY. (Juedes and Lutz [32]) (i) *If A is weakly complete for* E *then A is also weakly complete for* E_2.

(ii) *There exists a set $A \in \mathrm{E}$ which is weakly complete for* E_2 *but not for* E.

PROOF. (i) Since E is contained in E_2, this is immediate by Theorem 2.7.1.

(ii) By Corollary 2.4.5 (ii), let $B \in \mathrm{E}_2$ be p-random, and by padding B, let $A \in \mathrm{E}$ be a set with $A \equiv^p_m B$. Then A is weakly complete for E_2 by Theorem 2.7.1. However, by Corollary 2.5.3, $P_m(A) \cap \mathrm{E}$ does not contain any n^2-random set, so by Theorem 2.7.1 A is not weakly complete for E. □

By Corollary 2.7.2, we can extend Lutz's theorem on the existence of proper weakly complete sets from p-m-reducibility to p-btt-reducibility:

2.7.5. COROLLARY. *There is a weakly complete set for* E *which is not p-btt-complete for* E.

PROOF. By Corollary 2.4.5 there is an n^2-random set A in E and, by Corollary 2.7.2 and Corollary 2.5.2, A is weakly complete but not p-btt-complete for E. $\square$

We do not know whether there are weakly complete sets which are not p-tt-complete or even not p-T-complete. As our final result shows, however, to prove this it suffices to show that the classes of *in*complete sets under these reducibilities in E do not have p-measure 0.

2.7.6. COROLLARY. *$\mu_{\mathrm{p}}(\{A : A$ weakly complete for $\mathrm{E}\}) \neq 0$. In fact, $\mu(\{A : A$ weakly complete for $\mathrm{E}\}|\mathrm{E}) = 1$. Similarly, for the measure in E_2 we have $\mu(\{A : A$ weakly complete for $\mathrm{E}_2\}|\mathrm{E}_2) = 1$.*

PROOF. By Corollary 2.7.2 this follows from Corollary 2.4.5 and the fact that every p-random set is n^2-random. $\square$

Juedes [29] independently proved the first part of Corollary 2.7.6 (namely that the weakly complete sets for E do not have p-measure 0) using Lutz's martingale diagonalization technique.

Chapter 3

Measure on the Recursively Enumerable Sets

In this chapter we investigate a measure for the class of recursively enumerable sets RE. We define a class $\mathfrak{a}$ of Δ_2–martingales that mimic the asymmetry of the r.e. sets in such a way that for the resulting measure $\mu_{\mathfrak{a}}$ it holds that $\mu_{\mathfrak{a}}(\text{RE}) \neq 0$ and $\mu_{\mathfrak{a}}(\text{REC}) = 0$. We shall see that $\mu_{\mathfrak{a}}$ does not satisfy an effective analogue of the σ-additivity, so that $\mu_{\mathfrak{a}}$ is not an effective measure in the proper sense. However, it seems implausible that such a measure would exist, apart from trivial solutions such as measures induced by classes consisting of just one martingale, or classes constructed by direct diagonalization[1]. So instead of $\mu_{\mathfrak{a}}$ we consider its closure under finite unions $\bar{\mu}_{\mathfrak{a}}$. We study the weak completeness notions corresponding to this finitely additive measure and obtain a complete picture of the relations between these and the ordinary completeness notions. The proofs have also consequences for a question that is not related to measure theory, namely to what extent an incomplete set can resemble a complete set. This is discussed in Section 3.3.

3.1 The class of martingales $\mathfrak{a}$

3.1.1. Definition. The class $\mathfrak{a} \subset \Delta_2$ ('a' for 'asymmetric') consists of all martingales $d : 2^{<\omega} \to \mathbb{Q}$ such that

[1] It is possible to construct by direct diagonalization a class of martingales $\Delta \subset \Delta_2$ such that every martingale in Δ covers REC, but no martingale in Δ covers RE. Namely, for every nonrecursive r.e. set A, construct with help of the oracle $\emptyset'$ a martingale d_A that covers REC but not A, by letting d_A bet on a 0 only if the next bit of A is a 1, and conversely. Since every recursive set differs from A infinitely often we can still let d_A cover REC. Let Δ consist of all the d_A.

(i) there is a recursive function $f : \omega \times 2^{<\omega} \to \mathbb{Q} \cap [0,2]$ such that for every w,

- $f(s, w0)$ is monotonic nondecreasing in s and $(\exists s)(\forall t \geq s)[f(t, w0) = d(w0)/d(w)]$,
- $f(s, w1)$ is monotonic nonincreasing in s and $(\exists s)(\forall t \geq s)[f(t, w1) = d(w1)/d(w)]$.

(ii) d is *oblivious* [7], that is, for every $w, v \in 2^{<\omega}$ such that $|w| = |v|$ it holds that $d(w0)/d(w) = d(v0)/d(v)$ and $d(w1)/d(w) = d(v1)/d(v)$.

So for a martingale $d \in \mathfrak{a}$ the betting percentages $d(wi)/d(w)$ are monotonically recursively approximable, from below if $i = 0$ and from above if $i = 1$. Furthermore, the martingale d only bets on elements separately, without respect to previous outcomes. It is so to say 'oblivious' to what happened previously. Oblivious martingales were introduced by Ambos-Spies et al. in [7].

3.1.2. DEFINITION. We denote the measure induced by the class $\mathfrak{a}$ by $\mu_{\mathfrak{a}}$. So $\mu_{\mathfrak{a}}(\mathcal{A}) = 0$ if there is a martingale $d \in \mathfrak{a}$ such that $\mathcal{A} \subseteq S[d]$, where

$$S[d] = \{A : \limsup_{n \to \infty} d(A{\restriction}n) = \infty\}.$$

The closure of $\mu_{\mathfrak{a}}$ under finite unions is denoted by $\bar{\mu}_{\mathfrak{a}}$. So $\bar{\mu}_{\mathfrak{a}}(\mathcal{A}) = 0$ if there exist martingales $d_1, \ldots, d_k \in \mathfrak{a}$ such that $\mathcal{A} \subseteq \bigcup_{i=1}^{k} S[d_i]$.
We say that $\mathcal{A}$ has *measure zero in* RE if $\bar{\mu}_{\mathfrak{a}}(\mathcal{A} \cap \mathrm{RE}) = 0$, and that $\mathcal{A}$ has *measure one in* RE if $\bar{\mu}_{\mathfrak{a}}(\mathcal{A}^c \cap \mathrm{RE}) = 0$.

3.1.3. PROPOSITION. *The 'measure' $\mu_{\mathfrak{a}}$ is not closed under finite unions. Hence $\mu_{\mathfrak{a}} \neq \bar{\mu}_{\mathfrak{a}}$. Even $\mu_{\mathfrak{a}}{\restriction}\mathrm{RE} \neq \bar{\mu}_{\mathfrak{a}}{\restriction}\mathrm{RE}$.*

PROOF. Let d_0 and d_1 be the $\mathfrak{a}$–martingales defined by

$$\begin{array}{ll} d_0(w0) = 3/2 \cdot d_0(w) & d_1(w0) = 1/2 \cdot d_1(w) \\ d_0(w1) = 1/2 \cdot d_0(w) & d_1(w1) = 3/2 \cdot d_1(w) \end{array}$$

We prove that for every martingale $d \in \mathfrak{a}$ there exists an r.e. set A such that $A \in S[d_0] \cup S[d_1] - S[d]$. Let $I_n = [3^n, 3^{n+1} - 1]$, so that $|I_n| = 3 \cdot \sum_{m<n} |I_m|$. Let $d \in \mathfrak{a}$. Define A by the following r.e. procedure: As long as the approximation to $\prod_{i \in I_n} d(0^{i+1})/d(0^i)$ is smaller than $\prod_{i \in I_n} d(0^i 1)/d(0^i)$ let $A \cap I_n$ be empty, and let $A \cap I_n = I_n$ otherwise. Now A is an r.e. set because d is in $\mathfrak{a}$, and we claim that A satisfies the proposition.
Firstly, $A \notin S[d]$: It is enough to prove that d does not grow on A on every interval I_n. Fix n and suppose that for $i \in I_n$, $d(0^{i+1})/d(0^i) = 1 + \tau_i \varepsilon_i$, $\tau_i \in \{-1, 1\}$, $\varepsilon_i \in [0,1]$. Since $\prod(1 + \tau_i \varepsilon_i) \prod(1 - \tau_i \varepsilon_i) = \prod(1 - {\varepsilon_i}^2) \leq 1$ it holds that at least

one of the products $\prod(1+\tau_i\varepsilon_i)$, $\prod(1-\tau_i\varepsilon_i)$ is smaller than 1, which means that d does not grow on either the all-zero extension or the all-one extension of $A\lceil 3^n$ on I_n. Since A on I_n was defined accordingly it follows that $A \notin S[d]$.
Secondly, $A \in S[d_0] \cup S[d_1]$: Note that if $A \cap I_n = \emptyset$ then $d_0(A\lceil 3^{n+1}) \geq (1/2)^{c_n}(3/2)^{2c_n}$ where $c_n = \sum_{m<n}|I_m|$. Hence if $A \cap I_n = \emptyset$ for infinitely many n then d_0 succeeds on A. Likewise, if $A \cap I_n = I_n$ for infinitely many n then d_1 succeeds on A. Since one of these must be the case it follows that $A \in S[d_0] \cup S[d_1]$. □

To prove that REC has measure zero in RE we will need an r.e. set such that no recursive method can guess more than half of its elements. Does the halting set K have this property? A little reflection shows that this depends on the numbering of the recursive functions. It is possible to construct a numbering with all the usual properties such that K becomes very easy, from a probabilistic point of view. However, as we shall see in the proof of the next lemma, given any Gödel numbering it is possible to construct an r.e. set with the desired property. In Section 3.3 we will have more to say about the question how much an incomplete set can resemble a complete set.

3.1.4. LEMMA. *There exists an r.e. set M such that for every $A \in \Pi_1$,*

$$\liminf_{n\to\infty} \frac{\|\{i \leq n : A(i) = M(i)\}\|}{n} = 0$$

PROOF. Let $\{W_e\}_{e\in\omega}$ be the standard recursive enumeration of the r.e. sets and let $V_e = \overline{W}_e$. Enumerate M in stages as follows. Given the part M_s of M constructed at stage s we reserve the next $(s-1)|M_s|$ bits to make $\|\{i \leq s|M_s| : V_s(i) = M(i)\}\| \leq |M_s|$. Enumerate an element at one of these reserved positions if and only if this element is co-enumerated in V_s. This guarantees that M differs from V_s in this reserved block of length $(s-1)|M_s|$ at every position, so we have $\|\{i \leq s|M_s| : V_s(i) = M(i)\}\|/s|M_s| \leq 1/s$. Because every Π_1 set A has infinitely many codes s the r.e. set M is equal to the complement of A on infinitely many of the reserved intervals. □

Lynch [56] proved that there is a creative set M such that Lemma 3.1.4 holds for all $A \in$ REC. This answered a question posed by Meyer. Since being creative is equivalent with being m-complete, and because the set M from Lemma 3.1.4 is easily seen to be m-complete, Lemma 3.1.4 can be seen as a strengthening of Lynch's result.

3.1.5. LEMMA. *For every class $\mathcal{A} \subseteq 2^\omega$ it holds that if there is an r.e. set M and an $\varepsilon > 0$ such that for every $A \in \mathcal{A}$,*

$$\liminf_{n\to\infty} \frac{\|\{i \leq n : A(i) = M(i)\}\|}{n} \leq \frac{1}{2} - \varepsilon, \tag{3.1}$$

then $\mu_{\mathfrak{a}}(\mathcal{A}) = 0$.

PROOF. Let $\delta \in (0,1)$ be a rational number. Define a martingale d by

$$\begin{aligned} d(w0) &= \begin{cases} (1-\delta)d(w) & \text{if } |w| \notin M \\ (1+\delta)d(w) & \text{if } |w| \in M \end{cases} \\ d(w1) &= 2d(w) - d(w0). \end{aligned}$$

One can easily check that d is a martingale in $\mathfrak{a}$. Note that $d(wi) = (1-\delta)d(w)$ if and only if $M(|w|) = i$. It follows that A is a set satisfying (3.1) if and only if for infinitely many n it holds that

$$\|\{i \leq n : d(A\restriction n+1)/d(A\restriction n) = 1-\delta\}\| \leq (\frac{1}{2} - \varepsilon)n. \tag{3.2}$$

Because $(1-\delta)^{\alpha n}(1+\delta)^n$ grows to infinity if and only if $\alpha < \alpha_\delta = \frac{-\log(1+\delta)}{\log(1-\delta)}$, $A \in S[d]$ if and only if for infinitely many n,

$$\begin{aligned} \frac{\|\{i \leq n : d(A\restriction n+1)/d(A\restriction n) = 1-\delta\}\|}{n} &< \frac{\alpha_\delta}{\alpha_\delta + 1} \\ &= \frac{1}{1 - \frac{\log(1-\delta)}{\log(1+\delta)}} \end{aligned} \tag{3.3}$$

Since the limit for $\delta \downarrow 0$ of the right-hand side is $1/2$, by (3.2) and by taking δ small enough we can ensure that (3.3) holds for all $A \in \mathcal{A}$, and hence that $\mathcal{A} \subseteq S[d]$. □

3.1.6. THEOREM. (i) $\mu_{\mathfrak{a}}(\mathrm{RE}) \neq 0$.

(ii) $\mu_{\mathfrak{a}}(\mathrm{REC}) = \mu_{\mathfrak{a}}(\Pi_1) = 0$.

PROOF. To prove (i) let d be a martingale in $\mathfrak{a}$. Let f be the function approximating the betting percentages of d, as in Definition 3.1.1. Define an r.e. set V generated by the following algorithm. Enumerate x in V if there is an s such that $f(s, 0^{x+1}) \geq 1$. Then

$$\begin{aligned} x \in V &\Leftrightarrow (\exists s)[f(s, 0^{x+1}) \geq 1] \\ &\Leftrightarrow d(0^{x+1})/d(0^x) \geq 1 && (f \text{ monotonic}) \\ &\Leftrightarrow d((V\restriction x+1)0)/d(V\restriction x+1) \geq 1 && (|V\restriction x+1| = |0^x| \text{ and } d \\ & && \text{is oblivious}). \end{aligned}$$

So d bets on '$x \notin V$' if and only if $x \in V$, hence d does not succeed on V. Since V is r.e. it follows that $\mu_{\mathfrak{a}}(\mathrm{RE}) \neq 0$.

For (ii), let M be the r.e. set from Lemma 3.1.4. Then M witnesses that the class Π_1 satisfies the condition of Lemma 3.1.5, hence $\mu_{\mathfrak{a}}(\Pi_1) = 0$. Since REC is a subset of Π_1 we have immediately that $\mu_{\mathfrak{a}}(\mathrm{REC}) = 0$ also. □

Since we actually want to use $\bar{\mu}_{\mathfrak{a}}$ rather than $\mu_{\mathfrak{a}}$, because the last one is not finitely additive, we need a result that is stronger than Theorem 3.1.6, namely we need that $\bar{\mu}_{\mathfrak{a}}(\mathrm{RE}) \neq 0$. We prove this now.

3.1.7. LEMMA. *Let* $a_i \in \mathbb{R}$, $1 \leq i \leq k$, *be such that* $\sum_{i=1}^k a_i \leq k$, *and for every* i *let* $0 \leq \varepsilon_i \leq 1$ *and* $\tau_i \in \{-1,1\}$. *Then* $\sum_{i=1}^k (1+\tau_i\varepsilon_i)a_i \leq k$ *or* $\sum_{i=1}^k (1-\tau_i\varepsilon_i)a_i \leq k$.

PROOF. Suppose that $\sum_{i=1}^k (1+\tau_i\varepsilon_i)a_i > k$. Then $\sum \tau_i\varepsilon_i a_i > k - \sum a_i \geq 0$, so $\sum(1-\tau_i\varepsilon_i)a_i = \sum a_i - \sum \tau_i\varepsilon_i a_i \leq k - \sum \tau_i\varepsilon_i a_i < k$. □

3.1.8. THEOREM. $\bar{\mu}_{\mathfrak{a}}(\mathrm{RE}) \neq 0$, *that is,* RE *is not contained in any finite union* $\bigcup_{i=1}^k S[d_i]$ *of success sets of* $\mathfrak{a}$*–martingales* d_i.

PROOF. Let $d_1 \dots d_k$ be $\mathfrak{a}$–martingales. For every d_i, let f_i satisfy Definition 3.1.1 (with d_i instead of d). Without loss of generality we may assume that for every i, $d_i(\lambda) = 1$ and

$$(\forall s)(\forall w)[f_i(s,w0) + f_i(s,w1) = 2]. \tag{3.4}$$

Note that then for every $w > \lambda$ it holds that

$$\begin{aligned} d_i(w) &= \lim_{s\to\infty} \prod_{n=1}^{|w|} \frac{d_i(w\restriction n+1)}{d_i(w\restriction n)} \cdot d_i(\lambda) \\ &= \lim_{s\to\infty} \prod_{n=1}^{|w|} f_i(s, w\restriction n+1). \end{aligned}$$

Define $d_i^s(w) = \prod_{n=1}^{|w|} f_i(s, w\restriction n+1)$. We will enumerate an r.e. set X such that for every initial segment $\sigma \sqsubset X$ and every s and k we have $\sum_{i=1}^k d_i^s(\sigma) \leq k$. This guarantees that none of the d_i grows to infinity on X. We will in fact require that this holds for any $\sigma = X_s$, the part of X constructed by stage s, at every stage of the construction.
Stage $s=0$. Set $X_0 = \emptyset$. Note that $\sum_{i=1}^k d_i(X_0\restriction 0) = \sum_{i=1}^k d_i(\lambda) \leq k$.
Stage $s+1$. At this stage X_s is given, containing only elements less than or equal to s and such that $\sum_{i=1}^k d_i^s(X_s\restriction x) \leq k$ for every $x \leq s+1$. Without loss of generality there is at most *one* $x < s+1$ such that $f_i(s, X_s\restriction x+1) \neq f_i(s+1, X_s\restriction x+1)$. If there is no such x then do nothing, i.e. set $X_{s+1} = X_s$ and proceed to the next stage.

Otherwise, if $x < s+1$ is such that $f_i(s, X_s\restriction x+1) \neq f_i(s+1, X_s\restriction x+1)$, let $\tau_i \in \{-1,1\}$ and $\varepsilon_i \in \mathbb{Q} \cap [0,1]$ be the unique numbers such that $1+\tau_i\varepsilon_i = f_i(s+1, (X_s\restriction x)0)$. Note that by (3.4) we have that $f_i(s+1,(X_s\restriction x)1) = 1-\tau_i\varepsilon_i$.

If $x \in X_s$ then $f_i(s+1, X_s\restriction x+1) \leq f_i(s, X_s\restriction x+1)$ so $\sum_{i=1}^k d_i^{s+1}(X_s\restriction x') \leq \sum_{i=1}^k d_i^s(X_s\restriction x')$ for every $x' \leq s+1$. Do nothing and proceed to the next stage.

If $x \notin X_s$ and

$$\sum_{i=1}^k d_i^{s+1}((X_s\restriction x)0) = \sum_{i=1}^k (1+\tau_i\varepsilon_i) d_i^s(X_s\restriction x) > k$$

enumerate x in X, that is, set $X_{s+1} = X_s \cup \{x\}$. By Lemma 3.1.7 we then again have that $\sum_{i=1}^{k} d_i^{s+1}(X_{s+1}\restriction x+1) = \sum_{i=1}^{k}(1-\tau_i\varepsilon_i)d_i^s(X_s\restriction x) \leq k$. This concludes the construction of the r.e. set $X = \bigcup_s X_s$.

Since the sum of the martingales d_i, $i \leq k$, is bounded on X none of these martingales succeeds on X, hence X is not an element of $\bigcup_{i=1}^{k} S[d_i]$. □

Although Theorem 3.1.8 shows that it is safe to use $\bar{\mu}_{\mathfrak{a}}$ instead of $\mu_{\mathfrak{a}}$ on RE we cannot use the closure of $\mu_{\mathfrak{a}}$ under effective unions. Namely, RE is contained in a recursive infinite union of $\mathfrak{a}$-null sets: Consider first the following strategy of covering the r.e. set W_e. Initially set $f(w0) = 0$ and $f(w1) = 2$ for every w. When n is enumerated in W_e redefine $f(w0) = 0$ and $f(w1) = 2$ for all w of length n, and set $f(v) = 1$ for all v of length smaller than n that have not yet been redefined. This f defines an $\mathfrak{a}$–martingale d that succeeds on W_e if W_e is infinite. Furthermore, the definition of d depends on e in a uniform way. That is, there is a *recursive* sequence of $\mathfrak{a}$–martingales d_e such that for each e, if W_e is infinite the martingale d_e succeeds on W_e. This shows that RE is contained in a recursive infinite union of success sets of $\mathfrak{a}$–martingales, because there is also an $\mathfrak{a}$-martingale that succeeds on all the finite sets.

3.2 Weakly complete sets

3.2.1. Definition. Let $\leq_r$ be a reducibility relation. An r.e. set A is *weakly r-complete* for RE if $\bar{\mu}_{\mathfrak{a}}({}^{\leq_r}A \cap \mathrm{RE}) \neq 0$.

3.2.2. Theorem. *There exists a weakly m-complete set in* RE *that is not btt-complete (and hence not m-complete).*

Proof. We first prove the existence of an r.e. set A satisfying $\mu_{\mathfrak{a}}({}^{\leq_m}A) \neq 0$ that is not m-complete. Fix an enumeration $\{d_n\}$ of all the (approximations of) $\mathfrak{a}$-martingales and an enumeration of all partial recursive m-reductions f_n. (Note that we can not recursively list all total reductions, but this will not harm us since in the proof we diagonalize against every f_n that is defined on sufficiently large initial segments.) We construct r.e. sets A and B such that $B \not\leq_m A$, hence A is not m-complete, and such that for every d_n the set

$$A_n = \{x : \langle x, n\rangle \in A\}$$

is not in $S[d_n]$. Since the sets A_n clearly m-reduce to A this last property guarantees that for every $\mathfrak{a}$–martingale d there is a set m-below A on which d does not succeed, hence A is weakly m-complete. In the construction of A and B we satisfy the requirements

$$\begin{aligned} R_{2n}: &\quad A_n \notin S[d_n] \\ R_{2n+1}: &\quad (\exists x)[B(x) \neq A(f_n(x))]. \end{aligned}$$

Let us first look at the strategies for satisfying these requirements in isolation. The even requirements can be satisfied as in the proof of Theorem 3.1.6 (i), i.e. by always choosing the 'minimal side' of the martingale. For martingales in $\mathfrak{a}$ this is an r.e. process. The odd requirements can be satisfied by choosing a witness $x \in \omega$. If during the construction we see at some stage that $f_n(x)\downarrow$ (f_n is defined on x) we diagonalize as follows: If there exists $z < x$ such that $f_n(z)\downarrow = f_n(x)$ then we define $B(x) = 1 - B(z)$. This guarantees that f_n is not an m-reduction from B to A and moreover leaves the A-side free, that is, we do not have to make a commitment for $A(f_n(x))$. Otherwise, if $f_n(x) = y$, we set $B(x) = 0$ if y was already enumerated in A, and otherwise we set $B(x) = 1$ and restrain y from being enumerated in A. Only in this very last case do we make a commitment for $A(y)$, namely, in order for the diagonalization to be successful we are not allowed to enumerate y into A at a later stage of the construction.

Now these two strategies for satisfying the even and the odd requirements, respectively, may be in conflict with each other. The witnesses for the odd requirements are chosen from from different k-sections of ω in order to separate the actions for the odd requirements as much as possible. The even requirements are already tolerant among each other because they refer to disjoint sections of A. The idea for resolving the conflict between the R_{2m} and R_{2n+1} is the following. If we enumerate y into A for the sake of R_{2m}, thereby injuring R_{2n+1}, we give up the old witness for R_{2n+1} and pick a fresh one. But we do not injure R_{2n+1} as soon as R_{2m} wants to do this. We only do it when we gain enough by this action, namely if the enumeration of y brings down the value of d_m on A substantially. We then argue that if R_{2n+1} is injured very often by some R_{2m} it will finally get an opportunity to be satisfied.

We give the R_n the natural priority ranking, i.e. R_n has *higher priority* than R_m if and only if $n < m$.

Before we give the construction we define the following two notions. To *initialize* R_{2n+1} at stage s means to give up its current witness $x_{n,s}$ and to pick a fresh witness $x_{n,s+1} \in \omega^{[n]}$ that is bigger than any number used in the construction so far. If R_{2n+1} is initialized we say that the previous actions taken for it are *injured.* The requirement R_{2n+1} *requires attention* at stage s if for its current witness $x_{n,s}$ the value $f_n(x_{n,s})$ is defined. In the construction we will use numbers $\varepsilon_n \in \mathbb{Q} \cap [0,1]$ such that $\prod_{n=0}^{\infty} 1 + \varepsilon_n < \infty$ and $\prod_{i=0}^{\infty} 1 + \varepsilon_n = \infty$ for every n. Fix any such recursive sequence (e.g. $\varepsilon_n = 1/n^2$) and fix a recursive sequence of $k_n \in \omega$ such that $(1-\varepsilon_n)^{k_n} < 1/2$.

Stage $s = 0$. Set $A_0 = \emptyset$, $B_0 = \emptyset$. Define $x_{n,0} = \langle 0, n\rangle$ for every n.

Stage $s > 0$. At stage s we consider all R_{2m} with $2m < s$. In addition we consider one odd requirement R_{2n+1}, where $n < s$ is minimal such that R_{2n+1} requires attention at s. (If n does not exist we consider only even requirements.) Suppose that such n exists and that $f_n(x_{n,s}) = y \in \omega^{[n]}$.

(a) If there is a $z < x_{n,s}$ such that $f_n(z)\downarrow = y$ then define $B(x_{n,s}) = 1 - B(z)$.

(b) If the number y was used before by some requirement $R_{2n'+1} < R_{2n+1}$ then initialize R_{2n+1}. (This separates the actions for the odd requirements and will make the proof easier.)
If neither (a) nor (b) holds do the following.
(c) If $A_s(y) = 1$ restrain $x_{n,s}$ from B during the rest of the construction, thereby satisfying R_{2n+1}.
(d) If $A_s(y) = 0$ and $n < m$ we diagonalize by enumerating $x_{n,s}$ into B, i.e. we set $B_s(x_{n,s}) = 1$, and we say that y is *restrained* (indicating that we would like to keep y out of A during the rest of the construction).
(e) If $A_s(y) = 0$ and $m \leq n$ we guess that we will never have to put y into A so we put $x_{n,s}$ into B_s, and again we restrain y.
So far for the action taken at s for the odd requirements. We now describe the action for the even requirements. For every R_{2m} with $2m < s$:
(f) If $\langle x, m\rangle < s$ is unrestrained define $A_m(x) = 1$ if and only if

$$\frac{d_m(0^x)}{d_m(0^{x-1})} > 1$$

i.e. $A_s(\langle x, m\rangle) = 1$ if and only if the approximation to $d_m(0^x)/d_m(0^{x-1})$, after running s steps, is bigger than 1.
(g) If $\langle x, m\rangle < s$ is restrained by R_{2n+1} (there can be at most one odd requirement restraining $\langle x, m\rangle$, since if there were two the one of lower priority would be initialized by (b)), $2m < 2n+1$, and R_{2n+1} was initialized not more than k_n times by R_{2m} then enumerate $\langle x, m\rangle$ into A if and only if

$$\frac{d_m(0^x)}{d_m(0^{x-1})} > 1 + \varepsilon_n.$$

In case $\langle x, m\rangle$ is enumerated into A initialize R_{2n+1}. If R_{2n+1} was initialized by R_{2m} more than k_n times we keep $\langle x, m\rangle$ out of A.
(h) If $\langle x, m\rangle < s$ is restrained by R_{2n+1} and $2n+1 < 2m$ we do nothing, i.e. we keep $\langle x, m\rangle$ out of A.
This ends the description of the construction. The theorem follows from the following two lemma's.

LEMMA 1 *Each R_{2n+1} requires attention only finitely often and is eventually satisfied.*

PROOF. By induction assume that all $R_{2n'+1} < R_{2n+1}$ are satisfied by the end of stage s. If f_n is not total then R_{2n+1} is vacuously satisfied, so assume that f_n is total. If f_n is not injective then it is not an m-reduction from B to A by step (a). So assume in addition that f_n is 1-1. Suppose that R_{2n+1} is initialized only finitely often. This means that at the last stage t at which R_{2n+1} was initialized a witness $x_{n,t}$ was chosen such that the diagonalization

$B_t(x_{n,t}) \neq A_t(f_n(x_{n,t}))$ was never injured later. Hence in this case R_{2n+1} is satisfied. It remains to show that R_{2n+1} cannot be initialized infinitely often. Suppose for a contradiction that R_{2n+1} is initialized infinitely often. This can only happen because of the actions taken for the requirements $R_{2m} < R_{2n+1}$ or because of step (b). But because f_n is 1-1 the latter can happen only finitely often. It follows that there must be at least *one* $R_{2m} < R_{2n+1}$ that initializes R_{2n+1} infinitely often. But this is impossible by (g). Lemma 1 $\square$

LEMMA 2 *Each R_{2m} is satisfied.*

PROOF. Fix m. By steps (f), (g), and (h) of the construction, A_m diagonalizes against d_m directly, unless a restrained $\langle x, m\rangle$ is encountered. If $\langle x, m\rangle$ is restrained by $R_{2n+1} < R_{2m}$ then R_{2m} is not allowed to enumerate $\langle x, m\rangle$ into A by step (h). However, there are only finitely many odd requirements of priority higher than R_{2m}, and since these act only finitely often by Lemma 1 they can increase the value of d_m on A_m by only a constant. If $\langle x, m\rangle$ is restrained by R_{2n+1} with $2m < 2n+1$ then R_{2m} may only enumerate $\langle x, m\rangle$ if $d_m(0^x)/d_m(0^{x-1}) > 1+\varepsilon_n$ and R_{2n+1} was not initialized by R_{2m} more than k_n times. If R_{2m} indeed initializes R_{2n+1} k_n times and wants to enumerate $\langle x, m\rangle$ restrained by R_{2n+1} it may not do so, so d_m might double its value on $x \notin A_m$. However, R_{2n+1} is initialized k_n times, at the stages $s_1, \dots s_{k_n}$ say, so for x_i with $\langle x_i, m\rangle = f_n(x_{n,s_i})$ it holds that $d_m(0^{x_i})/d_m(0^{x_i-1}) > 1+\varepsilon_n$, so on these points d_m is diminished by at least $(1-\varepsilon_n)^{k_n} < 1/2$, which is small enough to compensate for the possible doubling of d_m on x.

If there are infinitely many R_{2n+1} with $2m < 2n+1$ such that R_{2n+1} restrains some $\langle x, m\rangle$ that is never enumerated into A then d_m can gain at most $\prod_{n=0}^{\infty} 1+\varepsilon_n$, which is a finite amount.

Summarizing, R_{2m} lets A_m diagonalize against d_m and is frustrated in this action by the higher priority odd requirements only finitely often, and if it is in conflict with some lower priority requirement it does not immediately injure it but waits until there is some compensation for the injury, so that after finitely many injuries to the same lower priority requirement this gets a chance to be satisfied by 'injuring' the higher priority requirement, which is not really injured because of the compensation that was built up in the previous stages.

Lemma 2 $\square$

This concludes the proof of the existence of r.e. sets A with $\mu_{\mathfrak{a}}(^{\leq_m}A) \neq 0$ that are not m-complete. We now indicate the modifications one can make to the construction above to obtain a weakly m-complete r.e. set that is not btt-complete. We have to improve on two things. Firstly, we have to use $\bar{\mu}_{\mathfrak{a}}$ rather than $\mu_{\mathfrak{a}}$. This can be taken care of using Lemma 3.1.7, just as in the proof of Theorem 3.1.8. The strategy for the n-th row of A is now, instead of diagonalizing against one martingale, to keep the sum of a finite number of $\mathfrak{a}$–martingales bounded. Apart from some horrible changes in notation the proof does not change essentially. Secondly, we have to make A btt-incomplete rather than just m-incomplete. This we can do by changing the numbers k_n used in the proof above. If we are diagonalizing against a btt-reduction using c queries, then we need compensation for a potential loss of $(1/2)^c$ rather than just $1/2$ as above. So if R_{2n+1} diagonalizes against c-tt-reduction f_n then we define k_n such that $(1-\varepsilon_n)^{k_n} < (1/2)^c$. This concludes the proof of Theorem 3.2.2. □

Theorem 3.2.2 is optimal in the sense that there does not exist a weakly m-complete set that is not tt-complete, as we prove in Theorem 3.4.4.

Theorem 3.2.2 shows that there exist r.e. sets that are weakly m-complete but not btt-complete. The next theorem shows that the two notions are incomparable. We first prove a lemma that will be useful later.

3.2.3. Lemma. *Define the intervals $I_m = [5^m, 5^{m+1})$. Suppose that for the class $\mathcal{A}$ and r.e. set M it holds that*

$$(\forall A \in \mathcal{A})(\exists^\infty m)[A \cap I_m \text{ and } M \cap I_m \text{ differ on at least } \frac{2}{3}|I_m| \text{ points}].$$

Then $\mu_{\mathfrak{a}}(\mathcal{A}) = 0$.

Proof. If A differs from M on I_m on at least $\frac{2}{3}|I_m|$ points then A differs from M on the initial segment $[0, 5^{m+1})$ on at least $\frac{2}{3}(5^{m+1}-5^m) = \frac{8}{3}5^m > \frac{1}{2}5^{m+1}$ points. So if $\mathcal{A}$ is as in the lemma then $\mathcal{A}$ satisfies the condition of Lemma 3.1.5. □

3.2.4. Theorem. *There exists an r.e. set that is btt-complete but not weakly m-complete.*

Proof. Fix the intervals $I^n_x = [5^{\langle n,x\rangle}, 5^{\langle n,x\rangle+1})$. Let $\{f_n\}_{n\in\omega}$ be an enumeration of all partial m-reductions. We use a finite injury construction to build r.e. sets A and M such that

$$x \in K \Leftrightarrow \{\langle x,0\rangle, \langle x,1\rangle, \langle x,2\rangle\} \cap A \neq \emptyset$$

and such that all the requirements

$$\begin{aligned} R_n:\quad f_n \text{ total } &\Rightarrow \exists^\infty x[f_n^{-1}(A) \cap I^n_x \text{ and } M \cap I^n_x \\ &\qquad \text{differ on at least } \tfrac{2}{3}|I^n_x| \text{ points}] \end{aligned}$$

are satisfied. The first condition guarantees that A is 3-tt-complete, and the requirements guarantee that there exist a martingale in $\mathfrak{a}$ that succeeds on ${}^{\leq_m}A$ by Lemma 3.2.3. We split the requirements R_n into

$$R_{\langle n,x\rangle}:\quad f_n \text{ total } \Rightarrow \exists y \geq x(\|(f_n^{-1}(A) \cap I^n_y) \triangle (M \cap I^n_y)\| \geq \tfrac{2}{3}|I^n_y|)$$

To *initialize* $R_{\langle n,x\rangle}$ at stage s means to pick $y_s \geq x$ such that $I^n_{y_s}$ was not yet used in the construction. The requirement $R_{\langle n,x\rangle}$ is *satisfied* at stage s if at some previous stage action was taken for $R_{\langle n,x\rangle}$ and $R_{\langle n,x\rangle}$ was not initialized later. We now describe the construction for A and M.

Stage $s = 0$. Set $A_0 = M_0 = \emptyset$. Assign to every $R_{\langle n,x\rangle}$ some $y_0 \geq x$, say $y_0 = \langle n, x\rangle$.

Stage $s+1$. Choose the highest priority requirement $R_{\langle n,x\rangle}$ (i.e. the one with the lowest index $\langle n, x\rangle$) that is not satisfied at s and such that for the y_s assigned to $R_{\langle n,x\rangle}$ at a previous stage $f_n{\restriction}5^{\langle n,y_s\rangle+1}$ is defined. Initialize all lower priority requirements. Say that $z \in \omega$ is *relevant* if $f_n(u) = \langle z, i\rangle$ for some $i \leq 2$ and $u \in I^n_{y_s}$, or if z was already relevant at a previous stage. For z relevant not in use for any higher priority requirement choose $i \leq 2$ such that the set of pre-images $f_n^{-1}(\langle z, i\rangle) \cap I^n_{y_s} = \{u \in I^n_{y_s} : f_n(u) = \langle z, i\rangle\}$ is minimal in size. We use this $\langle z, i\rangle$ for coding K, that is, we enumerate $\langle z, i\rangle$ into A_t if and only if $z \in K_t - K_{t-1}$, and in this case we initialize all lower priority requirements at stage t. Define M on $I^n_{y_s}$ as follows. If $f_n(u) = v$, enumerate $u \in I^n_{y_s}$ into M if and only if $A_s(v) = 0$. This ends the description of the construction.

To see that $K \leq_{btt} A$ it suffices to observe that *every* $z \in \omega$ is declared to be relevant at some stage of the construction (because there exists an m-reduction f_n such that whenever $R_{\langle n,x\rangle}$ is initialized and I^n_y is assigned to f_n, f_n converges on I^n_y to the smallest z that was not yet relevant), and hence that one of $\{\langle z, 0\rangle, \langle z, 1\rangle, \langle z, 2\rangle\}$ is enumerated into A as soon as z is enumerated into K. Since this is the only way that elements may enter A, $\{\langle z, 0\rangle, \langle z, 1\rangle, \langle z, 2\rangle\} \cap A$ is empty if $z \notin K$.

We now prove by induction that $R_{\langle n,x\rangle}$ is satisfied. Suppose that all higher priority requirements $R_m < R_{\langle n,x\rangle}$ are satisfied at all stages $t \geq s$. Suppose that f_n is total (otherwise $R_{\langle n,x\rangle}$ is trivially satisfied) and that $R_{\langle n,x\rangle}$ is not yet satisfied at s. Let $t > s$ be the first stage such that $K_t(z) = K(z)$ for all the relevant z that are in use for the requirements $R_m < R_{\langle n,x\rangle}$ and such that $f_n(I^n_{y_t})$ is defined, $I^n_{y_t}$ the interval assigned to $R_{\langle n,x\rangle}$ at stage t. Then $R_{\langle n,x\rangle}$ is never initialized after t, because this could only happen if either some higher priority requirement acted, which by assumption none of them does, or if some relevant z in use by a higher priority requirement is enumerated in K. Now M_t differs from $f_n^{-1}(A_t)$ on all the elements of $I^n_{y_t}$. This can only change later if some z is enumerated into K, for a relevant z that is in use by $R_{\langle n,x\rangle}$ at stage t. But $\langle z, i\rangle$ is enumerated in A for $i \leq 2$ only if $\|f_n^{-1}(\langle z, i\rangle)\|$ is minimal among $\|f_n^{-1}(\langle z, 0\rangle)\|$, $\|f_n^{-1}(\langle z, 1\rangle)\|$, and $\|f_n^{-1}(\langle z, 2\rangle)\|$. It follows that if $\{z_1, \ldots, z_k\}$ is the set of numbers relevant for $R_{\langle n,x\rangle}$ at t (so $k \leq |I^n_{y_t}|$), and if $\langle z_j, i_j\rangle$ are enumerated into A after stage t, then $\|\bigcup_{j=1}^k f_n^{-1}(\langle z_j, i_j\rangle)\| \leq \frac{1}{3}|I^n_{y_t}|$. Hence M differs from $f_n^{-1}(A)$ on at least 2/3-rd of the elements of $I^n_{y_t}$, and $R_{\langle n,x\rangle}$ is satisfied at all stages $t' \geq t$. □

Note that Theorem 3.2.4 improves on the known result that there is a btt-complete set that is not m-complete (Young [81]). Likewise, the next theorem improves on the result of Post [68] that there is a tt-complete set that is not btt-complete.

3.2.5. THEOREM. *There exists an r.e. set that is tt-complete but not weakly btt-complete.*

PROOF (SKETCH). The proof is actually the same as that of Theorem 3.2.4, except for some minor changes which we now describe. Instead of coding $z \in K$ on one of the three elements $\langle z, 0\rangle$, $\langle z, 1\rangle$, $\langle z, 2\rangle$, when dealing with a c-tt-reduction we now want to code z on one of $\langle z, i\rangle$, $i \leq 3c$. One obtains the simplest construction when for every $z \in K$, one codes z on one of the numbers $\langle z, i\rangle$, $i \leq z$. For a relevant z we pick i such that $\{u \in I^n_{y_s} : \langle z, i\rangle \text{ is in the query set of } f_n(u)\}$ is minimal in size. The rest of the argument remains virtually unchanged. $\square$

We summarize the content of Theorems 3.2.2, 3.2.4, and 3.2.5 in the following diagram. 'w-r' stands for 'weakly r-complete', and no other implications hold than the ones indicated. The implication *w-btt* $\to$ *tt* will be proved in Section 3.4.

$$\begin{tikzcd} m \arrow[r] \arrow[dr] & btt \arrow[r] & w\text{-}btt \arrow[r] & tt \\ & w\text{-}m \arrow[ur] & & \end{tikzcd}$$

3.3 ON THE STRUCTURAL DIFFERENCE BETWEEN COMPLETE AND INCOMPLETE R.E. SETS

It is a natural question to ask how much the characteristic sequence of an incomplete set can look like that of a complete set. On how many bits should the incomplete set differ from the complete set in order to stay incomplete? As we will see this depends very much on the completeness notion under consideration. First we make precise what we mean by "A resembles M". Define the function $Res : 2^\omega \times 2^\omega \to [0, 1]$ by

$$Res(A, M) = \liminf_{n\to\infty} \frac{\|\{i \leq n : A(i) = M(i)\}\|}{n}.$$

So Res takes real values from $[0, 1]$, and the higher $Res(A, M)$, the more A resembles M. Lynch [56] proved that for some Gödel numberings ϕ the halting set K_ϕ becomes very easy: There exist ϕ and a recursive set A such that $Res(A, K_\phi) = 1$. She also showed that there is an m-complete set C such that for all recursive sets A it holds that $Res(A, C) = 0$ (which we improved from REC to Π_1 in Lemma 3.1.4), and used this to show that there is a Gödel numbering ϕ such that $Res(A, K_\phi) = 0$ for any recursive A. Furthermore, she showed that for any recursive real $r \in [0,1]$ there is a Gödel numbering ϕ such that there is a recursive set A with $Res(A, K_\phi) = r$ and that no recursive set can do better.

In this section we consider sets that are incomplete with respect to the various reducibilities, rather than just the recursive sets. As everywhere in this thesis, *the results in this section are independent of the Gödel numbering chosen.*

3.3.1. THEOREM. *For every btt-complete r.e. set M there exists an r.e. set $A <_{btt} M$ such that $Res(A, M) = 1$.*

PROOF. Let M be a btt-complete r.e. set. We imitate Posts original construction of a simple set. Let $Q = \{e^2 : e \in \omega\}$. Define the partial recursive function ψ by letting $\psi(e)$ be the first element enumerated into W_e that is bigger than $4e^2$. Clearly $A = (M - Q) \cup \mathrm{rng}(\psi)$ is an r.e. set that intersects every infinite r.e. set W_e. Furthermore, since below n at most $\frac{1}{2}\sqrt{n}$ elements of $\mathrm{rng}(\psi)$ are enumerated into A and $\|\overline{M - Q} \cap [0,n]\| \geq \sqrt{n}$, the complement of A is infinite, and we have for every n

$$\frac{\|\{i \leq n : A(i) = M(i)\}\|}{n} \geq \frac{n - \frac{1}{2}\sqrt{n} - \sqrt{n}}{n} \to 1.$$

Hence A satisfies $Res(A, M) = 1$, and A is simple, and therefore not btt-complete (Post [68]). □

So for m- and btt-reducibility an incomplete set can look very much like a complete set. The next theorem shows that we cannot have Theorem 3.3.1 for tt-reducibility.

3.3.2. THEOREM. *There exists an m-complete r.e. set M such that for any set A satisfying $Res(A, M) > \frac{1}{2}$ it holds that $M \leq_{tt} A$.*

PROOF. Inductively define a recursive series of disjoint successive intervals I_e such that $\bigcup_e I_e = \omega$ and $|I_e| > e \cdot \sum_{i<e} |I_i|$. Now define the m-complete r.e. set M by

$$(\forall x \in I_e)[M(x) = K(e)].$$

If A is a set that satisfies the condition of the theorem then there exists $\varepsilon > 0$ such that $Res(A, M) > \frac{1}{2} + \varepsilon$. Then A disagrees with M on at most $(\frac{1}{2} - \varepsilon)\sum_{i \leq e} |I_i|$ points of $\bigcup_{i \leq e} I_i$, so in particular A disagrees with M on at most $(\frac{1}{2} - \varepsilon)\sum_{i \leq e} |I_i|$ points of I_e. Let e_0 be so large that for all $e \geq e_0$ it holds that $e > (\frac{1}{2\varepsilon} - 1)$. Then we have that for all $e \geq e_0$,

$$|I_e| > e \cdot \sum_{i<e} |I_i| > (\frac{1}{2\varepsilon} - 1)\sum_{i<e} |I_i|.$$

By multiplying with ε and adding $(\frac{1}{2}-\varepsilon)|I_e|$ we obtain

$$\frac{1}{2}|I_e| > (\frac{1}{2}-\varepsilon)\sum_{i\leq e}|I_i|.$$

It follows that A disagrees with M on less than $\frac{1}{2}|I_e|$ points of I_e hence we can decide '$e \in K$' by asking about the majority of the values $A(x)$, for $x \in I_e$. This constitutes a tt-reduction from K to A. $\square$

As a corollary to the proof of Theorem 3.4.1 in the next section we have the following result.

3.3.3. THEOREM. *There exists an m-complete r.e. set M such that for all r.e. sets A with $A <_{wtt} M$ it holds that $Res(A, M) = 0$.*

So for the complete set M from Theorem 3.3.3, no T-incomplete set can resemble M even a little bit.

3.4 The weak completeness notions for the weak reducibilities

In this section we prove the coincidence of the weak completeness notions with the ordinary completeness notions for the weak reducibilities T, wtt, and tt. We actually prove that the set of incomplete r.e. sets is of measure zero in RE.

3.4.1. THEOREM. (i) $\mu_{\mathfrak{a}}(\{A : (\exists B \text{ r.e. not T-complete})[A \leq_T B]\}) = 0.$

(ii) $\mu_{\mathfrak{a}}(\{A : \exists B \text{ r.e. not wtt-complete } (A \leq_{wtt} B)\}) = 0.$

PROOF. For a proof of (i) fix the intervals $I_n = [3^n, 3^{n+1}-1]$. We define an r.e. set M in such a way that for every $e, d \in \omega$,

(1) $\{d\}^{W_e}$ not total

or

(2) $(\exists^\infty x)[\{d\}^{W_e} \cap I_{\langle e,d,x\rangle} = \overline{M} \cap I_{\langle e,d,x\rangle}]$

or

(3) $K \leq_T W_e$.

The theorem then follows from Lemma 3.1.5. We define M on $I_{\langle e,d,x\rangle}$ as follows. As long as $x \notin K_s$ or $\{d\}^{W_{e,s}}$ is not defined on all of $I_{\langle e,d,x\rangle}$ we let M be empty on $I_{\langle e,d,x\rangle}$. If a stage s is found such that $x \in K_s$ and $\{d\}_s^{W_{e,s}}\restriction I_{\langle e,d,x\rangle}\downarrow$ then we define M on $I_{\langle e,d,x\rangle}$ to be the complement of $\{d\}^{W_{e,s}}$. We now verify that for every e and d one of (1), (2), or (3) holds. Suppose $\{d\}^{W_e}$ is total.

First assume that for infinitely many $x \in K$, if s is the least stage such that $x \in K_s$ and $\{d\}_s^{W_{e,s}}\restriction I_{\langle e,d,x\rangle}\downarrow$, and u is the use of this computation, then $W_{e,s}\restriction u = W_e\restriction u$. Then M on $I_{\langle e,d,x\rangle}$ is the complement of $\{d\}_s^{W_{e,s}} = \{d\}^{W_e}$ so (2) is satisfied.

In the complementary case, for almost every $x \in K$ it holds that if s is the least stage such that $x \in K_s$ and $\{d\}_s^{W_{e,s}}\restriction I_{\langle e,d,x\rangle}\downarrow$ with use u then $W_{e,s}\restriction u \neq W_e\restriction u$. We then can reduce K to W_e as follows. Given x, by totality of $\{d\}^{W_e}$ there exists an s such that $\{d\}_s^{W_{e,s}}\restriction I_{\langle e,d,x\rangle}\downarrow$, with use u say. With oracle W_e compute $t > s$ such that $W_{e,t}\restriction u = W_e\restriction u$. Then $x \in K$ if and only if $x \in K_t$: If x enters K at stage t', then by assumption $W_{e,t'}\restriction u \neq W_e\restriction u$, hence $t' < t$. This procedure works for almost every x, so $K \leq_T W_e$.

The proof of item (ii) is very similar to the proof just given, so we describe only the essential changes. Instead of considering all reductions $\{d\}^{W_e}$ we now consider pairs $(\{d\}^{W_e}, \varphi_c)$ consisting of a (possible) Turing-reduction and a potential recursive bound on its use function. The analogues of (1) and (3) above are

(1′) $(\{d\}^{W_e}, \varphi_c)$ is not a wtt-reduction

(3′) $K \leq_{wtt} W_e$.

As soon as we find that φ_c is not such a bound we discard the pair. Now the proof proceeds along the same lines as above, but now, if we are dealing with a true wtt-reduction, if W_e changes below the use u and the computation $\{d\}^{W_e}\restriction I_{\langle e,d,c,x\rangle}$ becomes defined again later, *the use u of this new computation is the same as the old one.* Hence, in order to decide K as above, we only have to query W_e on elements below $\varphi_c(x)$. That is, if for almost every $x \in K$, if x enters K at stage s then $W_{e,s}\restriction\varphi_c(x) \neq W_e\restriction\varphi_c(x)$. So $K \leq_{wtt} W_e$. (This is called 'Yates permitting', see Soare [72, V.3]. The element x is only allowed to enter K if the set W_e *permits* it by changing below $\varphi_c(x)$ later.) □

3.4.2. Corollary. *For any r.e. set A,*

(i) *either* $\mu_{\mathfrak{a}}(^{\leq_T}A) = 0$ *or* $K \leq_T A$,

(ii) *either* $\mu_{\mathfrak{a}}(^{\leq_{wtt}}A) = 0$ *or* $K \leq_{wtt} A$.

Proof. Immediate from the theorem. □

3.4.3. COROLLARY. *For any r.e. set A,*

(i) *A is weakly T-complete $\Leftrightarrow$ A is T-complete,*

(ii) *A is weakly wtt-complete $\Leftrightarrow$ A is wtt-complete,*

PROOF. The implications from right to left follow from Theorem 3.1.6 (i). For the reverse implications, if $r \in \{wtt, T\}$ and A is not r-complete, then by Corollary 3.4.2 the r-lower cone of A has $\mathfrak{a}$–measure zero, so in particular the r-lower cone intersected with RE has $\mathfrak{a}$–measure zero. Hence A is not weakly r-complete. $\square$

3.4.4. THEOREM. $\bar{\mu}_{\mathfrak{a}}(\{W_e : W_e$ *not tt-complete* $\}) = 0$. *Hence every r.e. set is weakly tt-complete if and only if it is tt-complete.*

PROOF. Fix the intervals $I_n^e = [5^{\langle e,n\rangle}, 5^{\langle e,n\rangle+1})$. We define two $\mathfrak{a}$–martingales d_0 and d_1 such that $\{W_e : W_e \text{ not tt-complete}\} \subseteq S[d_0] \cup S[d_1]$. Define the r.e. set M_0 by

$$\begin{array}{lcl} n \notin K & \Rightarrow & M_0 \cap I_n^e = \emptyset \\ n \in K & \Rightarrow & M_0 \cap I_n^e = I_n^e. \end{array}$$

and define the r.e. set M_1 by

$$M_1 \cap I_n^e = \begin{cases} \emptyset & \text{if } \|W_e \cap I_n^e\| \leq 1/3 \cdot |I_n^e| \ , \\ \overline{W}_{e,s} \cap I_n^e & \text{for } s \text{ minimal such that } \|W_{e,s} \cap I_n^e\| > 1/3 \cdot |I_n^e|. \end{cases}$$

Define the corresponding martingales ($i \in \{0,1\}$)

$$\begin{array}{rcl} d_i(w0) & = & \begin{cases} 1/2 \cdot d_i(w) & \text{if } |w| \notin M_i \\ 3/2 \cdot d_i(w) & \text{if } |w| \in M_i \end{cases} \\ d_i(w1) & = & 2d_i(w) - d_i(w0). \end{array}$$

As before we have that $d_i \in \mathfrak{a}$. We now distinguish the following cases.
(a) $\exists^\infty n(1/3 \cdot |I_n^e| < \|W_e \cap I_n^e\| < 2/3 \cdot |I_n^e|)$.
In this case M_1 differs from W_e on I_n^e on more than 2/3-rd of the elements, for infinitely many n. Hence d_1 succeeds on W_e by (the proof of) Lemma 3.2.3.
(b) $\forall^\infty n(\|W_e \cap I_n^e\| \leq 1/3 \cdot |I_n^e| \vee \|W_e \cap I_n^e\| \geq 2/3 \cdot |I_n^e|)$.
In this case W_e satisfies either
(b1) $\exists^\infty n((n \notin K \wedge \|W_e \cap I_n^e\| \geq 2/3 \cdot |I_n^e|) \vee (n \in K \wedge \|W_e \cap I_n^e\| \leq 1/3 \cdot |I_n^e|))$
or
(b2) $\forall^\infty n((n \notin K \Rightarrow \|W_e \cap I_n^e\| < 2/3 \cdot |I_n^e|) \wedge (n \in K \Rightarrow \|W_e \cap I_n^e\| > 1/3 \cdot |I_n^e|))$.
In case (b1) the martingale d_0 succeeds on W_e. By (b) the case (b2) is equivalent to

(b2′) $\forall^\infty n((n \notin K \Rightarrow \|W_e \cap I_n^e\| \leq 1/3 \cdot |I_n^e|) \wedge (n \in K \Rightarrow \|W_e \cap I_n^e\| \geq 2/3 \cdot |I_n^e|))$. In this case we have that $K \leq_{tt} W_e$ via a finite variant of the tt-reduction

$$n \in K \Leftrightarrow \|W_e \cap I_n^e\| \geq 2/3 \cdot |I_n^e|. \qquad \square$$

Combining the results of this section with those of Section 3.2 we get the following complete picture of the completeness notions:

$$\begin{tikzcd}
m \arrow[r, shift right] \arrow[d] & btt \arrow[l, shift right, "/" marking] \arrow[r, shift right] \arrow[d] \arrow[dl, shift left, "|" marking] & tt \arrow[l, shift right, "/" marking] \arrow[r, shift right] \arrow[d, leftrightarrow] \arrow[dl, "|" marking] & wtt \arrow[l, shift right, "/" marking] \arrow[r, shift right] \arrow[d, leftrightarrow] & T \arrow[l, shift right, "/" marking] \arrow[d, leftrightarrow] \\
w\text{-}m \arrow[r] \arrow[ur, shift left, "|" marking] & w\text{-}btt \arrow[r] & w\text{-}tt \arrow[r] & w\text{-}wtt \arrow[r] & w\text{-}T
\end{tikzcd}$$

For proofs of the strictness of the implications in the first row we refer to Odifreddi [64, p341].

3.5 Coding

There are several theorems on the measure of upper cones of sets. (We will discuss this topic in Section 6.1.) For example, for every set not in P the p-m-upper cone of this set has Lebesgue measure zero (Ambos-Spies), and for every nonrecursive set in Δ_2 the Turing-upper cone has Δ_2–measure zero (Theorem 6.2.1). The corresponding fact for $\mathfrak{a}$–measure in RE is not true: For every $\mathfrak{a}$–martingale we can code a set in the upper cone of a given set on which the martingale does not succeed.

3.5.1. Theorem. *For every r.e. set C and every $\mathfrak{a}$–martingale d, there exists an r.e. set A such that $C \leq_m A$ (even $C \leq_1 A$) and $A \notin S[d]$.*

Proof. Define the set

$$V = \{x \in \omega : \frac{d(0^{x+1})}{d(0^x)} > 1\}.$$

Note that V is r.e. We consider four cases. In each of these cases we will argue that there exists an infinite recursive set X such that, if we diagonalize outside of X with A against d, like in Theorem 3.1.6 (i), we can safely code C on X, i.e. if $X = \{x_0 < x_1 < \dots\}$ then $n \in C \Leftrightarrow x_n \in A$. Note that this gives a one-one-reduction from C to A.
Case I. The set V is infinite and $\prod_{x \in V} \frac{d(0^{x+1})}{d(0^x)} = \infty$. In this case V contains an infinite recursive subset X as described above: We can wait for the recursive approximation to $\frac{d(0^{x+1})}{d(0^x)}$ from below to grow bigger than 1, and since the approximation is monotone we have a secure lower bound on its value. Since we

are in Case I we know that this event occurs infinitely often and, moreover, that the product of the betting values becomes arbitrarily large, so we can diagonalize against d until its value is certainly below $1/2$ (and maybe smaller when we approximate better), so that we can safely code a bit from C. In the worst case d doubles its value on this particular bit, but since its value was guaranteed to be smaller than $1/2$ it is still bounded by 1.

Case II. The set V is infinite and $\prod_{x \in V} \frac{d(0^{x+1})}{d(0^x)} < \infty$. In this case we can let X be any infinite recursive subset of V.

Case III. The set V is finite and

$$(\forall n)(\exists x \notin V)[\frac{d(0^{x+1})}{d(0^x)} > 1 - \frac{1}{n^2}].$$

In this case we recursively choose an increasing sequence x_n such that for every n this x_n satisfies the formula, and we let X consist of all these x_n. Since $\prod_{n \in \omega}(1 + 1/n^2)$ is finite we can use X for coding without any danger of letting d grow to infinity on X.

Case IV. The set V is finite and

$$(\exists n)(\forall x \notin V)[\frac{d(0^{x+1})}{d(0^x)} \leq 1 - \frac{1}{n^2}].$$

This case is similar to Case I. In this case we know with certainty that if we use x for diagonalizing against d that we win at least $1 + \frac{1}{n^2}$, for some fixed n. So if $(1 - \frac{1}{n^2})^m < 1/2$ then we can just let X be equal to $\{mx : x \in \omega\}$. □

It follows from Theorem 3.5.1 that a Small Span Theorem, like the one for btt-reducibility in E or for Turing-reducibility in Δ_2, fails for m-reducibility in RE (and hence also fails for all the weaker reducibilities). Namely, by Theorem 3.2.2, take any weakly m-complete set that is not m-complete. By definition of weak completeness this set has an m-lower cone of non-zero measure, and by Theorem 3.5.1 the same holds for its upper cone.

Recall that an r.e. set is simple if it has infinite complement and intersects every r.e. set. At this point we cannot prove that the simple sets do not have $\bar{\mu}_{\mathfrak{a}}$–measure zero, but we can prove that this holds for the class consisting of all simple sets plus all finite sets:

3.5.2. THEOREM. *For each martingale $d \in \mathfrak{a}$ there exists a simple set S and a finite set D such that $S \notin S[d]$ or $D \notin S[d]$. In fact, the class of all simple sets plus all finite sets is not of $\bar{\mu}_{\mathfrak{a}}$–measure zero.*

PROOF (SKETCH). We first sketch the proof that no $\mathfrak{a}$–martingale succeeds simultaneously on all simple sets and all finite sets. Suppose that $d \in \mathfrak{a}$ succeeds on every finite set. Recall that with an r.e. set we can diagonalize against a $\mathfrak{a}$–martingale (as in the proof of Theorem 3.1.6(i)). The following observations are sufficient for the proof.

(i) For any initial segment w, we can make the value of d arbitrarily high by appending to w sufficiently many zeros.

(ii) Under (i), we can *see* when we have appended enough zeros, by the using the recursive approximation of d (cf. Definition 3.1.1(i)).

Now the strategy for constructing A simple with $A \notin S[d]$ is the following. If W_e appears to be infinite and we want to enumerate an element $x \in W_e$ into A, the martingale d may grow on A because it bets on x. By (i) and (ii) above, we can compensate this by 'waiting' long enough, that is, by appending a large interval of zeros I to the finite part of A constructed so far, until we see that d has grown big on I, and subsequently by diagonalizing against d on the interval I. The large gain that d has on I is thus turned into a big loss for d on I, compensating for the possible gain of d on x. A moment of thought will convince the reader that (i) and (ii) give us enough control to make A simple and simultaneously keep the value of d on A bounded by $2d(\lambda)$.

Virtually the same proof gives the improvement of $\mu_{\mathfrak{a}}$–measure to $\bar{\mu}_{\mathfrak{a}}$–measure. □

Finally we note that the e-generic sets of Jockusch [28] have $\mathfrak{a}$–measure zero: If A is e-generic, or even just weakly e-generic, then

$$\exists^\infty n(A \cap [n, n^2] = [n, n^2]),$$

so the martingale d_1 that always bets 3/2 of its capital on the 1-side succeeds on A.

CHAPTER 4

ARITHMETICAL MEASURE

In this chapter we develop arithmetical measure theory along the lines of Lutz [50]. This yields the same notion of "measure 0 set" as considered before by Martin-Löf, Schnorr, and others. We prove that the class of sets constructible by r.e.-constructors, a direct analogue of the classes Lutz devised his resource bounded measures for in [50], is not equal to RE, the class of r.e. sets, and we locate this class exactly in terms of the common recursion-theoretic reducibilities below K. We note that the class of sets that bounded truth-table reduce to K has r.e.-measure 0, and show that this cannot be improved to "truth-table." For Δ_2-measure the borderline between measure zero and measure nonzero lies between weak truth-table reducibility and Turing reducibility to K. It follows that there exists a Martin-Löf random set that is tt-reducible to K, and that no such set is btt-reducible to K.

4.1 R.E.-MEASURE

4.1.1. DEFINITION. Let A be a countable set recursively isomorphic to ω. Let B be a countable set totally ordered by $\leq_B$ such that $(B, \leq_B)$ is recursively isomorphic to $(\omega, \leq)$. A function $f : A \to B$ is *recursively enumerable (r.e.)*[1] if the set $\{(x, y) : y \leq_B f(x)\}$ is an r.e. set. Similarly, f is *co-r.e.* or Π_1 if $\{(x, y) : f(x) \leq_B y\}$ is r.e.

In Chapter 1 (page 12) we defined the class of martingales all. Define the class of martingales

$$\text{r.e.} = \{f \in \text{all} : f \text{ is r.e.}\}.$$

[1] R.e.-functions have also been called *enumerable*, *semi-computable* (not to be confused with the notion of semi-recursiveness), and *limitwise monotonic.*

Throughout this chapter, the standard notion of a recursively enumerable function will play an important role. Note that a function f is r.e. if and only if there exists a recursive approximation f_s such that for all x, $f_s(x) \leq f_{s+1}(x)$ and $(\exists s)(\forall t \geq s)[f_t(x) = f(x)]$. We will often make use of this last fact, namely that the recursive approximation f_s actually attains its limit. Clearly a function which is both r.e. and co-r.e. is recursive. As before, the class of (total) recursive functions is denoted by rec.

We will make use of the following characterization of $\leq_{tt}$.

4.1.2. LEMMA. (Carstens [18]) *For every $A \in 2^\omega$, $A \leq_{tt} K$ if and only if there exist recursive functions g and h such that for every $x \in \omega$, $\lim_s g(s,x) = A(x)$ with $|\{s : g(s,x) \neq g(s+1,x)\}| \leq h(x)$.*

For $r \in \{m, btt, tt, wtt, T\}$, ${}^{\leq_r}A$ denotes the class of sets that are r-reducible to the set A.

Σ_n, Π_n, and Δ_n denote the classes from Kleene's arithmetical hierarchy. They are also used to denote the corresponding function classes, defined in Section 4.4.

Recall that a supermartingale is a function $d : 2^{<\omega} \to \mathbb{R}^+$ with the property

$$d(w0) + d(w1) \leq 2d(w)$$

for every $w \in 2^{<\omega}$. Functions with this property are called *super*martingales, as opposed to martingales which have the property $d(w0) + d(w1) = 2d(w)$. Lutz [50, p239] remarks that for the classes that he considers it makes no difference whether one uses supermartingales or martingales. In our setting it will make a huge difference. We will consider supermartingales which are (approximable by) r.e.-functions, and one can easily check that r.e.-martingales with the property $d(w0) + d(w1) = 2d(w)$ are always recursive in the value $d(\lambda)$. The difference is then clear from Corollary 4.3.2.

4.1.3. DEFINITION. A supermartingale d is an *r.e.-supermartingale* if there is an r.e.-function $\hat{d} : \omega \times 2^{<\omega} \to \mathbb{Q}^+$ such that

$$(\forall k \in \omega)(\forall w \in 2^{<\omega})[|d(w) - \hat{d}(k,w)| \leq 2^{-k}].$$

The function $\hat{d}$ is called an *r.e.-computation* of d.

Now a class $\mathcal{A}$ has *r.e.-measure zero*, denoted $\mu_{\text{r.e.}}(\mathcal{A}) = 0$, if there exists an r.e.-supermartingale that succeeds on $\mathcal{A}$. $\mathcal{A}$ has *r.e.-measure one* if $\mathcal{A}^c = \{X : X \notin \mathcal{A}\}$ has r.e.-measure zero.

The following lemma is proved in exactly the same way as Proposition 1.5.5 (iii).

4.1.4. LEMMA. *Let d be an r.e.-supermartingale. Then there is a supermartingale $\tilde{d} : 2^{<\omega} \to \mathbb{Q}^+$ which is r.e. such that $S[\tilde{d}] \supseteq S[d]$.*

Having defined r.e.-measure using the approach of Lutz [50] we now prove that this yields the same notion of measure as considered before by Martin-Löf, Schnorr, and others. The proof is a simple extension of the work of Schnorr [73].

4.1.5. DEFINITION. (Martin-Löf [59], Kautz [34]) A class $\mathcal{A}$ of Lebesgue measure 0 is Σ_n^C*-approximable* if there is a recursive sequence of Σ_n^C-classes $\{\mathcal{S}_i\}_{i\in\omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$ and $\mathcal{A} \subseteq \bigcap_i \mathcal{S}_i$. A set A is *n-random* if $\{A\}$ is not Σ_n-approximable. The 1-random sets are also called Martin-Löf random. A sequence of classes $\mathcal{S}_i$ as above is called a *sequential test.*

Schnorr [73, Satz 5.3] calls a martingale g subcomputable ("subberechenbar") if it has a recursive approximation g_s satisfying $g_s(w) \leq g_{s+1}(w)$ for every $s \in \omega$ and $w \in 2^{<\omega}$, and such that $\lim_s g_s(w) = g(w)$. Note that in this definition it is not required that there is an $s \in \omega$ such that $g_s(w) = g(w)$. The equivalence of items (i) and (ii) in the next theorem was proved by Schnorr [73, Satz 5.3]

4.1.6. THEOREM. *For a class $\mathcal{A} \subseteq 2^\omega$, the following statements are equivalent.*

(i) *$\mathcal{A}$ is Σ_1-approximable,*

(ii) *there is a subcomputable martingale that succeeds on $\mathcal{A}$,*

(iii) $\mu_{\text{r.e.}}(\mathcal{A}) = 0$.

PROOF. (i)$\Rightarrow$(ii). Suppose that $\{\mathcal{S}_i\}_{i\in\omega}$ is a sequential test. Without loss of generality we may assume that the $\mathcal{S}_i$ are prefix-free. For every $w \in 2^{<\omega}$ define the martingale

$$f_w(x) = \begin{cases} 2^{|w|-|x|} & \text{if } x \sqsubseteq w \\ 1 & \text{if } w \sqsubset x \\ 0 & \text{otherwise.} \end{cases}$$

Define the martingale $F_{\mathcal{S}_i}(x) = \sum_{w\in\mathcal{S}_i} f_w(x)$, and note that $F_{\mathcal{S}_i}$ is subcomputable since $\mathcal{S}_i$ is r.e. Finally, define the martingale $F(x) = \sum_{i\in\omega} F_{\mathcal{S}_i}(x)$. F is again subcomputable because all the $F_{\mathcal{S}_i}$ are, and because the $\mathcal{S}_i$ are prefix-free we have $F(x) \leq \sum_i \mu(\mathcal{S}_i) \leq \sum_i 2^{-i} < \infty$. Clearly, if A is in infinitely many $\mathcal{S}_i$ then F succeeds on A.

(ii)$\Rightarrow$(iii). Given a subcomputable martingale g with recursive approximation g_s, we define an r.e.-supermartingale d with $S[d] \supseteq S[g]$ as follows. Define d through a recursive approximation d_s: For every $w \in 2^{<\omega}$, $d_0(w) = 0$, and if $g_{s+1}(w) > d_s(w) - 2^{-|w|-1}$ then put $d_{s+1}(w) = g_{s+1}(w) + 2^{-|w|}$, and put $d_{s+1}(w) = d_s(w)$ otherwise. It is immediate from the definition of d_s that

$$(\forall w \in 2^{<\omega})(\exists s)(\forall t \geq s)[d_t(w) = d_s(w)],$$

hence $d_s(w)$ reaches a limit $d(w)$ after a finite number of steps. It holds for every $w \in 2^{<\omega}$ that $g(w) + 2^{-|w|-1} \leq d(w) \leq g(w) + 2^{-|w|}$, hence it follows from the martingale property of g that

$$\begin{aligned} d(w0) + d(w1) &\leq g(w0) + g(w1) + 2 \cdot 2^{-(|w|+1)} \\ &= 2(g(w) + 2^{-|w|-1}) \\ &\leq 2d(w), \end{aligned}$$

whence d is indeed an r.e.-supermartingale.
(iii)⇒(i). Suppose that $\mu_{\text{r.e.}}(\mathcal{A}) = 0$. By Lemma 4.1.4 we may assume that there is an r.e.-function d that is a martingale and that succeeds on $\mathcal{A}$. Define $\mathcal{S}_i = \{x : d(x) \geq 2^k\}$. Then $\mu(S_i) \leq 2^{-i}$ by Lemma 1.3.4, and $\mathcal{S}_i$ is clearly r.e., so $\{\mathcal{S}_i\}_{i\in\omega}$ is a sequential test. Moreover, if $A \in S[d]$ then for every i there is an x such that $A{\restriction}x \in \mathcal{S}_i$. □

4.1.7. COROLLARY. *A class $\mathcal{A}$ has r.e.-measure 0 if and only if $\mathcal{A}$ does not contain a Martin-Löf random set.*

PROOF. From the existence of a universal Martin-Löf-test (Martin-Löf [59], see Theorem 5.1.3) it follows that

$$\mu_{\text{r.e.}}(\{A \in 2^\omega : A \text{ is Martin-Löf random}\}) = 1.$$

Note that this is stronger than merely saying that the class of Martin-Löf random sets has Lebesgue measure 1 (Schnorr [73, Korollar 4.7]). Hence if $\mathcal{A}$ contains no Martin-Löf random set then $\mu_{\text{r.e.}}(\mathcal{A}) = 0$. The converse is true by the definition of Martin-Löf random set. □

In particular, we can use all the known facts about Martin-Löf randomness in the study of r.e.-measure.

4.2 THE CLASS R(r.e.)

Recall the definition of $R(\Delta)$ from Definition 1.5.3. Lutz [50] observed that

$$R(\text{rec}) = \text{REC}$$

$$\mu_{\text{rec}}(\text{REC}) \neq 0.$$

R(r.e.) is the set of all $R(\delta)$ for r.e. δ, the results of constructors $\delta : 2^{<\omega} \to 2^{<\omega}$ for which the set $\{(x, y) : y \leq \delta(x)\}$ is r.e., where $\leq$ denotes the usual lexicographic ordering on $2^{<\omega}$.

4.2.1. Definition. We say that $A \in 2^\omega$ is *right-limit* of the infinite set of initial segments $X \subseteq 2^{<\omega}$ if $(\forall \sigma \in X)[\sigma \leq A \restriction |\sigma|]$ and $(\forall n)(\exists m \geq n)[A \restriction m \in X]$.
We say that $\sigma \in 2^{<\omega}$ is *right-limit* of the (possibly infinite) set of initial segments $X \subseteq 2^{<\omega}$ if $(\forall \tau \in X)[\tau \sqsubseteq \sigma \vee \exists n(\tau \restriction n < \sigma \restriction n)]$.

It will be useful to have the following characterization. R(r.e.) is the class of sets that are right-limits of r.e. sets of initial segments in $2^{<\omega}$. Equivalently, $A \in R$(r.e.) if and only if there is a recursive function $\phi : \omega \times \omega \to \{0,1\}$ such that $\lim_k \phi(k,n) = A(n)$, and for every $k, n \in \omega$, $\phi(k)\restriction n \leq \phi(k+1)\restriction n$ and

$$(\forall m < n)[\phi(k,m) = A(m)] \wedge \phi(k,n) = 1 \Rightarrow A(n) = 1 \tag{4.1}$$

(if A is the right-limit of an r.e. set $X \subseteq 2^{<\omega}$, with recursive enumeration $\{X_s\}_{s\in\omega}$, we get ϕ as in (4.1) by putting $\phi(k,n) = \tau_k(n)$, where τ_k is the right-limit of X_s).

From this characterization it follows immediately that RE $\subseteq R$(r.e.) $\subseteq \Delta_2$ (the second inclusion follows from the Limit Lemma [72, III.3.3]). In the next theorem we locate the class R(r.e.) more precisely. We denote by DRE the class of differences $W_e - W_d$ of r.e. sets (the *d.r.e.* sets, see [72, p57]). This is precisely the class of sets with a recursive approximation that changes at most two times for every argument (starting with 0).

4.2.2. Theorem. *The inclusions* RE $\subseteq R$(r.e.) $\subseteq {}^{\leq_{tt}}K$ *are both proper. Furthermore,* R(r.e.) $\not\subseteq {}^{\leq_{btt}}K$ *and* DRE $\not\subseteq R$(r.e.).

$$\begin{array}{ccccccc} & & \text{DRE} & \subsetneq & {}^{\leq_{btt}}K & & \\ & \subsetneq & & & & \subsetneq & \\ \text{RE} & & & & & & {}^{\leq_{tt}}K \\ & \subsetneq & & & & \subsetneq & \\ & & & R(\text{r.e.}) & & & \end{array}$$

Proof. Note that indeed we have the inclusion R(r.e.) $\subseteq {}^{\leq_{tt}}K$ by the characterization in Lemma 4.1.2, where we can take $h(x) = 2^x$. That the two inclusions RE $\subseteq R$(r.e.) $\subseteq {}^{\leq_{tt}}K$ are both proper will follow from R(r.e.) $\not\subseteq {}^{\leq_{btt}}K$ and DRE $\not\subseteq R$(r.e.), since RE $\subsetneq$ DRE $\subsetneq {}^{\leq_{btt}}K \subsetneq {}^{\leq_{tt}}K$.

One can prove R(r.e.) $\not\subseteq {}^{\leq_{btt}}K$ directly by a diagonalization construction, but the result will also follow from Theorem 4.3.1 (iii). So we do not give a proof here.

That DRE $\not\subseteq R$(r.e.) follows from results in [72] that require a finite injury argument, but it can also be proved directly, as we show now. To prove that

DRE $\not\subseteq R$(r.e.) we use a finite injury argument to construct a d.r.e. set A such that for every e the requirement

> R_e: If $\varphi_e : \omega \times \omega \to \{0,1\}$ is a recursive approximation of A satisfying $\phi(k)\restriction n \leq \phi(k+1)\restriction n$ for every $k, n \in \omega$, then there is an n such that condition (4.1) is not satisfied.

is satisfied. We define A as a limit of recursive sets A_s, such that $|\{s : A_s(x) \neq A_{s+1}(x)\}| \leq 2$ for every $x \in \omega$. For every requirement R_e we will have a witness x_e satisfying it, i.e., x_e is such that if $(\exists k)(\forall n < x_e)[\varphi_e(k,n)\downarrow = A(n)]$ and $\varphi_e(k, x_e)\downarrow$ then $A(x_e) = 1 - \varphi_e(k, x_e)$. We give the requirements a priority ranking: R_e has *higher priority than* R_d if $e < d$. A requirement R_d is *injured* by a higher priority requirement R_e at a given stage s if action is taken to satisfy R_e at s. Since this may cause us to undertake action several times to satisfy R_d we may have to choose several witnesses to satisfy it. So we will have witnesses x_e^s for R_e for every stage s. Since it is crucial that the witnesses x_e^s for R_e are not used for any other requirement, we reserve the e-th section $\omega^{[e]}$ for R_e. As is typical for finite injury arguments, we will only have to pick a new witness a finite number of times, and will be able to satisfy R_d for ever after. R_e *requires attention* at stage s if

$$(\forall t \leq s)(\forall n \leq x_e^s)[\varphi_e(t,n)\downarrow \in \{0,1\}], \tag{4.2}$$

$$(\forall t < s)[\varphi_{e,s}(t)\restriction(x_e^s + 1) \leq \varphi_{e,s}(t+1)\restriction(x_e^s+1)], \text{ and} \tag{4.3}$$

$$(\forall n \leq x_e^s)[\varphi_{e,s}(s,n)\downarrow = A_s(n)]. \tag{4.4}$$

So R_e requires attention at stage s if at s all the computations $\varphi_e(t,n)$, with $t \leq s$ and $n \leq x_e^s$, have converged and show that φ_e up to stage s is a monotonic approximation of $A\restriction x_e^s + 1$, as in (4.1). Define $A_0 = \emptyset$ and $x_e^0 = \langle 0, e\rangle$. At stage $s+1$ of the approximation, choose the least $e \leq s$ such that R_e requires attention. If such e does not exist, define $A_{s+1} = A_s$ and $x_e^{s+1} = x_e^s$ for every e. Otherwise, define $A_{s+1}(x_e^s) = 1 - \varphi_{e,s}(s, x_e^s)$, and $A_{s+1}(x) = A_s(x)$ for all $x \neq x_e^s$, and say that R_e *is active.* In this case define new witnesses for the lower priority requirements: For all $d \leq e$, $x_d^{s+1} = x_d^s$, and for all $d > e$, recursively define x_d^{s+1} to be the least number in $\omega^{[d]}$ greater than x_{d-1}^{s+1}. This ends the description of the construction of A.

LEMMA 4.2.2.1 *Suppose that R_e requires attention at stage s, that $A_s\restriction x_e^s = A\restriction x_e^s$, $(\forall t > s)[x_e^t = x_e^s]$, and that $\varphi_{e,s}(s, x_e^s) = 1$, Then R_e does not require attention at any stage $t > s$.*

PROOF. Suppose R_e requires attention at some stage larger than s, and let t be the first such stage. By (4.4) above, $A_{s+1}(x_e^s) = 1 - \varphi_{e,s}(s, x_e^s) = 0$, hence $\varphi_{e,t}(t, x_e^s) = 0$. Since by assumption $A_s{\restriction}x_e^s = A{\restriction}x_e^s$, by (4.4)

$$\varphi_{e,t}(t){\restriction}x_e^t + 1 = (A_t{\restriction}x_e^t)0 < (A_t{\restriction}x_e^t)1 = (A_s{\restriction}x_e^s)1 = \varphi_{e,s}(s){\restriction}x_e^s + 1,$$

contradicting that R_e requires attention at t. This proves Lemma 4.2.2.1. $\square$

By induction on e we prove that every R_e requires attention only finitely often, and is satisfied forever after the last stage that it required attention. In particular $A(x) = \lim_s A_s(x)$ exists for every x. Suppose for $e \in \omega$ that after stage s no R_d with $d < e$ requires attention. Then in particular no R_d with $d < e$ is active after stage s, hence $A_s{\restriction}x_e^s = A{\restriction}x_e^s$, since for every $t > s$ and $f \geq e$, $x_f^t \geq x_f^s \geq x_e^s$. If R_e never requires attention after stage s then one of (4.2), (4.3), or (4.4) above is never satisfied for $t \geq s$. Hence φ_e is not a monotonic recursive approximation of A, and R_e is vacuously satisfied. Now suppose that R_e requires attention at some stage greater than s, and let $t \geq s$ be the least such stage. Then $(\forall n \leq x_e^s)[\varphi_{e,t}(t,n)\downarrow = A_t(n)]$, and by the construction $A_{t+1}(x_e^s) = 1 - \varphi_{e,t}(t, x_e^s)$ (note that since no R_d with $d < e$ is active after stage s, for every $t \geq s$, $x_e^t = x_e^s$). If R_e never requires attention after stage t then it is satisfied by the action taken at this stage. Now suppose R_e requires attention at some stage greater than t, and let $t' > t$ be the least such stage. Then $\varphi_{e,t'}(t', x_e^s)$ equals $1 - \varphi_{e,t}(t, x_e^s)$. It now follows from Lemma 4.2.2.1 that R_e can never require attention at any stage later than t': since R_e required attention two times since stage s (at stage t and at stage t'), $\varphi_e(x_e^s)$ has taken the value 1 at one of these stages. By Lemma 4.2.2.1, R_e does not require attention after this stage. Hence R_e is satisfied forever after stage t'. From the above analysis we see in particular that each R_e is active at most two times after the last stage at which it was injured. Since every time R_e is injured a new witness is being chosen, for every witness x_e, $A_s(x_e)$ changes at most two times. Hence A is in DRE. This concludes the proof of DRE $\not\subseteq R$(r.e.), and the proof of Theorem 4.2.2. $\square$

4.3 The measure of some basic classes

One may now wonder what the r.e.-measure is of classes such as REC, RE, and R(r.e.). One can easily prove that $\mu_{\text{rec}}(\text{REC}) \neq 0$ by constructing for each recursive martingale d a recursive set A such that d does not succeed on A: Given $A{\restriction}n$ define $A(n) = 1$ iff $d((A{\restriction}n)1) \leq d((A{\restriction}n)0)$. However, this argument fails if d is an r.e.-supermartingale since the set A constructed as above will only be Δ_2. The following theorem shows that indeed the analogous result is not true at all.

Let $\mathcal{B}$ be the smallest Boolean algebra containing all the r.e. sets, i.e. $\mathcal{B}$ is the closure of the class of r.e. sets under complementation, union, and intersection

($\mathcal{B}$ is Ershov's *Boolean*, or *difference hierarchy*, see Odifreddi [64]). $\mathcal{B}$ is exactly the "lower cone" of sets that are bounded truth-table reducible to K ([64, Prop. III.8.7]).

4.3.1. THEOREM. (i) $\mu_{\text{r.e.}}(\text{RE}) = 0$.
(ii) $\mu_{\text{r.e.}}({}^{\leq_{btt}}K) = 0$.
(iii) $\mu_{\text{r.e.}}(R(\text{r.e.})) \neq 0$.

PROOF. (i) follows from the well-known fact that no Martin-Löf random set is r.e. (In fact, every Martin-Löf random set is bi-immune [37].) (ii) follows from (i) and the fact that for every set A in $\mathcal{B}$ either A or $\overline{A}$ contains an infinite r.e. set (Jockusch and others [72, III.3.10]).
(iii) Let d be any r.e.-supermartingale. We have to show that there is an element of $R(\text{r.e.})$ on which d does not succeed. Let A be the leftmost path in 2^ω such that $d(A\restriction n) \leq 1$ for every n. Note that A exists since for any martingale d with $d(\lambda) = 1$, $\{B \subseteq \omega : \forall n(d(B\restriction n) \leq 1\}$ is nonempty, in fact, has positive Lebesgue measure (cf. Lemma 6.1.2). It is easy to see, using that d is r.e., that A is the right-limit of an r.e. set of initial segments, and hence that $A \in R(\text{r.e.})$. □

4.3.2. COROLLARY. *The measures μ_{rec} and $\mu_{\text{r.e.}}$ are unequal.*

PROOF. $\mu_{\text{r.e.}}(\text{REC}) = 0$ by Theorem 4.3.1 (i), but $\mu_{\text{rec}}(\text{REC}) \neq 0$ by Theorem 1.5.6. □

Theorem 4.3.1 shows that r.e.-measure is not suited for the quantitative study of RE, hence Lutz's approach [50], that worked for classes like 2^ω, REC, and the linear and polynomial deterministic exponential time and space classes, does not work here.

It follows from Theorem 4.3.1 that if A is r.e. then $\mu_{\text{r.e.}}(\{A\}) = 0$. (For an individual r.e. set A it even holds that $\mu_{\text{rec}}(\{A\}) = 0$ since every infinite r.e. set contains an infinite recursive subset.) The converse is certainly not true: it is easy to construct a martingale which succeeds on all nondense sets (i.e. sets with a characteristic string that contains significantly more zeros than ones), and among those are sets of arbitrary high complexity.

Corollary 4.3.2 shows that the approach using martingales d with the property $d(w0) + d(w1) = 2d(w)$ instead of our supermartingales with the weaker property $d(w0)+d(w1) \leq 2d(w)$ does indeed make a difference (cf. the discussion at page 62).

It follows from Theorem 4.4.4 that $\mu_{\text{r.e.}}(\Delta_2) \neq 0$. Δ_2 is exactly the "lower cone" of sets that are Turing reducible to K, the halting set. In our notation: $\Delta_2 = {}^{\leq_T}K$. Results on the measure of cones form a classical topic in the intersection of measure theory and computability theory. As a corollary to Theorem 4.3.1 we have a stronger result than $\mu_{\text{r.e.}}({}^{\leq_T}K) \neq 0$, saying that for truth-table reducibility the lower cone ${}^{\leq_{tt}}K$ does not have r.e.-measure 0.

4.3.3. COROLLARY. $\mu_{\text{r.e.}}({}^{\leq_{tt}}K) \neq 0$.

PROOF. In Theorem 4.2.2 we saw that $R(\text{r.e.}) \subset {}^{\leq_{tt}}K$, so the result follows from Theorem 4.3.1 (iii). $\square$

Theorem 4.3.1 (ii), in conjunction with Corollary 4.3.3, gives a precise border between r.e.-measure zero and r.e.-measure nonzero in terms of the common recursion-theoretic reducibilities below K.

4.4 ARITHMETICAL MEASURE

We have the following function classes corresponding to the various levels of the arithmetical hierarchy.

4.4.1. DEFINITION. The class of (total) Σ_n-functions, $n \geq 1$, is defined to be

$$\{f : 2^{<\omega} \to \mathbb{Q}^+ : \{(x,y) : f(x) \geq y\} \text{ is } \Sigma_n\}.$$

Similarly, the class of (total) Π_n-functions, $n \geq 1$, consists of $\{f : 2^{<\omega} \to \mathbb{Q}^+ : \{(x,y) : f(x) \leq y\} \text{ is } \Sigma_n\}$. The function classes Δ_n are defined as $\Delta_n = \Sigma_n \cap \Pi_n$.

Note that the Σ_n-functions are those that are Δ_n-approximable from below, by a Δ_n-function that attains its limit value. Similarly Π_n-functions can be approximated from above in the same manner. Therefore, the Δ_n-functions coincide with the functions computable recursively in $\emptyset^{(n)}$.

The measures μ_{Σ_n}, μ_{Π_n}, and μ_{Δ_n}, with $n \geq 1$, are defined exactly as the measures $\mu_{\text{r.e.}}$ and μ_{rec}. So, for example, $\mu_{\Pi_n}(\mathcal{A}) = 0$ if there is no supermartingale with a computation in Π_n that succeeds on $\mathcal{A}$. Again, as in the case of rec-measure, it is clear that in the definition of μ_{Δ_n} we may use martingales instead of supermartingales (Proposition 1.5.5).

Now Lemma 4.1.4 is proved exactly as before, and the proof of Theorem 4.3.1 relativizes.

4.4.2. THEOREM. *For all $n \geq 1$, $\mu_{\Sigma_n}({}^{\leq_{btt}}(\emptyset^{(n)})) = 0$.*

PROOF. Relativize the proof of Theorem 4.3.1 (ii) to the oracle $\emptyset^{(n)}$. $\square$

4.4.3. COROLLARY. *For all $n \geq 1$, $\mu_{\Sigma_n} \neq \mu_{\Delta_n}$.*

In the next theorem we find an exact border between Δ_2-measure zero and Δ_2-measure nonzero in terms of the reducibilities $\leq_{wtt}$ and $\leq_T$ below K.

4.4.4. THEOREM. *Let $n \geq 2$. For every $A \in \Delta_n$ it holds that $\mu_{\Delta_n}({}^{\leq_{wtt}}A) = 0$. Hence, for every $n \geq 2$, $\mu_{\Delta_n}({}^{\leq_{wtt}}(\emptyset^{(n-1)})) = 0$. For every $n \geq 1$, $\mu_{\Delta_n}(\Delta_n) \neq 0$.*

PROOF. We prove that for every $A \in \Delta_2$ it holds that $\mu_{\Delta_2}({}^{\leq_{wtt}}A) = 0$. It suffices to uniformly K-compute d_i such that d_i covers $B \leq_{wtt} A$, if $B = \{e\}^A$ with use bounded by φ_n, $i = \langle e, n\rangle$. Then a standard sum-argument (cf. Lemma 2.2.2) shows that there is one martingale that succeeds on ${}^{\leq_{wtt}}A$, cf. Lemma 2.2.2. That such a sequence of martingales d_i exists easily follows from the claim that ${}^{\leq_{wtt}}A$ is uniformly K-computable. To prove this claim we show how to compute the x-th bit $B(x)$ of the $\langle e, n\rangle$-th set B in ${}^{\leq_{wtt}}A$. First K-compute whether $\varphi_n(x)$ converges. If not, then the bound on the use φ_n is not total and the wtt-reduction $\langle e, n\rangle$ is a fake. If $\varphi_n(x)\downarrow$ then K-compute $A\restriction\varphi_n(x)$. Finally, K-compute $\{e\}^{A\restriction\varphi_n(x)}(x)$. If this is not defined then $\langle e, n\rangle$ is again not a wtt-reduction, and if $\langle e, n\rangle$ indeed codes a wtt-reduction we K-compute in this fashion all the bits of $B = \{e\}^A$. This proves the claim and the first part of the theorem.

The last part, $\mu_{\Delta_n}(\Delta_n) \neq 0$, is proved exactly as $\mu_{\text{rec}}(\text{REC}) \neq 0$ (cf. discussion preceding Theorem 4.3.1 or Theorem 1.5.6). □

Note that in the above argument it is crucial that we consider wtt-reductions and not Turing-reductions: if a Turing-reduction to the set A becomes defined, and later it becomes undefined because of a change in the oracle, it may happen that if at a later stage it becomes defined again the use of this last computation is bigger than the use of the first defined computation. Hence, for some 'fake' reduction, the process in the proof above may not terminate.

Note that analogous to Theorem 4.2.2 we have that $\Sigma_n \subsetneq R(\Sigma_n) \subsetneq \Delta_{n+1}$ and $\Pi_n \subsetneq R(\Pi_n) \subsetneq \Delta_{n+1}$. From Theorem 4.4.4 it follows that for every $n \geq 1$, $\mu_{\Delta_{n+1}}(R(\Sigma_n)) = 0$ and $\mu_{\Delta_{n+1}}(R(\Pi_n)) = 0$. In contrast to this result we have the generalized version of Theorem 4.3.1 (iii) (the proof of the second part is completely symmetric):

4.4.5. THEOREM. *For all $n \geq 1$, $\mu_{\Sigma_n}(R(\Sigma_n)) \neq 0$ and $\mu_{\Sigma_n}(R(\Pi_n)) \neq 0$.*

To complete the picture of inclusions, note that $\Delta_n = R(\Delta_n) = R(\Sigma_n) \cap R(\Pi_n)$. It follows that $\Sigma_n \not\subseteq R(\Pi_n)$ since otherwise $\Sigma_n \subseteq R(\Sigma_n) \cap R(\Pi_n) = \Delta_n$, a contradiction. In particular $R(\Sigma_n) \not\subseteq R(\Pi_n)$. So the only inclusion relations are $\Delta_n \subsetneq \Sigma_n \subsetneq R(\Sigma_n) \subsetneq \Delta_{n+1}$, $\Delta_n \subsetneq \Pi_n \subsetneq R(\Pi_n) \subsetneq \Delta_{n+1}$, and no other inclusions hold.

In Corollary 4.4.3 we have seen that the measure induced by the function class Σ_n differs from the measure induced by Δ_n. We now prove that, surprisingly, the measure induced by Π_n equals the latter. Whence the measure μ_{Σ_n} is stronger (more sets have measure 0) than the measure μ_{Π_n}. The reason for this asymmetry lies in the asymmetry of the supermartingale property $d(w0) + d(w1) \leq 2d(w)$, which makes Σ_n-supermartingales more powerful than Π_n-supermartingales.

4.4.6. THEOREM. *For all $n \geq 1$, $\mu_{\Pi_n} = \mu_{\Delta_n}$.*

PROOF. We give the proof for $n = 1$. The proof for arbitrary $n \in \omega$ is obtained by relativizing the following proof to the oracle $\emptyset^{(n)}$. It suffices to prove that if a class has Π_1-measure 0 then it has rec-measure 0. Let d be a Π_1-supermartingale, with nonincreasing recursive approximation d_s say. We prove that there exists a rec-supermartingale d' with $S[d'] \supseteq S[d]$. (This suffices by Proposition 1.5.5.) Without loss of generality $d(\lambda) = 1$. Define $d'(\lambda) = 1$. Suppose now that $d'(w)$ has been defined and that $d'(w) \geq d(w)$. Choose the least $s \in \omega$ such that $d_s(w0) + d_s(w1) \leq 2d_s(w)$. Note that s exists since d is a supermartingale and $(\exists s)(\forall t \geq s)[d_t(w) = d(w)]$. Define $d'(wi) = d_s(wi)$, for $i \in \{0, 1\}$. Then $d'(wi) \geq d(wi)$ because $d_s(wi)$ is nonincreasing in s. For d' thus defined we have that $d'(w0) + d'(w1) \leq 2d_s(w) \leq 2d'(w)$, so d' is a supermartingale. Clearly d' is recursive, and $S[d'] \supseteq S[d]$ because for every $w \in 2^{<\omega}$, $d'(w) \geq d(w)$. □

4.5 NOTES

Kautz [34, p26] has shown that a class $\mathcal{A}$ is Σ_n^C-approximable if and only if there is a $C^{(n-1)}$-recursive sequence (rather than just recursive) of Σ_n^C-classes $\{\mathcal{S}_i\}_{i \in \omega}$ with $\mu(\mathcal{S}_i) \leq 2^{-i}$ and $\mathcal{A} \subseteq \bigcap_i \mathcal{S}_i$. It follows that we may relativize the result of Theorem 4.1.6 to the oracle $\emptyset^{(n-1)}$ to obtain

4.5.1. THEOREM. *For every class $\mathcal{A} \subseteq 2^\omega$ and every $n \geq 1$, $\mathcal{A}$ is Σ_n-approximable if and only if $\mu_{\Sigma_n}(\mathcal{A}) = 0$.*

Schnorr [73, Satz 7.6] proves that there is a 1-random (Martin-Löf random) set A in Δ_2. A natural example of such a set is Chaitins Halting Probability Ω [46, p187]. For more on Martin-Löf random sets in Δ_2 see M. van Lambalgen [42]. The results from the previous sections show

4.5.2. THEOREM. (i) *There is a Martin-Löf random set in ${}^{\leq_{tt}}K$.*
(ii) *There is no Martin-Löf random set in ${}^{\leq_{btt}}K$.*

PROOF. This is immediate from Corollary 4.1.7, Corollary 4.3.3 and Theorem 4.3.1 (ii). □

With a similar proof one can actually show, using Theorem 4.5.1, that for any $C \in 2^\omega$, there exists a C-n-random set in ${}^{\leq_{tt}}C^{(n)}$ (even in $R(\Sigma_n^C)$) but not in ${}^{\leq_{btt}}(C^{(n)})$.

It is known that the Martin-Löf random set of Theorem 4.5.2 (i) cannot have the same tt-degree as K (Bennett [13], Juedes, Lathrop, and Lutz [30]).

Note that there is not an analogue of the result $\mu_{\text{r.e.}}(\{A \in 2^\omega : A \text{ is Martin-Löf random}\}) = 1$ for the case of Δ_n-measure. In fact, it is easy to see that

$\mu_{\Delta_n}(\{A \in 2^\omega : A \text{ is } \Delta_n\text{-random}\}) \neq 1$. Namely, for every Δ_n-martingale d there exists $A \in \Delta_n$ such that $A \notin S[d]$. Hence d does not succeed on all the non-Δ_n-random sets. However, the class of Δ_n-random sets does of course have Lebesgue measure 1.

It follows from results of Arslanov and Kučera [37] that if A is an r.e. set that is Turing-incomplete then $\mu_{\text{r.e.}}({\leq_T} A) = 0$.

Chapter 5

Randomness and Lowness

In this chapter we study sets that are *low* for the class $\mathcal{R}$ of Martin-Löf random reals and the class $\mathcal{S}$ of Schnorr random reals. A set A is low for a class $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}^A$. In Section 5.2 we prove that there is a nonrecursive r.e. set that is low for $\mathcal{R}$, thereby answering a question raised by M. van Lambalgen and D. Zambella. In Section 5.4 we prove that there are uncountably many sets that are low for the class of Schnorr random reals $\mathcal{S}$. We give a purely recursion theoretic characterization of these sets and show that they all have Turing degree incomparable to $0'$, the degree of the halting problem. This contrasts with the case of $\mathcal{R}$.

5.1 Introduction

The first three sections of this chapter are concerned with the notion of randomness as originally defined by P. Martin-Löf in [59]. Recall Definition 4.1.5. A set is Martin-Löf-random, or 1-random for short, if it cannot be approximated in measure by a Martin-Löf test. These sets have played a central role in the study of algorithmic randomness. One can relativize this definition of randomness to an arbitrary oracle. Relativized randomness has been studied by several authors. The intuitive meaning of "A is 1-random relative to B" is that A is independent of B. A justification for this interpretation is given by M. van Lambalgen [44]. In this introduction we review some of the basic properties of sets which are 1-random and we state the main problem.

The e-th r.e. set W_e can be both interpreted as a set of numbers $W_e \subseteq \omega$ or a set of initial segments $W_e \subseteq 2^{<\omega}$. In the last case W_e defines the Σ^0_1 class $\mathrm{Ext}(W_e) = \{A \in 2^\omega : (\exists \sigma \in W_e)\,[\sigma \sqsubset A]\}$. The distinction will always be clear from the context. Instead of $\mu(\mathrm{Ext}(W_e))$ we also write $\mu(W_e)$.

5.1.1. DEFINITION. (Martin-Löf [59], Kautz [34]) A class $\mathcal{A}$ is $\Sigma_1^{0,A}$-*approximable* if there is an A-recursive function f such that for the $\Sigma_1^{0,A}$ classes $W^A_{f(i)}$ it holds that $\mu(W^A_{f(i)}) < 2^{-i}$ and $\mathcal{A} \subseteq \bigcap_i W^A_{f(i)}$. A set C is Martin-Löf-random relative to A, or A-1-*random* for short, if $\{C\}$ is not $\Sigma_1^{0,A}$-approximable. The class of A-1-random sets is denoted by $\mathcal{R}^A$. If A is recursive we write $\mathcal{R}$ instead of $\mathcal{R}^A$.

5.1.2. DEFINITION. A set A is *low for* a class $\mathcal{C}$ if the relativized version $\mathcal{C}^A$ of $\mathcal{C}$ satisfies $\mathcal{C} = \mathcal{C}^A$. The class of sets that are low for $\mathcal{C}$ is denoted by $\mathrm{Low}(\mathcal{C})$.

For example, the ordinary low sets from recursion theory are the sets that are low for the class of T-complete sets, and a set is low for the class of recursive sets if and only if it is recursive. For a class $\mathcal{C}$, the class $\mathrm{Low}(\mathcal{C})$ consists of the oracles that are not 'helpful' for $\mathcal{C}$ in the sense that they do not alter $\mathcal{C}$. A set $A \in \mathrm{Low}(\mathcal{C})$ either contains no information that is useful for $\mathcal{C}$, or the information in it is coded in such a way that elements from $\mathcal{C}$ cannot retrieve it. In this chapter we are interested in sets that are low for $\mathcal{R}$ and $\mathcal{S}$. We first consider the case of $\mathcal{R}$. In Section 5.4 we consider the case of $\mathcal{S}$.

Motivated by the work in [42], M. van Lambalgen and D. Zambella formulated the question whether there exist nontrivial examples of sets A such that every random set is already random relative to A. (The question is first explicitly stated in Zambella [82].) This question was raised in the context of a comparison between randomness properties in classical dynamic systems (specifically, Bernoulli sequences) and recursion theoretic randomness. A famous result of Kamae [33] showed that the infinite binary sequences that have no information about Bernoulli sequences (normal sequences) are precisely the sequences with zero entropy. The question was whether a similar characterization exists for sets that have no information about Martin-Löf random sequences. This motivates the question whether every element of $\mathrm{Low}(\mathcal{R})$ has to be recursive.

First, it is not immediately clear that there is a 1-random set that has a nonrecursive set Turing-below it in which it is 1-random. That this situation at least can occur was proved by Kučera in [38] by consideration of diagonally nonrecursive functions. He also proved that if a nonrecursive set A admits an A-1-random set above it (which is the case when $A \in \mathrm{Low}(\mathcal{R})$, cf. the proof of Corollary 5.3.4) then A is not too complex in the sense that A is generalized low (GL_1), i.e. $A \oplus \emptyset' \equiv_T A'$, see Corollary 5.3.4. In Section 5.2 we prove that indeed the class $\mathrm{Low}(\mathcal{R})$ contains nonrecursive sets, thereby answering the above question.

Next we prove some facts that will be useful later. A recursive sequence of Σ_1^0 classes such as in Definition 5.1.1 is called a *sequential test*. The next theorem shows that there are sequential tests that are *universal* in the sense that they cover all the sets that are covered by some sequential test.

5.1.3. THEOREM. (Martin-Löf [59]) *There exists a universal sequential test. That is, there is a recursive sequence of Σ_1^0 classes $\mathcal{U}_0, \mathcal{U}_1, \ldots$ such that*

- $\mathcal{U}_0 \supseteq \mathcal{U}_1 \supseteq \ldots$
- $\forall n\, (\mu(\mathcal{U}_n) < 2^{-n})$
- *for any Σ_1^0-approximable class $\mathcal{A}$ we have $\mathcal{A} \subseteq \bigcap_n \mathcal{U}_n$.*

PROOF. For every n construct an r.e. set $U_n \subseteq 2^{<\omega}$ as follows. For every $e > n$, U_n enumerates all the elements of $W_{\{e\}(e)}$ (where we take this set to be empty if $\{e\}(e)$ is undefined) as long as $\mu(W_{\{e\}(e)}) < 2^{-e}$. Define $\mathcal{U}_n = \mathrm{Ext}(U_n)$. Then $\mu(\mathcal{U}_n) < \sum_{e>n} 2^{-e} = 2^{-n}$, and if $\{e\}$ defines a sequential test then for every n there exists by padding $i \in \omega$ (in fact, infinitely many i) such that $W_{\{e\}(i)} \subseteq \mathcal{U}_n$ (if $i > n$ is an alternative code for e then $W_{\{e\}(i)} = W_{\{i\}(i)} \subseteq U_n$), so $\bigcap_i W_{\{e\}(i)} \subseteq \bigcap_n \mathcal{U}_n$. □

5.1.4. DEFINITION. For every n, denote by $\mathcal{U}_n$ the Σ_1^0 class from the above proof. Define $\mathcal{P}_n$ to be the complement of $\mathcal{U}_n$.

Define the *left shift* $T : 2^\omega \to 2^\omega$ by $T(C)(n) = C(n+1)$. Let T^k denote the k-iteration of T.

5.1.5. LEMMA. *For every $C \in \mathfrak{R}$ there exists $k \in \omega$ such that $T^k(C) \in \mathcal{P}_0$.*

PROOF. For a set of initial segments Σ and a class $\mathcal{A}$ define $\Sigma\hat{\ }\mathcal{A} = \{\sigma\hat{\ }A : \sigma \in \Sigma \wedge A \in \mathcal{A}\}$, where $\sigma\hat{\ }A$ denotes the concatenation of σ with the characteristic sequence of A. Fix an r.e. set U_0 that defines the Σ_1^0 class $\mathcal{U}_0$. By induction define $\mathcal{U}_0^1 = \mathcal{U}_0$ and $\mathcal{U}_0^{k+1} = U_0\hat{\ }\mathcal{U}_0^k$.
Now by $q = \mu(\mathcal{U}_0) < 1$ there is an $l \in \omega$ such that $q^l < 1/2$, so $\mu(\mathcal{U}_0^{kl}) = q^{kl} < 2^{-k}$. It follows that the sequence

$$\mathcal{U}_0,\ \mathcal{U}_0^l,\ \mathcal{U}_0^{2l},\ \ldots$$

constitutes a sequential test. Therefore, if $C \in \mathfrak{R}$ then either $C \notin \mathcal{U}_0$, i.e. $C \in \mathcal{P}_0$ and we are done, or for some $k > 0$ we have $C \in \mathcal{U}_0^{kl}$ and $C \notin \mathcal{U}_0^{(k+1)l}$. But the latter means that $T^{k'}(C) \notin \mathcal{U}_0$ for some k', so $T^{k'}(C) \in \mathcal{P}_0$. □

When we relativize the concept of a sequential test to an oracle A it makes no difference if we relativize the function that gives the indices of the levels of the test or not, as the following standard lemma shows.

5.1.6. LEMMA. *Let f be an A-recursive function. Then there is a recursive function g such that $W^A_{f(n)} = W^A_{g(n)}$ for every n.*

PROOF. Suppose that e is a code for f, i.e. $f(n) = \{e\}^A(n)$ for every n. Let d be a code of a partial recursive function such that for every oracle X, $\{d\}^X(n, x) = \{\{e\}^X(n)\}^X(x)$. Now apply the S^m_n-theorem to obtain g with code $S^1_1(d)$. □

5.2 A nonrecursive set that is low for the Martin-Löf random sets

A set which is low for $\mathcal{R}$ is computationally weak in the sense that it cannot detect any regularity in any 1-random sequence. Clearly every recursive set is in Low($\mathcal{R}$). This section is devoted to a proof that also nontrivial examples of such sets exist.

5.2.1. THEOREM. *There exists a nonrecursive r.e. set A that is low for $\mathcal{R}$.*

PROOF. We make A simple to guarantee nonrecursiveness. That is, during the construction we want to satisfy the requirements

$$R_z : \quad W_z \text{ is infinite } \Rightarrow \; W_z \cap A \neq \emptyset.$$

By Theorem 5.1.3 and Lemma 5.1.6, let f be a recursive function that defines the universal sequential test relative to X for any set X. That is, for every i, $f(i)$ is an index of the i-th level $W^X_{f(i)}$ of the universal sequential test relative to X. So for every oracle X and every i it holds that $\mu(W^X_{f(i)}) < 2^{-i}$. Simultaneously with A we describe a program coded by e ($e > 0$) such that

$$W_{\{e\}(e)} \supseteq W^A_{f(e+1)} \tag{5.1}$$

and such that $\mu(W_{\{e\}(e)}) < 2^{-e}$. By the recursion theorem we may assume that we know the number e in advance. Note that by construction of the first level $\mathcal{U}_0$ of the universal sequential test (see above) the equation (5.1) implies that the $(e+1)$-st level of the universal sequential test relative to A is included in $\mathcal{U}_0$. In particular $\mathcal{R}^A \supseteq \mathcal{P}_0$. So if $C \in \mathcal{R}$, then $T^k(C)$ is in $\mathcal{P}_0$ for some k by Lemma 5.1.5, hence $T^k(C) \in \mathcal{R}^A$, and therefore $C \in \mathcal{R}^A$. So (5.1) guarantees that A is low for $\mathcal{R}$.

Let A_s denote the (finite) set of elements of A enumerated by the end of stage s. To be able to satisfy (5.1) we want to make sure that whenever y enters A at stage s for the sake of R_z, the 'mistake' we have made, that is, the amount of measure enumerated up to stage s on the basis of '$A(y) = 0$', is small, so that we can correct it without danger of enumerating too much in total. Given y and s, let M^s_y be the set of all strings $\sigma \in \bigcup_{t<s} W^{A_t}_{f(e+1),t}$ such that for some $t < s$ with $y \notin A_t$ the computation $\{f(e+1)\}^{A_t}_t(\sigma)$ converges and has use bigger than y, and such that there is no $\tau \sqsubseteq \sigma$ such that $(\exists t < s)[\text{use}(\{f(e+1)\}^{A_t}_t(\tau)\downarrow) < y]$. That is, M^s_y is the set of strings σ that contribute to the measure of $W_{\{e\}(e)}$ (this set will be defined below) on the basis of '$A(y) = 0$', and that were not yet enumerated (or implicitly enumerated because some initial segment was enumerated) on the basis of some other computation before stage s that did not

use the bit $A(y)$. We think of M_y^s as the potential mistake we make, which may become a real mistake when we enumerate y into A, thereby changing the bit $A(y)$ from 0 to 1. Note that we do not require different mistakes to be disjoint and that mistakes may be counted more than once. The (finite) set M_y^s defines a Σ_1^0 class of which we can compute the measure.
We say that R_z *requires attention* at stage s if

$$(\exists y \leq s)\,[y \in W_{z,s} \wedge y \geq 2z \wedge W_{z,s} \cap A_s = \emptyset \wedge \mu(M_y^s) \leq 2^{-z-e-2}]. \qquad (5.2)$$

The construction of A is now easily described:
Stage $s = 0$. Define $A_0 = \emptyset$.
Stage $s > 0$. For every $z \leq s$ such that R_z requires attention at s, pick some y witnessing this, say the smallest y satisfying (5.2), and enumerate y into A_s.
The number $\{e\}(e)$ is defined to be an index of a Σ_1^0 class such that whenever σ is enumerated into $W_{f(e+1),s}^{A_s}$ then σ is enumerated into $W_{\{e\}(e)}$, i.e., $W_{\{e\}(e)}$ is defined as

$$W_{\{e\}(e)} = \bigcup_{s\in\omega} W_{f(e+1),s}^{A_s}\,.$$

Note that when the oracle A changes, say because y enters A at stage s, no further string is enumerated into $W_{\{e\}(e)}$ using the 'wrong' bit $A(y) = 0$ because of the 's' occurring in the subscript.

LEMMA 1 $\mu(W_{\{e\}(e)}) < 2^{-e}$.
PROOF. The measure of $W_{\{e\}(e)}$ is by definition equal to the measure of $W_{f(e+1)}^A$ plus the amounts of measure $\mu(M_y^s)$ enumerated by 'mistake' because the approximation to A was changed. Because the approximation to A is only changed for the sake of R_z if this mistake is not bigger than 2^{-z-e-2} and every R_z requires attention at most once we have

$$\mu(W_{\{e\}(e)}) < 2^{-(e+1)} + \sum_{z\in\omega} 2^{-z}\cdot 2^{-e-2} = 2^{-e}$$

□ Lemma 1

LEMMA 2 *R_z is satisfied for every z.*
PROOF. Suppose W_z is infinite and that for all $y \geq 2z$ with $y \in W_{z,s}$ it holds that $\mu(M_y^s) > 2^{-z}\cdot 2^{-e-2}$. First observe that for every y and s there exist $y' > y$ and $s' > s$ such that for every $v \geq y'$ and every $t \geq s'$ we have $\mathrm{Ext}(M_v^t) \cap \mathrm{Ext}(M_y^s) = \emptyset$. To see that y' and s' exist, define the downward closure

$$\mathrm{downcl}(M_y^s) = \{\tau \in 2^{<\omega} : \exists\sigma \in M_y^s(\tau \sqsubseteq \sigma)\}.$$

Let $t_0 \in \omega$ be so large that

$$\{\tau \in \mathrm{downcl}(M_y^s) : \exists t(\{f(e+1)\}_t^{A_t}(\tau)\downarrow)\} = \\ \{\tau \in \mathrm{downcl}(M_y^s) : \exists t \leq t_0(\{f(e+1)\}_t^{A_t}(\tau)\downarrow)\}.$$

Consider the maximum

$$\max\{\mathrm{use}(\{f(e+1)\}_t^{A_t}(\tau)) : t \leq t_0 \wedge \tau \in \mathrm{downcl}(M_y^s)\}.$$

Note that this maximum exists because the set $\mathrm{downcl}(M_y^s)$ is finite. Now if y' is chosen above this maximum and $s' > t_0$ then for every $v \geq y'$ and $t \geq s'$ we have $M_v^t \cap \mathrm{downcl}(M_y^s) = \emptyset$ and $M_y^s \cap \mathrm{downcl}(M_v^t) = \emptyset$, so $\mathrm{Ext}(M_v^t) \cap \mathrm{Ext}(M_y^s) = \emptyset$. It follows from our assumption and from the above that there are infinitely many pairs y and s such that $y \in W_{z,s}$ with $\mu(M_y^s) > 2^{-z-e-2}$ and such that all the Σ_1^0 classes $\mathrm{Ext}(M_y^s)$ are disjoint. Because for every y and s it holds that $M_y^s \subseteq W_{\{e\}(e)}$ we then have $\mu(W_{\{e\}(e)}) = \infty$, a contradiction. So the assumption from the beginning of our proof cannot be true, and it follows with (5.2) that for infinite W_z, R_z requires attention at some stage and is satisfied at that same stage. (For finite W_z the requirement R_z is vacuously satisfied.) □ Lemma 2

From the construction we see that the set A is r.e. By the clause '$y \geq 2z$' in (5.2) it has infinite complement and by Lemma 2 it intersects every infinite r.e. set, so A is simple. By Lemma 1 and the definition of $\mathcal{U}_0$ we have the inclusion $W_{f(e+1)}^A \subseteq W_{\{e\}(e)} \subseteq \mathcal{U}_0$, so (5.1) is satisfied. This concludes the proof of Theorem 5.2.1. □

We conclude this section with some remarks. Zambella (private communication) has shown that the use of the recursion theorem in the above proof is not essential. It is unknown exactly how complex sets in $\mathrm{Low}(\mathcal{R})$ can be. The nonrecursive example constructed above is still r.e. Are there sets in $\mathrm{Low}(\mathcal{R})$ that are outside of Δ_2^0? And if so, are there uncountably many such sets? In Section 5.4 we will see that there are uncountably many nonrecursive sets that are low for the class of Schnorr random sequences, and that these sets are *all* outside of Δ_2^0. This contrasts the situation for the 1-random sequences above.

5.3 Some limitations

In this section we make some remarks on the complexity of sets that are low for $\mathcal{R}$. Since every nonrecursive r.e. set bounds a 1-generic set we immediately have the existence of 1-generic sets that are low for $\mathcal{R}$. However, if $A \in \mathrm{Low}(\mathcal{R})$ then A cannot be 1-random, since this would imply that A is A-1-random, which is impossible. More generally, we have

5.3.1. Proposition. *If A is 1-random and $A \leq_T B$ then B is not A-1-random.*

Proof. M. van Lambalgen [44] (see also Kautz [34]) proved that if A is 1-random and B is A-1-random, then $A \oplus B$ is 1-random and A is B-1-random. In particular $A \,|_T B$. □

Another limitation comes from the next theorem. We first prove a lemma.

5.3.2. Lemma. (Kučera [38]) *For every nonrecursive set A there exists a function $g \leq_T A \oplus \emptyset'$ such that for all x*

$$\mu(\{C : (\exists i \leq x)[\{i\}^C \supseteq A\restriction g(x)]\}) < 2^{-x}.$$

Proof. It is easy to see that for any $\sigma \in 2^{<\omega}$, x, $j \in \omega$ it is possible to find $\emptyset'$-recursively a rational number q such that

$$|q - \mu(\{C : (\exists i \leq x)[\{i\}^C \supseteq \sigma]\})| < 2^{-j}.$$

The lemma now follows immediately from Theorem 1.4.2. □

5.3.3. Theorem. (Kučera [38]) *If $A \leq_T B$ and B is A-1-random then $A \in \mathrm{GL}_1$.*

Proof. Let A and B be as in the theorem. Let H be the function partial recursive in A that on argument e outputs the least s such that $\{e\}^A_s(e)$ is defined, and that is undefined if such s does not exist. Let $\{\mathcal{C}_e\}_{e\in\omega}$ be a recursive sequence of Σ^A_1-classes such that

$$\mathcal{C}_e = \begin{cases} \{C : (\exists i \leq e)[\{i\}^C \supseteq A\restriction H(e)]\} & \text{if } e \in A' \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to transform the recursive sequence $\{\mathcal{C}_e\}_{e\in\omega}$ into a recursive sequence $\{\mathcal{B}_e\}_{e\in\omega}$ such that for every e we have that $\mathcal{B}_e \subseteq \mathcal{C}_e$, $\mu(\mathcal{B}_e) < 2^{-e}$, and $\mathcal{B}_e = \mathcal{C}_e$ whenever $\mu(\mathcal{C}_e) < 2^{-e}$. (For the definition of $\mathcal{B}_e$, enumerate $\mathcal{C}_e$ as long as the measure of $\mathrm{Ext}(\mathcal{C}_e)$ is smaller than 2^{-e}.) Now let g be a function satisfying the condition of Lemma 5.3.2. We claim that for almost every e, if $H(e)$ is defined then $H(e) \leq g(e)$. Namely, suppose that this fails. Then every set C such that $A \leq_T C$ belongs to the class $\mathcal{B}_e$ for infinitely many e. It follows that the upper cone $A^{\leq_T} = \{C : C \leq_T A\}$ is Σ^A_1-approximable and therefore contains no A-1-random set, contradicting the assumption on A. So the function g dominates the function H, and since $g \leq_T A \oplus \emptyset'$ we see that $A' \leq_T A \oplus \emptyset'$, that is, $A \in \mathrm{GL}_1$. □

5.3.4. Corollary. *If A is low for $\mathcal{R}$ then $A \in \mathrm{GL}_1$.*

Proof. Since every set has a 1-random set above it ([37, 24]), if A is low for $\mathcal{R}$ then in particular A has a set above it that is A-1-random, and the corollary immediately follows from Theorem 5.3.3. □

Next we prove a limitation that shows that all (partial) functions that are of degree that is low for $\mathcal{R}$ can be uniformly dominated by a function recursive in $\emptyset'$. First we give two definitions. We say that a function g *dominates* a partial function f if there is a $k \in \omega$ such that whenever $f(n)$ is defined for some $n \geq k$ it holds that $g(n) \geq f(n)$. For strings τ and σ we say that τ is *to the left* of σ, denoted $\tau <_{\mathrm{L}} \sigma$, if there is a string ρ such that $\rho\hat{\ }0 \sqsubseteq \tau$ and $\rho\hat{\ }1 \sqsubseteq \sigma$.

5.3.5. THEOREM. *There exists a function* $g \leq_T \emptyset'$ *that dominates every function in the class of partial functions* $\{\{e\}^A : A \in \mathrm{Low}(\mathcal{R})\}$.

PROOF. Let R be the leftmost path in $\mathcal{P}_0$, $\mathcal{P}_0$ as defined in Definition 5.1.4. Then R is 1-random, being an element of $\mathcal{P}_0$, and it is easy to see that $R \leq_T \emptyset'$ (even $R \leq_{tt} \emptyset'$). Denote by V the set of strings to the left of R, i.e.

$$V = \bigcup_{i \in \omega} \{\tau \in 2^{<\omega} : \tau <_{\mathrm{L}} R \upharpoonright i\}.$$

Note that V is an r.e. set since $\mathcal{U}_0$ is Σ^0_1. Let $\{V_s\}_{s\in\omega}$ be a recursive enumeration of V. To every set $A \in 2^\omega$ and every partial A-recursive function $\{e\}^A$ we can assign an A-recursive sequential test $\{\mathcal{B}_i^{e,A}\}_{i\in\omega}$ as follows. If $\{e\}^A(i)\downarrow$ let τ be the first string of length i to the right of the rightmost (in the sense of the ordering $<_{\mathrm{L}}$ defined above) string of length i in $V_{\{e\}^A(i)}$ if such a string exists, and let τ be 0^i otherwise. Now let $\mathcal{C}_i^{e,A}$ be the basic open set defined by the string τ, i.e. $\mathcal{C}_i^{e,A} = \{B : \tau \sqsubset B\}$. If $\{e\}^A(i)\uparrow$ let $\mathcal{C}_i^{e,A}$ be empty. Finally, let

$$\mathcal{B}_i^{e,A} = \bigcup_{j>i} \mathcal{C}_j^{e,A}.$$

Now define

$$g(i) = (\text{least } s)(\forall \tau <_{\mathrm{L}} R\upharpoonright i)[\,|\tau| = i \rightarrow \tau \in V_s].$$

We claim that g satisfies the statement of the theorem. Clearly we have $g \leq_T R \leq_T \emptyset'$. Let $A \in \mathrm{Low}(\mathcal{R})$. Suppose that there are infinitely many $i \in \omega$ such that $\{e\}^A(i)$ is defined and greater than or equal to $g(i)$. For every such i it holds that $R \in \mathcal{B}_i^{e,A}$. It follows that R is not A-1-random. Since $A \in \mathrm{Low}(\mathcal{R})$ we then also have that R is not 1-random, a contradiction. $\square$

Zambella has shown, using ideas of a totally different nature, that Theorem 5.3.5 can be improved considerably, namely that the values of the functions in $\{\{e\}^A : A \in \mathrm{Low}(\mathcal{R})\}$ can be *traced*, in the sense of the next section, in a uniform way by an r.e. trace. These ideas will be used in the next section to characterize the degrees that are low for $\mathcal{S}$.

5.4 TRACEABILITY AND LOWNESS FOR SCHNORR'S NOTION

Recall that for a finite binary sequence σ, we denote by C_σ the set of reals that extend σ. These sets form a basis of clopens for the usual discrete topology on 2^ω. With every set $U \subseteq 2^{<\omega}$ we associate the open set $\bigcup_{\sigma\in U} C_\sigma$. When it is convenient, we confuse U with the open set associated to it, in particular we write μU for the measure of the open set corresponding to U. We use the following abbreviation for the measure conditioned to σ:

$$\mu_\sigma U = \frac{\mu(U \cap C_\sigma)}{\mu C_\sigma}$$

We recall the definition of Martin-Löf test so that the reader can easily compare this definition to that of Schnorr.

5.4.1. DEFINITION. A *Martin-Löf test* [59] is a recursive set $U \subseteq \omega \times 2^{<\omega}$ such that $\mu U_n \leq 2^{-n}$, where $U_n = \{x : \langle x, n\rangle \in U\}$. A *Schnorr test* [73] has in addition the property that $\mu U_n = 2^{-n}$ or, alternatively, that there is a recursive enumeration of U such that $\mu(U_n - U_{n,s}) \leq 2^{-s}$ for all n and s, where $U_{n,s}$ are the elements enumerated into U_n before stage s. This latter is a more flexible notion of test but defines the same Schnorr null sets. A set of reals is *Martin-Löf/Schnorr null* if it is contained in the G_δ set

$$\bigcap_{n\in\omega} \bigcup_{\sigma \in U_n} C_\sigma$$

for some Martin-Löf/Schnorr test U. We concisely write $\bigcap U$ for the null set above. A real $R \in 2^\omega$ is *Martin-Löf/Schnorr random* if it does not belong to any Martin-Löf/Schnorr null set. We denote the set of Martin-Löf random reals by $\mathcal{R}$ and the set of Schnorr random reals by $\mathcal{S}$.

Martin-Löf tests give essentially only positive information and Schnorr tests give both positive and negative information. For this reason Martin-Löf/Schnorr randomness is sometimes called "recursively enumerable/recursive randomness".

The definitions above relativize naturally to an arbitrary parameter. When U is recursive in A, we shall speak of tests and null sets *relative to* A. The relativized notion of randomness has first been considered by van Lambalgen [42] in his work on formalizations of the notion of "stochastic independence". Some more details on effective null sets are given below as they are needed. For a more complete account we refer the reader to the literature (e.g. [42]).

A set A is *low for* $\mathcal{R}$/*low for* $\mathcal{S}$ if every Martin-Löf/Schnorr null set relative to A is contained in a Martin-Löf/Schnorr null set. Clearly every recursive set is low for $\mathcal{R}$ and $\mathcal{S}$. The existence of nonrecursive sets that are low for $\mathcal{R}$ was proved in [39]. We want to prove the existence of nonrecursive sets that are low for $\mathcal{S}$ and study their complexity.

We introduce some recursion-theoretic notions that we use to characterize sets that are low for $\mathcal{S}$. A set $T \subseteq \omega \times \omega$ is called a *trace* if all its sections $T^{[k]}$ are finite. If the function mapping k to the canonical code of $T^{[k]}$ is a recursive function, we call T a *recursive trace.* Let $g : \omega \to \omega$ be any function. We say that T *traces* or *captures* g if $g(k) \in T^{[k]}$ for every k.[1] A *bound* is a function $h : \omega \to \omega$ that is nondecreasing and has infinite range. We say that a trace T *has bound* h if $\|T^{[k]}\|$ (the cardinality of $T^{[k]}$) is less than $h(k)$ for all k. Trivially, every recursive trace has a recursive bound, but there is no uniform recursive bound for all recursive traces.

5.4.2. Definition. A set A is *recursively traceable* if there is a recursive bound h such that all (total) functions $g \leq_T A$ have a recursive trace bounded by h.

The following easy fact says that all recursive bounds are essentially equivalent. This has interesting consequences for traceable degrees.

5.4.3. Proposition. *Let h be a recursive bound. Suppose that every function recursive in A has a recursive trace T with bound h. Then for an arbitrarily small recursive bound h', every function recursive in A has a recursive trace T with bound h'.*

Proof (sketch). Identify in a canonical way each sequence $\sigma \in \omega^{<\omega}$ with a natural number. When $k < |\sigma|$ we write $\sigma(k)$ for the k-th digit of the sequence σ. Let $g \leq_T A$. Let f be an increasing recursive function to be specified below (roughly: an inverse of h). Let T be a recursive trace with bound h that captures the function $i \mapsto g\restriction f(i)$ (the string that codes the first $f(i)$ values of g). Let S be the set defined by

$$S^{[k]} = \left\{\sigma(k) : \sigma \in T^{[i_k]}\right\},$$

where i_k is least i such that $|\sigma| = f(i) > k$. Clearly, S is a recursive trace. The cardinality of $S^{[k]}$ is easily computed and is bounded by $h(i_k)$. So, the faster f grows, the slower the cardinality of $S^{[k]}$ grows. It is easy to design an f that makes S attain a given recursive bound. □

Recall the following definition.

5.4.4. Definition. (Miller and Martin [62]) A set A has *hyperimmune-free* degree if every function (total) recursive in A is majorized by a recursive function, i.e. for every $f \leq_T A$ there is a recursive function g such that $\forall x(f(x) \leq g(x))$.

From Proposition 5.4.3 it follows immediately that recursively traceable degrees are hyperimmune-free: If A is recursively traceable then every function $g \leq_T A$ is bounded by the recursive function $\max T^{[k]}$ where T is a recursive trace of g. (Incidentally, observe that being dominated by a recursive function is trivially

[1] Similar notions of approximability exist in complexity theory. For example, in [11] a set A is called *(m, n)-verbose* if there exists a uniformly recursive procedure to enumerate for any n-tuple $x_1 \dots x_n$ a set of at most m possibilities for the characteristic string $A(x_1) \dots A(x_n)$, one of which is correct.

equivalent to being captured by a recursive trace, but this does not make every hyperimmune-free degree recursively traceable, because a uniform bound may not exist. In Section 5.5 we prove that indeed the two notions do not coincide.) Miller and Martin's construction of a nonrecursive hyperimmune-free degree [62] is easily adapted to yield traceable degrees —actually, a continuum of such degrees. The proof is postponed to Section 5.5. Finally, recall that nonrecursive hyperimmune-free degrees are incomparable to $0'$ (see e.g. [64, p495]), so the same holds for nonrecursive recursively traceable degrees.

The next theorem gives a "measure-theoretical" characterization of recursively traceable degrees that is inspired by the combinatorics used in a proof of Raisonnier [69].

5.4.5. THEOREM. *A set is recursively traceable if and only if it is low for* $\mathcal{S}$.

PROOF. For the "only if" direction, let A be a recursively traceable set. Let U be a given Schnorr test recursive in A. We want to construct a Schnorr test V such that $\bigcap V \supseteq \bigcap U$. We can approximate the set U with an A-computable function that yields the finite sets $U_{n,s}$. By hypothesis, $U_{n,s}$ (that is, the function mapping $\langle n, s\rangle$ to the canonical code of the finite set $U_{n,s}$) has a recursive trace T. This T we use to enumerate V. In order to enumerate not too much measure into V_n, we have to make sure that the bulk of U_n is approximated by $U_{n,s}$ fast, that is, while T is still informative. After all, the longer we wait, the worse T gets. The following will suffice: we require $\mu U_{n,s} > 2^{-n} - 2^{-s}$. We also have to fix a bound h for T that is sharp enough.

Recapitulate. We fix a recursive trace T with bound h (for convenience this h will be specified below) and such that $U_{n,s} \in T^{[\langle n,s\rangle]}$ for all n, s. (Finite sets are identified with their code.) Now we prune T to eliminate elements that are not a candidate for $U_{n,s}$: define $\hat{T}$ as follows. Let $\hat{T}^{[\langle n,s\rangle]}$ be the set of those $D \in T^{[\langle n,s\rangle]}$ such that D is a finite subset of $2^{<\omega}$ and

$$2^{-n} - 2^{-s} \leq \mu D \leq 2^{-n} \quad \text{and} \quad C \subseteq D \text{ for some } C \in \hat{T}^{[\langle n,s-1\rangle]}$$

(for $s = 0$ the second clause is assumed to be empty). Observe that $\hat{T}$ is still a recursive trace that captures $U_{n,s}$. Finally, define

$$V_{n,r} = \bigcup_{s<r} \hat{T}^{[\langle n,s\rangle]} \qquad \text{and} \qquad V_n = \bigcup_{r\in\omega} V_{n,r}$$

Observe that

$$\mu V_n \leq 2^{-n} \cdot \|\hat{T}^{[\langle n,0\rangle]}\| + \sum_{s\in\omega} 2^{-s} \cdot \|\hat{T}^{[\langle n,s\rangle]}\|.$$

So, we can make μV_n recursively converge to 0, by choosing the bound h of $\|\hat{T}^{[\langle n,s\rangle]}\|$ small enough. To see that $\mu V_{n,r}$ recursively converges to μV_n observe that

$$\mu \bigcup_{s>r} \hat{T}^{[\langle n,s\rangle]} < \sum_{s>r} 2^{-s} \cdot \|\hat{T}^{[\langle n,s\rangle]}\|.$$

Again it is just a question of choosing a bound h that is sufficiently small.

We now prove the "if" direction. Let $B_{k,l}$ for $k, l \in \omega$ be the clopens

$$B_{k,l} = \left\{\tau * 1^k \ : \ \tau \in 2^{<\omega} \ \ \& \ \ |\tau| = l\right\}$$

where 1^k is a string of 1's of length k and $*$ denotes concatenation. Note that $\mu B_{k,l} = 2^{-k}$ for all l. Now, given a function $g \leq_T A$, we define the test U^g by stipulating that

$$U_n^g \ := \ \bigcup_{k>n} B_{k,g(k)}.$$

It is easy to see that μU_n^g can be approximated recursively in A, so U^g is a Schnorr test relative to A. By lowness of A, we can find a Schnorr test that contains $\bigcap U^g$. A fortiori we can find a recursive set $V \subseteq 2^{<\omega}$ and a recursive enumeration of V such that μV_s converges recursively to $\mu V < 1/4$, where V_s is the set of the elements of V enumerated before stage s. To simplify the proof below we also require that V satisfies a technical assumption. Namely that for every k and l,

$$\mu(B_{k,l} - V) \neq 2^{-(l+3)} \tag{5.3}$$

We leave to the reader to check that, if necessary, one can always enlarge V to some V' satisfying (5.3). (Hint. If at stage s for some $\langle k, l\rangle < s$ the difference between the two numbers above appears to be "small", add to V' a fraction of $B_{k,l}$ that ensures that equality will never obtain. One can ensure that in the end $\mu(V' - V) < \varepsilon$ for an arbitrarily small ε and that $\mu V'$ can still be approximated recursively.)

The construction below is simpler if we assume that $\mu(U_n^g - V) = 0$ for some n. So, we make this provisional assumption, and we shall eliminate it later. We define a trace T for g (to be precise, a set T such that $g(k) \in T^{[k]}$ for $k > n$, so a trace of g is obtained immediately from T). First we define T and show that it is recursive. Then we show that there is a recursive upper bound on the largest element of $T^{[k]}$. This is enough to conclude that $\|T^{[k]}\|$ is a recursive function of k. Define

$$T^{[k]} := \left\{ \ l : \mu(B_{k,l} - V) < 2^{-(l+3)} \right\} \tag{5.4}$$

By assumption $\mu(U_n^g - V) = 0$, so T traces g, with a possible exception of the first n values. It is evident that T is recursively enumerable. We show how to enumerate the complement of T. Let $s_0 = 0$ and define s_{i+1} and ε_i such that

$$\varepsilon_i := \mu(B_{k,l} - V_{s_i}) - 2^{-(l+3)} \qquad \text{and} \qquad \mu V_{s_{i+1}} > \mu V - \frac{\varepsilon_i}{2}.$$

Suppose $l \notin T^{[k]}$. Then $\varepsilon_i > 0$ for all i. It is clear that ε_i converges to a limit ε and, by the assumption (5.3) above we have that $\varepsilon > 0$. So, $\varepsilon_i/2 < \varepsilon$ for some i. Therefore $\varepsilon_i/2 < \varepsilon_{i+1}$ for some i. So, enumerating V up to stage s_{i+1} we know with certainty that $l \notin T^{[k]}$. So $\overline{T}$ is r.e., hence T is recursive. To show that T is a recursive trace it remains to show that we can compute $\|T^{[k]}\|$. It suffices to show that we can effectively find an l_k such that $l \notin T^{[k]}$ for all $l > l_k$. Find a stage s such that $\mu V_s > \mu V - 2^{-(k+2)}$. Let l_k be larger than k and larger than the length of all strings in V_s. From the definition of $B_{k,l}$ it is clear that V_s and $B_{k,l}$ are independent[2] for every $l > l_k$. This implies immediately that $\mu(B_{k,l} - V_s) = 2^{-k}(1 - \mu V_s) > 2^{-k}(3/4)$. Consequently, we cannot have that $\mu(B_{k,l} - V) < 2^{-(k+2)}$ and a fortiori that $\mu(B_{k,l} - V) < 2^{-(l+3)}$.

Now, note that l_k depends on the recursive enumeration of V and, indirectly, on g, so we still have to show that there is an uniform bound on $\|T^{[k]}\|$. We claim that $\|T^{[k]}\| < 2^k k$ for every k. Observe that (5.4) above guarantees that

$$\sum_{l \in T^{[k]}} \mu(B_{k,l} - V) \;<\; \frac{1}{4}$$

so,

$$\mu \bigcup_{l \in T^{[k]}} B_{k,l} - \mu V \;\leq\; \mu \bigcup_{l \in T^{[k]}} (B_{k,l} - V) \;\leq\; \frac{1}{4}.$$

We obtain that

$$\mu \bigcup_{l \in T^{[k]}} B_{k,l} \leq \frac{1}{2}.$$

As observed above $\mu B_{k,l} = 2^{-k}$ and, for k fixed, the $B_{k,l}$'s are mutually independent as soon as the l's are taken sufficiently far apart. So,

$$1 - \left(1 - 2^{-k}\right)^{\frac{\|T^{[k]}\|}{k}} \;\leq\; \mu\left(2^\omega - \bigcap_{l \in T^{[k]}} (2^\omega - B_{k,l})\right) \;\leq\; \frac{1}{2}.$$

From the inequality above we obtain $\|T^{[k]}\| \leq 2^k k$. So, as required, we have a recursive bound independent of g.

To complete the proof we show that the hypothesis that $\mu(U_n^g - V) = 0$ for some n can be weakened to: $\mu_\sigma(U_n^g - V) = 0$ for some σ and some n such that $\mu_\sigma V < 1/4$. (Recall that μ_σ is the measure conditioned to C_σ.) Then we show

[2] Sets A and B are *independent* if $\mu(A \cap B) = \mu A \cdot \mu B$

that this latter hypothesis is indeed true. So, suppose first that $\mu_\sigma(U_n^g - V) = 0$ and $\mu_\sigma V < 1/4$. For a set of strings W we use the notation

$$W|\sigma = \{\tau \in 2^{<\omega} : C_{\sigma\hat{}\tau} \subseteq \mathrm{Ext}(W)\}.$$

We may assume that $g(k) > k$ for every k because a trace for $g(k) + k$ immediately gives a trace for g. Clearly we can also assume that $n > |\sigma|$. We claim that $\mu(U_n^{\tilde{g}} - \tilde{V}) = 0$ where $\tilde{V} = V|\sigma$ and $\tilde{g}$ is the translation of g defined by $k \mapsto g(k) \dot{-} |\sigma|$. Namely, if $l > |\sigma|$ then $B_{k,l}|\sigma = B_{k,l-|\sigma|}$. Since $g(k) > k$ and $n > |\sigma|$ we have that $U_n^g|\sigma = U_n^{\tilde{g}}$, so $\mu(U_n^{\tilde{g}} - \tilde{V}) = \mu_\sigma(U_n^g - V) = 0$. This proves the claim. Now, it is clear that $\mu\tilde{V} < 1/4$ has also a recursively approximable measure. So the proof given above is valid when $\tilde{V}$ and $\tilde{g}$ are substituted for V and g and ensures the existence of a recursive trace for $\tilde{g}$. But from a trace of $\tilde{g}$ we immediately obtain a trace for g.

Now, suppose that no σ and n exist such that $\mu_\sigma(U_n^g - V) = 0$ and $\mu_\sigma V < 1/4$. We shall obtain a contradiction by constructing a real in $\bigcap U^g - V$. Let σ_0 be the empty string and assume we have defined σ_n such that $\mu_{\sigma_n} V < 1/4$. By absurd hypothesis $\mu_{\sigma_n}(U_n^g - V) > 0$, so there is a $\tau \in U_n^g$ such that $\mu_{\sigma_n}(C_\tau - V) > 0$. In particular $\tau \sqsupseteq \sigma_n$ and $\mu_\tau V < 1$. Apply the Lebesgue density theorem to find $\sigma_{n+1} \sqsupseteq \tau$ such that $\mu_{\sigma_{n+1}} V < 1/4$. Let R be the real that extends all σ_n's constructed in this way. Since $C_{\sigma_{n+1}} \sqsubseteq U_n^g$ for all n we have that $R \in \bigcap U^g$. But $C_{\sigma_n} \not\sqsubseteq V$ for every n, so, since V is open, $R \notin V$. This contradiction completes the proof of Theorem 5.4.5. □

The following corollary contrasts with Theorem 5.2.1. It is worthwhile to note that it is is unknown whether there are low sets for $\mathcal{R}$ that are not below $\emptyset'$.

5.4.6. Corollary. *There are $2^{\aleph_0}$ many sets that are low for $\mathcal{S}$. Nonrecursive degrees that are low for $\mathcal{S}$ are incomparable with $0'$.*

Proof. This follows immediately from the discussion below Proposition 5.4.3, Theorem 5.4.5 above and Theorem 5.5.3 below. □

5.5 A continuum of traceable sets

In this section we prove the existence of nonrecursive recursively traceable degrees. In fact, we prove that there are uncountably many recursively traceable degrees. We merely check that the construction of Miller and Martin [62] produces such degrees.

5.5.1. Definition. A *tree* is a partial function $T : \omega^{<\omega} \to \omega^{<\omega}$ such that

$$T(\sigma)\downarrow \wedge \tau \sqsubseteq \sigma \Rightarrow T(\tau)\downarrow \wedge T(\tau) \sqsubseteq T(\sigma).$$

Node σ is *on* T if σ is in the range of T. The set of infinite branches of T is denoted by $[T]$. For $A \in [T]$ we also say that A is on T. That Q is a *subtree* of T, denoted by $Q \subseteq T$, means that every node on Q is also on T. The *full subtree* of T above a node σ is the subtree of T that consists of every string on T that extends σ. A tree is finitely branching if every node on it has only finitely many extensions on the tree.

In the following we will deal with finitely branching *recursive* trees, that is, trees that are recursive as (total) functions, when $\omega^{<\omega}$ is properly identified with the natural numbers. Note that, being total functions, recursive trees have no isolated branches and no finite branches. To recall the method we first prove

5.5.2. PROPOSITION. *There exists a nonrecursive set that is recursively traceable.*

PROOF (SKETCH). A (nonrecursive) sequence of perfect recursive binary trees $T_0 \supseteq T_1 \supseteq \dots T_n \supseteq \dots$ is defined in such a way that

$$\forall A \in [T_{2e}](A \neq \{e\})$$

and

> there is a recursive function f such that for every $A \in [T_{2e+1}]$, if $\{e\}^A$ is total then $\{e\}^A$ is dominated by f.

To construct T_{2e} from T_{2e-1}, note that one of $T_{2e-1}(0)$ and $T_{2e-1}(1)$ must disagree with $\{e\}$. If $T_{2e-1}(i)$ is such, we can let T_{2e} be the full subtree above it. To construct T_{2e+1} from T_{2e} we first noneffectively decide whether there is a point $\sigma \in T_{2e}$ such that for all oracles $A \sqsupset \sigma$ the function $\{e\}^A$ is not total. If this is the case we let T_{2e+1} be the full subtree of T_{2e} above σ. Otherwise, given x, we can extend every $\sigma \in T_{2e}$ to some $\tau \in T_{2e}$ such that $\{e\}^\tau(x)\downarrow$. Thus we can recursively define a perfect subtree T_{2e+1} of T_{2e} for which $\{e\}^{T_{2e+1}}(\tau)(n)$ is defined for every n and σ of length n. If we let A be a set in the intersection $\bigcap_n T_n$, then the even trees force that $A \notin \mathrm{REC}$, and the odd trees force that if $\{e\}^A$ is total, then all the functions $\{e\}^B$ are total for $B \in [T_{2e+1}]$, and these functions are dominated by a recursive function. Also, they are recursively traced by a trace with bound $h(k) = 2^k$. $\square$

Proposition 5.5.2 is now easily improved to

5.5.3. THEOREM. *There are $2^{\aleph_0}$ recursively traceable degrees.*

PROOF. We write T^τ for the full subtree of T above τ. We construct a chain of trees $T_0 \supseteq \cdots \supseteq T_s \supseteq \dots$ such that the set $\bigcap_s [T_s]$ is perfect and contains only sets that are recursively traceable with bound $h(k) = 2^k k$.

Let $T_0 = 2^{<\omega}$. For every minimal string ν of T_{2e}, if for some $\tau \sqsupseteq \nu$ all branches B of T_{2e}^{τ} are such that $\{e\}^B$ is not total, replace T_{2e}^{ν} with T_{2e}^{τ} in T_{2e}. Otherwise, replace T_{2e}^{ν} with the trees S defined by the following recursive procedure: let $S_0 = \{\nu\}$, then, for each maximal string in S_k enumerate in S_{k+1} two incomparable extensions $\sigma_0, \sigma_1 \in T_{2e}$ such that $\{e\}^{\sigma_i}(k)\downarrow$ for both $i = 0, 1$. Finally, to make $\bigcap_s [T_s]$ perfect, construct T_{2e+2} from T_{2e+1} by erasing from T_{2e+1} all minimal elements.

We check that all branches in $\bigcap_s [T_s]$ are recursively traceable. Let $\nu_1, \dots, \nu_i$ be those minimal elements of T_{2e+1} for which $\{e\}^B$ is total for all branches B that extend them. By construction, $\{e\}^B(k)$ attains at most 2^{k+e} different values as B is one of the branches above (note there are 2^e minimal elements in T_{2e+1}). Branches that do not extend one of $\nu_1, \dots, \nu_i$, make $\{e\}^B$ nontotal and may be considered "irrelevant". Each T_{2e+1} is a recursive set, so all relevant values of $\{e\}^B(k)$ are computable. □

Finally, we show that the notions of recursive traceability and hyperimmune-freeness do not coincide.

5.5.4. Theorem. *If a set is recursively traceable then it is of hyperimmune-free degree. The converse is not true in general.*

Proof (sketch). If a function g is recursively traced by T then clearly with T we can compute a function dominating g by taking the maximal values in $T^{[k]}$. So it follows immediately from the Definition 5.4.2 that every function recursive in a recursively traceable set is dominated by a recursive function.

To see that the converse implication does not hold we construct a function f such that $\deg_T(f)$ is hyperimmune-free and such that f is not recursively traceable with bound $h(k) = k$. It then follows from Proposition 5.4.3 that f is not recursively traceable with any bound. We construct f using the method of Miller and Martin as described in the proof of Proposition 5.5.2. In the current proof, we do not force with recursive binary trees as above, but with recursive trees T such that $T(\sigma)$ is finitely branching, and such that $T(\sigma)$ has at least $n+1$-branches if $|\sigma| = n$. Since these trees are compact, we can let f be an element of $\bigcap_n T_n$. Nonrecursiveness of f is forced as before with the trees T_{3e}. Hyperimmune-freeness is forced as before with the trees T_{3e+1}. The only difference is that now we have to ensure that in every node we have enough branches, whereas before finding two incompatible extensions for every node was sufficient. The only extra ingredient in the proof is that with T_{3e+2} we force that if the e-th partial recursive function φ_e defines a recursive trace T with bound $h(k) = k$, then for every $f \in [T_{3e+2}]$, f is not traced by T. Since at a node $T_{3e+1}(\sigma)$, $|\sigma| = n$, we have at least $n+1$ possibilities of extending the node, and T can use only n of these possibilities (which we can compute), we can choose for f a possible extension not covered by T. □

Chapter 6

Recursive Martingales

In this chapter we consider various questions about recursive measure theory. Recall that the recursive measure μ_{rec} is defined using recursive martingales. In Section 6.2 we consider the measure μ_{Δ_2} that we already encountered in Chapter 4. Note that μ_{Δ_2} is defined by martingales that are recursive in $\emptyset'$.

6.1 Reducibility to random sequences and the measure of upper cones

This section is devoted to the classical topic of the measure of upper cones. We will consider the recursive measure. First we review related results about other measures. Sacks' Theorem 1.4.2 shows that for every set A,

$$A \notin \text{REC} \quad \Leftrightarrow \quad \mu(A^{\leq_T}) = 0.$$

Ambos-Spies [2] proved that for the polynomial reducibilities we have the analogues

$$\begin{aligned} A \notin \text{P} \quad &\Leftrightarrow \quad \mu(A^{\leq^p_m}) = 0 \\ A \notin \text{BPP} \quad &\Leftrightarrow \quad \mu(A^{\leq^p_T}) = 0. \end{aligned} \tag{6.1}$$

This second item is implicit in Bennett and Gill [14]. In fact, Ambos-Spies [2], and independently Tang and Book [74] proved that

$$\begin{aligned} A \notin \text{P} \quad &\Leftrightarrow \quad \mu(A^{\leq^p_{btt}}) = 0 \\ A \notin \text{BPP} \quad &\Leftrightarrow \quad \mu(A^{\leq^p_{tt}}) = 0. \end{aligned}$$

Kučera [37], and independently Gács [24] proved that every set T-reduces to a Martin-Löf-random set. By Corollary 4.1.7 this is equivalent to

$$\forall A \in 2^{\omega} \;\; \mu_{\text{r.e.}}(A^{\leq_T}) \neq 0.$$

In fact, from the proofs in [37] and [24] one sees that

$$\forall A \in 2^\omega \ \ \mu_{\text{r.e.}}(A^{\leq_{wtt}}) \neq 0.$$

Bennett [13] indicated that this is not true for tt-reducibility, namely the halting problem K does not reduce to a Martin-Löf-random set. (A precise proof of this fact can be found in Juedes, Lathrop, and Lutz [30].)

For recursive measure the situation is different in the sense that it is not true that a class $\mathcal{A}$ has rec-measure zero if and only if it contains a rec-random set. By definition, if $\mathcal{A}$ contains a rec-random set then $\mu_{\text{rec}}(\mathcal{A}) \neq 0$, but for example the class REC does not have rec-measure zero and it does not contain a rec-random set. So in the case of rec-measure, given a set A and a reducibility $\leq_r$, we can ask two questions: does the r-upper cone of A have rec-measure zero and does A r-reduce to a rec-random set?

It is easy to see that if A is r.e. and nonrecursive and $A \leq_m B$ then B is not recursively random. Namely, in this situation B contains the infinite r.e. set $\{f(x) : x \in \omega\}$. But every r.e. set has rec-measure zero; if the set is finite this is clear and if it is infinite we can wait for some large element to appear in the enumeration. If this happens we can double our martingale value at the one-side at that point. When we perform this action infinitely often it is clear that we succeed in growing to infinity. The next proof is an elaboration of this argument.

6.1.1. Proposition. *No nonrecursive r.e. set btt-reduces to a rec-random set.*

Proof. Suppose A is r.e. and nonrecursive and $A \leq_{btt} B$ via a reduction $f(g_1 \dots g_k)$ of norm k. So for every x, f_x is a Boolean function with k variables and $x \in A$ if and only if $f_x(B(g_1(x)), \dots, B(g_k(x))) = 1$. Without loss of generality we may assume that the queries $g_i(x)$ are always strictly ordered: $g_1(x) < \dots < g_k(x)$. Consider the set

$$\begin{aligned} G = \{g_k(x) : f_x(B(g_1(x)), \dots, B(g_k(x))) = 1 \wedge \\ f_x(B(g_1(x)), \dots, B(g_{k-1}(x)), 1 - B(g_k(x))) = 0\}. \end{aligned}$$

If G is finite then by deleting almost all queries $g_k(x)$ in the reduction we see that actually $A \leq_{(k-1)-tt} B$. Since A is nonrecursive we know that $A \not\leq_{0-tt} B$, hence by minimizing k we may assume that G is infinite. Note that if we know that $x \in A$ and the answers to the first $k-1$ queries are given then we can recursively compute whether $g_k(x) \in G$, and if so recursively compute $B(g_k(x))$. Using this and the fact that A is infinite and r.e. it is easy to show that $\mu_{\text{rec}}(\{B\}) = 0$. □

Lutz (email August 1996) pointed out to us that from results in Juedes, Lathrop, and Lutz [30] and Fenner, Lutz, and Mayordomo [19] it follows that K does not tt-reduce to a rec-random set. However, it is not clear whether this result holds for every nonrecursive r.e. set. Note that every set wtt-reduces to a rec-random set by the result of Kučera and Gács quoted above, stating that every set wtt-reduces to a Martin-Löf-random set. In Section 6.2 we will prove that for every set $A \in \Delta_2$ it holds that $\mu_{\Delta_2}(A^{\leq_T}) = 0$, from which it follows that no nonrecursive set in Δ_2 T-reduces to a Δ_2-random set.

We now turn to the first type of question, about the measure of upper cones. We prove that for every set its upper cone is not small with respect to rec-measure. This holds for all reducibilities because it holds for the strongest one, namely 1-reducibility.

6.1.2. LEMMA. (Schnorr [73]) *For any martingale d and $w \in 2^{<\omega}$, if $d(w) \leq 1$ then $\mu(\{A \in C_w : \forall x \geq |w|(d(A\restriction x) \leq 1)\}) > 0$.*

PROOF. Fix $w \in 2^{<\omega}$ with $d(w) \leq 1$ and let V be the set of all $v \sqsupseteq w$ of minimal length such that $d(v) > 1$. Then V is prefix-free, so by Lemma 1.3.3 we have that

$$2^{-|w|} \geq 2^{-|w|}d(w) \geq \sum_{v \in V} 2^{-|v|}d(v) > \sum_{v \in V} 2^{-|v|} = \mu(\bigcup_{v \in V} C_v),$$

so $\mu(\{A \in C_w : \forall x(d(A\restriction x) \leq 1)\}) = \mu(C_w - \bigcup_{v \in V} C_v) > 0$. □

One can show that Lemma 6.1.2 is optimal, i.e. that the measure of the set in the lemma can be arbitrarily close to 0, but we do not need this here.

By Sacks' Theorem 1.4.2, whenever A is nonrecursive there is a martingale that succeeds on every B with $A \leq_T B$. On the other hand, we have

6.1.3. PROPOSITION. *For every set A and every martingale d there exists a set $B \notin S[d]$ such that $A \leq_T B \oplus d$.*

PROOF. We sketch this proof since this will help to understand the proof of Theorem 6.1.5. Fix a recursive martingale d. We may assume that $d(\lambda) = 1$. We d-recursively construct a set B in stages and during this construction we code A into B. Suppose that we are given B_s, the finite part of B constructed by stage s, and that for this string we have that for every $\sigma \sqsubseteq B_s$, $d(\sigma) \leq 1$. We then look for the first $m > |B_s|$, where $|B_s|$ denotes the length of the string B_s, such that there are at least two extensions $\tau \sqsupset B_s$ of length m with $\forall \sigma \sqsubseteq \tau(d(\sigma) \leq 1)$. By Lemma 6.1.2 such an m always exists. Now code A into B by letting B_{s+1} be the leftmost such τ if $A(s) = 0$ and the rightmost such τ if $A(s) = 1$. Since at every stage the number m is found recursively in d the set A can be retrieved recursively from $B \oplus d$. Furthermore, it is clear from the definition of B_{s+1} that $d(B_{s+1}) \leq 1$ for every s, hence $B \notin S[d]$. □

In particular, when d is a recursive martingale there is a set $B \notin S[d]$ with $A \leq_T B$. So for every A, $\mu_{\text{rec}}(A^{\leq_T}) \neq 0$, something we knew already from the results of Kučera and Gács. In the following theorem we improve this from Turing-reducibility to one-one-reducibility. (Recall that a one-one-reduction is an injective many-one-reduction.)

6.1.4. Lemma. *Suppose $\{w_i : i \leq n\}$ is a finite set of initial segments with $d(w_i) \leq 1$ for every i. Then there exist $m \in \omega$ and extensions $v_i \sqsupseteq w_i$ such that $|v_i| = m$, $d(v_i) \leq 1$, $d(v_i 0) \leq 1$, and $d(v_i 1) \leq 1$.*

Proof. If $d(w) \leq 1$ then by Lemma 6.1.2, $\mu(\{A \in C_w : \forall x(d(A\restriction x) \leq 1)\}) > 0$. Hence, by the Lebesgue Density Theorem there exist $A_i \in C_{w_i}$ such that

$$\lim_{m\to\infty} \mu(\{A \in C_{A_i\restriction m} : \forall x \geq |w_i|(d(A\restriction x) \leq 1)\})2^m = 1.$$

Let m be so large that $\mu(\{A \in C_{A_i\restriction m} : \forall x \geq |w_i|(d(A\restriction x) \leq 1)\})2^m > 1/2$ for every $i \leq n$. Then $\forall i \leq n(d(A_i\restriction m) \leq 1 \wedge d((A_i\restriction m)0) \leq 1 \wedge d((A_i\restriction m)0) \leq 1)$. □

The next theorem shows that for every set the rec-measure of even the smallest upper cone is not zero.

6.1.5. Theorem. *For every $A \in 2^\omega$, $\mu_{\text{rec}}(A^{\leq_1}) \neq 0$.*

Proof. Let A be arbitrary and let d be a recursive martingale. Without loss of generality we may assume that $d(\lambda) = 1$. Code A into B such that $B \notin S[d]$ as follows. Let $n_0 = 1$. Suppose n_s is defined. By Lemma 6.1.4 we can define n_{s+1} to be the smallest m such that

$$\forall w \in \{0,1\}^{n_s+1}(d(w) \leq 1 \to \exists v \sqsupseteq w(|v| = m \wedge \\ d(v) \leq 1 \wedge d(v0) \leq 1 \wedge d(v1) \leq 1)).$$

Now define B as follows. $B\restriction n_0 = \lambda$. Let $B\restriction n_{s+1}$ be the first string $v \sqsupseteq B\restriction n_s + 1$ of length n_{s+1} such that $d(x) \leq 1$ for $x \in \{v, v0, v1\}$. Define $B(n_{s+1}) = A(s)$. From this definition it is clear that $\forall x(d(B\restriction x) \leq 1)$, so $B \notin S[d]$. Also, $A \leq_1 B$ since $s \in A \Leftrightarrow n_{s+1} \in B$ and by recursiveness of d the sequence $\{n_s\}$ is recursive. □

In Section 6.2 we prove the analogue of Sacks' theorem for measure in Δ_2, namely that for nonrecursive sets A in Δ_2 the Δ_2-measure of $A^{\leq_T}$ is zero. The next proposition shows that this is not true for $\leq_m^p$ and E. We do not know whether the analogue of (6.1) holds for measure *in* E, i.e. whether for every set $A \in \mathrm{E}$ it holds that $A \notin \mathrm{P} \Leftrightarrow \mu(A^{\leq_m^p}|\mathrm{E}) = \mu_{\mathrm{p}}(A^{\leq_m^p} \cap \mathrm{E}) = 0$.

6.1.6. Proposition. *There exists $A \in \mathrm{E} - \mathrm{P}$ such that $\mu_{\mathrm{p}}(A^{\leq_m^p}) \neq 0$.*

PROOF. Define the sum-martingale

$$d(x) = \sum_{i\in\omega} 2^{-i} d_i(x),$$

where $\{d_i\}_{i\in\omega}$ is an enumeration of the polynomial time martingales such that d is in DTIME($n \log n$). Define a set B that directly diagonalizes against d, i.e. inductively define B by

$$z_n \in B \Leftrightarrow d((B\restriction z_n)1) \leq d((B\restriction z_n)0).$$

Then $B \in \mathrm{DTIME}(2^{n^2})$ and $\mu_{\mathrm{p}}(\{B\}) \neq 0$. Define $A = \{0^n : z_n \in B\}$. Then $A \in \mathrm{E}$ and $A \leq_m^p B$, so $\mu_{\mathrm{p}}(A^{\leq_m^p}) \neq 0$. □

Allender and Strauss [1] have proved that for almost every $A \in \mathrm{E}$ it holds that $\mathrm{BPP}^A = P^A$. It follows that if $A \in \mathrm{BPP}$ then the p-T-upper cone of A has measure 1 in E. Hence, if BPP $\neq$ P, then there is a set $A \in \mathrm{E} - \mathrm{P}$ such that $\mu(A^{\leq_T^p}|\mathrm{E}) = 1$.

6.2 MEASURE IN Δ_2

In this section we use martingales that are recursive in $\emptyset'$ (like in Chapter 4).

In Theorem 1.4.2 we saw that for every nonrecursive set A the Lebesgue measure of the class $A^{\leq_T} = \{B : A \leq_T B\}$ is zero. We now prove that if A is in Δ_2 this can be improved from Lebesgue measure to Δ_2-measure. Let $A\restriction n = \{e\}^B\restriction n$ denote that $(\forall x < n)[\{i\}^B(x)\downarrow = A(x)]$.

6.2.1. THEOREM. *For every nonrecursive set A in Δ_2 the Δ_2-measure of $A^{\leq_T}$ is zero.*

PROOF. Fix A as in the theorem and define for every i and n

$$\mathcal{E}_{i,n} = \{B : A\restriction n = \{i\}^B\restriction n\},$$

$$d_{i,n}(\sigma) = \begin{cases} 2^{|\sigma|} \cdot \Pr(C_\sigma|\mathcal{E}_{i,k_n}) & \text{if } k_n{:=}(\text{the first number } k \text{ such} \\ & \text{that } 0 < \mu(\mathcal{E}_{i,k}) \leq 2^{-2n}) \text{ exists,} \\ 1 & \text{otherwise.} \end{cases}$$

Note that the sets $\mathcal{E}_{i,n}$ are Lebesgue measurable, being a countable intersection of open sets. We think of them as approximations of the class $A^{\leq_T}$. The functions $d_{i,n}$ are martingales because for any class $\mathcal{A}$, $\Pr(C_{\sigma 0}|\mathcal{A}) + \Pr(C_{\sigma 1}|\mathcal{A}) = \Pr(C_\sigma|\mathcal{A})$. Furthermore, every $d_{i,n}$ is $\emptyset'$-approximable. We prove this after we have finished the main argument. It follows from the approximability of the $d_{i,n}$ that the function

$$d(\sigma) = \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} 2^{-i-n} d_{i,n}(\sigma).$$

is a Δ_2-martingale. Note that for every σ, $d(\sigma) \leq 2^{|\sigma|+2}$ because by the martingale property $d_{i,n}(\sigma) \leq 2^{|\sigma|} d_{i,n}(\lambda) \leq 2^{|\sigma|}$. We prove that d covers $\bigcap_n \mathcal{E}_{i,n}$. Suppose that $A = \{i\}^B$. Then for every n there is a $\sigma_n \sqsubset B$ such that $C_{\sigma_n} \subseteq \mathcal{E}_{i,n}$, hence $\mu(\mathcal{E}_{i,n}) > 0$ for every n. From Theorem 1.4.2 it follows that for every i, $\lim_k \mu(\mathcal{E}_{i,k}) = 0$ (here we use that A is nonrecursive) so for every n there is a k such that $0 < \mu(\mathcal{E}_{i,k}) \leq 2^{-2n}$. It now follows from the definition of $d_{i,n}$ that

$$d_{i,n}(\sigma_n) = 2^{|\sigma_n|} \frac{\mu(C_{\sigma_n})}{\mu(\mathcal{E}_{i,k_n})} = \frac{1}{\mu(\mathcal{E}_{i,k_n})} \geq 2^{2n}.$$

So we see that

$$\begin{aligned} \limsup_{n\to\infty} d(B{\restriction}n) &\geq \limsup_{n\to\infty} 2^{-i-n} d_{i,n}(\sigma_n) \\ &\geq \lim_{n\to\infty} 2^{-i-n} 2^{2n} = \infty. \end{aligned}$$

Hence $\bigcap_n \mathcal{E}_{i,n} \subseteq S[d]$ for every i, so d covers $\bigcup_i \bigcap_n \mathcal{E}_{i,n} = A^{\leq_T}$.

It remains to show that the $d_{i,n}$ are $\emptyset'$-approximable. First note that for any initial segment σ the clause

$$(\exists \tau \sqsupseteq \sigma)[C_\tau \subseteq \mathcal{E}_{i,n}] \tag{6.2}$$

is $\emptyset'$-recursive because it is equivalent to $(\exists \tau \sqsupseteq \sigma)[A{\restriction}n = \{e\}^\tau{\restriction}n]$. (We are not quantifying over A here since $A{\restriction}n$ is just a finite string that we can $\emptyset'$-compute since A is in Δ_2.) It easily follows from this that it is $\emptyset'$-decidable whether $\mu(\mathcal{E}_{i,n}) > 0$. With a little more effort one sees that it is $\emptyset'$-decidable whether $\mu(\mathcal{E}_{i,n}) > q$, for any rational number q. (Instead of asking about the existence of one τ satisfying (6.2) ask for a finite number of incomparable τ_i satisfying (6.2) such that $\sum |\tau_i| > q$.) Now the existence of k_n in the definition of $d_{i,n}$ can be decided recursively in $\emptyset'$, and k_n can be $\emptyset'$-recursively found if it exists because the sets $\mathcal{E}_{i,n}$ form a chain with the measure of its elements tending to zero. From the $\emptyset'$-decidability of "$\mu(\mathcal{E}_{i,n}) > q$" it also follows that $\mu(C_\sigma \cap \mathcal{E}_{i,n})$ and $\mu(\mathcal{E}_{i,n})$ are $\emptyset'$-approximable from below for any σ. We show that if $\mu(\mathcal{E}_{i,n}) > 0$ we can, given k, $\emptyset'$-recursively find i such that if we approximate $a = \mu(C_\sigma \cap \mathcal{E}_{i,n})$ and $b = \mu(\mathcal{E}_{i,n})$ from below within error bound 2^{-i} then we have approximated $\Pr(C_\sigma|\mathcal{E}_{i,n}) = \frac{a}{b}$ within error bound 2^{-k}. Find $\emptyset'$-recursively a rational number $\varepsilon > 0$ such that $b > \varepsilon$ (if $b = 0$ there is nothing to prove). Let i be so large that $2^{-i+1} \leq \varepsilon$ and $i \geq k + 1 - 2\log\varepsilon$. Then $-\log(\varepsilon - 2^{-i}) \leq -\log\frac{1}{2}\varepsilon$ so $i \geq -\log(\varepsilon - 2^{-i}) - \log\varepsilon + k$, hence

$$\left|\frac{a}{b} - \frac{a - 2^{-i}}{b - 2^{-i}}\right| = \left|\frac{2^{-i}(b-a)}{b(b-2^{-i})}\right| \leq \frac{2^{-i}}{\varepsilon(\varepsilon - 2^{-i})} \leq 2^{-k}$$

This concludes the proof that the $d_{i,n}$ are $\emptyset'$-approximable, and the proof of the theorem. □

From Theorem 6.2.1 we immediately obtain the following effectivization of Sacks' theorem:

6.2.2. COROLLARY. *For every set $A \in \Delta_2$ it holds that*

$$A \notin \text{REC} \Leftrightarrow \mu(A^{\leq_T} | \Delta_2) = \mu_{\Delta_2}(A^{\leq_T} \cap \Delta_2) = 0.$$

Since the degree of a set is a subset of its upper cone we also have the

6.2.3. COROLLARY. *Every Turing degree in Δ_2 has Δ_2-measure zero.*

Another immediate corollary is that K does not T-reduce to a Δ_2-random set. This shows that the Gács-Kučera result that every set wtt-reduces to a Martin-Löf random (=r.e.-random) set is optimal:

6.2.4. COROLLARY. *No nonrecursive set in Δ_2 T-reduces to a Δ_2-random set.*

PROOF. By definition a set is Δ_2-random if no Δ_2-martingale succeeds on it. If A is in Δ_2 then by Theorem 6.2.1 there is a Δ_2-martingale that succeeds on all the sets above A, hence no such set is Δ_2-random. □

It follows from Theorem 6.2.1 that two random elements of Δ_2 are Turing-incomparable, i.e. that the class $\{A \in \Delta_2 : A_0 |_{\mathrm{T}} A_1\}$ has measure one in Δ_2. However, this we already knew from the fact that every Martin-Löf-random set A has the property that $A_0 |_{\mathrm{T}} A_1$ (see e.g. [34, Thm. III.1.4]) and the fact that ML-RAND has r.e.-measure one.

We note that not every set in Δ_2 has the property that $\mu_{\Delta_2}({}^{\leq_T}A) = 0$ since ${}^{\leq_T}K = \Delta_2$ does not have Δ_2-measure zero. (It follows from results of Kučera and Arslanov that if A is r.e. and Turing-incomplete then $\mu_{\text{r.e.}}({}^{\leq_T}A) = 0$, [37]). We also note that not *every* nonrecursive set has the property that $A^{\leq_T}$ has Δ_2-measure 0:

6.2.5. PROPOSITION. *There exists a nonrecursive set A such that $\mu_{\Delta_2}(A^{\leq_T})$ is not zero.*

PROOF. For every martingale d, $S[d] \cup \text{REC}$ is a nullset, so in particular it has a nonempty complement. By taking for d a weighted sum of all the Δ_2-martingales,

$$d(x) = \sum_i \{2^{-i} d_i(x) : d_i \text{ the } i\text{-th } \Delta_2\text{-martingale}\},$$

one sees that there is a nonrecursive A with $\mu_{\Delta_2}(\{A\}) \neq 0$. □

In Chapter 4, Theorem 4.4.4 we have seen that almost every set in Δ_2 does not wtt-reduce to K. That is, the lower wtt-cone of K has Δ_2-measure zero.

6.3 Recursive randomness and Kolmogorov complexity

This section is devoted to relations between rec-randomness and Kolmogorov complexity. For an introduction to the theory of Kolmogorov complexity we refer the reader to Li and Vitányi [46]. We also follow the notation of that book. In particular, C is the plain Kolmogorov complexity and K is the prefix-free (self-delimiting) Kolmogorov complexity. Recall that an infinite sequence is recursively random if no recursive martingale succeeds on it. For notational convenience, in this section we denote the initial segment of length n of an infinite sequence $x \in 2^\omega$ is by x_n. We abbreviate the phrases 'infinitely often' and 'almost everywhere' by 'i.o.' and 'a.e.', respectively.

In the 1970's the following important characterization of Martin-Löf-randomness (Definition 4.1.5) was proved.

6.3.1. Theorem. (Schnorr) *A sequence $x \in 2^\omega$ is Martin-Löf-random if and only if there is a constant c such that $K(x_n) \geq n - c$ for every n.*

Ko [36] investigated the relations between polynomial time and space bounded versions of Martin-Löf randomness. His notion of pspace-randomness is obtained by defining a sequence to be non-pspace-random if it is covered by a pspace-computable Martin-Löf test *sufficiently fast.* The extra condition on the speed with which the set is covered is necessary, since otherwise the defined notion equals that of Martin-Löf. The following sufficient condition for pspace-randomness was proved by Ko.

6.3.2. Theorem. (K.-I. Ko [36]) *Let p be a polynomial. Let C^s denote the s-space bounded generating complexity. If for all polynomials q it holds that $C^q(x_n) > n - p(\log n)$ a.e. then x is pspace-random.*

We now turn our attention to recursive randomness. The next lemma is analogous to Claim 2.2 [46, p122], with the time bounded Kolmogorov complexity instead of the plain Kolmogorov complexity.

6.3.3. Lemma. *For any infinite sequence $x \in 2^\omega$ the following are equivalent:*

(i) *For every recursive function t there is a constant c such that $C^t(x_n) \geq n - c$ for infinitely many n.*

(ii) *For every recursive function t there is a constant c such that $C^t(x_n|n) \geq n - c$ for infinitely many n.*

PROOF. The implication from (ii) to (i) is trivial since always $C^t(x|n) \le C^t(x)$. For the other implication note that

$$C^{2t}(x_n) \le C^t(x_n|n) + 2 \cdot C^t(n - C^t(x_n|n)) + O(1).$$

For if we have a minimal program p that generates x_n given n, and a program q for $n - C^t(x_n|n)$, then we can reconstruct n from the length of p together with q. If p and q run both in time t then this new program takes time $2t + O(1)$.

Now fix any recursive t. From (i) it follows that infinitely often $n - c \le C^{2t}(x_n)$ for some constant c. For all the n for which this holds we then have

$$\begin{aligned} n - c &\le C^t(x_n|n) + 2 \cdot C^t(n - C^t(x_n|n)) + O(1) \\ &\le C^t(x_n|n) + 2 \cdot |n - C^t(x_n|n)| + O(1). \end{aligned}$$

Hence $n - C^t(x_n|n) \le 2 \cdot |n - C^t(x_n|n)| + O(1)$, but this is only possible if $n - C^t(x_n|n)$ is bounded, i.e. for the infinitely many n such that the above inequalities hold we must have that $n - C^t(x_n|n) \le c'$ for some constant c'. □

The characterization of Martin-Löf randomness (Theorem 6.3.1) was obtained by considering prefix-complexity instead of plain Kolmogorov complexity. We now give an 'infinitely often' criterion for recursive randomness. Since we use 'i.o.' rather than 'a.e.' we have no need for prefix-complexity at this point.

6.3.4. THEOREM. *Let x be an infinite sequence. If for every recursive function t there is a constant c such that it holds that infinitely often $C^t(x_n) \ge n - c$ then x is recursively random.*

PROOF. We prove this by contraposition. Suppose that x is a sequence that is not recursively random. By Lemma 6.3.3 it suffices to prove that there is a recursive t such that for every constant c it holds that $C^t(x_n|n) \le n - c$ a.e. First we prove that for every constant c there exists a recursive function t such that $C^t(x_n|n) \le n - c$ a.e. At the end of the proof we show that this proof can be made uniform. Fix c and let d be a recursive martingale such that $x \in S[d]$ and $d(\lambda) = 1$. Without loss of generality we may assume that $d(x_n) \ge c$ a.e. We have that

$$\mu(\{w \in 2^{<\omega} : w \text{ of minimal length such that } d(w) \ge c\}) \le 2^{-c}, \tag{6.3}$$

see [73, p41]. Let M be a machine that, given i and n, outputs the i-th initial segment w of length n with $d(w) \ge c$, or outputs zero if such w does not exist. Let $t_c(i,n)$ be the number of computation steps in the computation $M(i,n)$. Fix n such that $d(x_n) \ge c$. Let i be such that x_n is the i-th string w of length

n with $d(w) \geq c$. Note that by (6.3) there can be at most $2^{-c}/2^{-n}$ strings w of length n with $d(w) \geq c$, so $i \leq 2^{n-c}$, and thus $|i| \leq n - c$. Therefore,

$$\begin{aligned} C^{t_c}(x_n|n) &\leq |i| + C(c) + C(d) + |M| \\ &\leq n - c + \log c + C(d) + |M| \end{aligned}$$

Here by $C(d)$ we mean the Kolmogorov complexity of a program for the recursive martingale d. So for every c there is a recursive function t_c such that $C^{t_c}(x_n|n) \leq n - (c - \log c - b)$ a.e., where $b = C(d) + |M|$. Hence, given a constant c' we can choose c so large that $c - \log c - b \geq c'$ to obtain $C^{t_c}(x_n|n) \leq n - (c - \log c - b) \leq n - c'$ a.e.

It is now easy to see that the above construction can be done *uniformly*, yielding a function t that works for all c. We simply set $t(i,n) = \max_{c \leq n} t_c(i,n)$ and note that this t majorizes every t_c. □

Recently Lathrop and Lutz have proved the following two results relating rec-randomness and Kolmogorov complexity. The first result shows that rec-random sequences have very high time-bounded Kolmogorov complexity, and the second result shows that this is no longer true in the absence of time bounds.

6.3.5. THEOREM. (Lathrop and Lutz [45]) *Suppose that $x \in 2^\omega$ is rec-random and that $t, g : \omega \to \omega$ are recursive functions with g nondecreasing and unbounded. Then $K^t(x_n) > n - g(n)$ for infinitely many n.*

Call a sequence $x \in 2^\omega$ *ultracompressible* if for every nondecreasing unbounded recursive function $g : \omega \to \omega$ it holds that $K(x_n) < K(n) + g(n)$ for almost every n.

6.3.6. THEOREM. (Lathrop and Lutz [45]) *There exists a rec-random sequence x that is ultracompressible.*

It would be interesting to find a precise characterization of the notion of rec-randomness in terms of Kolmogorov complexity.

6.4 Two recursive measures

In this section we consider two notions of measure zero used by Schnorr in [73]. The first measure is μ_{rec}, the notion that we have defined in Section 1.5. The second measure, that we denote by μ_{mod}, 'mod' for 'modulated', was introduced by Schnorr as a more constructive version of Martin-Löf's measure defined using statistical tests (see Chapters 4 and 5). The measure μ_{mod} was also used by Freidzon [23].

6.4.1. DEFINITION. For a class $\mathcal{A}$, $\mu_{\text{mod}}(\mathcal{A}) = 0$ if and only if there is a recursive martingale d and a monotonic unbounded recursive function h such that $\mathcal{A} \subseteq S_h[d]$, where

$$S_h[d] = \{X : \limsup_{n\to\infty} \frac{d(X\restriction n)}{h(n)} \geq 1\}.$$

This is equivalent to the definition given in Definition 5.4.1, see [73, Satz 9.4, 9.5]. In [73] this definition is motivated and compared to Martin-Löf's definition of randomness.

It is immediate from the definition that $\mu_{\text{mod}}(\mathcal{A}) = 0$ implies that $\mu_{\text{rec}}(\mathcal{A}) = 0$. So for the corresponding notions of rec-randomness and μ_{mod}-randomness we have that the first is stronger than the latter. Wang [79] proved that rec-randomness is strictly stronger than μ_{mod}-randomness by constructing a sequence X and a recursive martingale d such that d succeeds on X, and such that no recursive martingale succeeds on X recursively fast. Note that such an X is necessarily nonrecursive, since no recursive sequence is random in either sense. Therefore, this separation of these two notions of randomness leaves open the question whether the measures μ_{rec} and μ_{mod} coincide on REC. The next theorem shows that this is not the case.

6.4.2. THEOREM. *There exists a subset $\mathcal{A}$ of* REC *such that $\mu_{\text{rec}}(\mathcal{A}) = 0$ but $\mu_{\text{mod}}(\mathcal{A}) \neq 0$.*

PROOF. For every number $n = \langle e, f\rangle$ recursively define a set of numbers $\{\langle y^n_m, z^n_m\rangle : m \in \omega\}$ as follows. Let $y^n_0 = z^n_0 = n$. If $\langle y^n_{m-1}, z^n_{m-1}\rangle$ is defined let $\langle y^n_m, z^n_m\rangle$ be the smallest number such that $y^n_m \geq z^n_m$, and

$$2^{y^n_{m-1}+1} \leq \varphi_{f,y^n_m}(z^n_m)\downarrow$$

if this exists and let $\langle y^n_m, z^n_m\rangle$ be undefined otherwise.
Call a number $\langle e, f\rangle$ a Schnorr-test if φ_e is a martingale with $\varphi_e(\lambda) = 1$ and φ_f is a monotonic unbounded function. For every Schnorr-test $n = \langle e, f\rangle$ define a recursive set A_n that escapes n: Let $A_n\restriction n = \emptyset\restriction n$ and define $A_n(n) = 1$, so that $0^n1 \sqsubset A_n$ and the number n is coded in this way into the initial segment of A_n. Since φ_e is a martingale and φ_f is unbounded we can recursively define the rest of A_n as follows. Given $A_n\restriction y^n_{m-1}$, define $A_n\restriction y^n_m$ to be the (lexicographically) first string σ of length y^n_m such that $(A_n\restriction y^n_{m-1})1 \sqsubseteq \sigma$ and

$$\forall j(y^n_{m-1} < j < |\sigma| - 1 \rightarrow \varphi_e(\sigma\restriction j) \leq \varphi_e(\sigma\restriction j+1))$$

i.e. the martingale φ_e does not grow along σ. Note that for every A_n defined in this way we have $\forall i > n(\varphi_e(A_n\lceil i) \leq \varphi_f(i))$, hence $A_{\langle e,f\rangle} \notin S_{\varphi_f}[\varphi_e]$:

$$\begin{array}{rcll} \varphi_e(A_n\lceil y^n_m) & \leq & \varphi_e((A_n\lceil y^n_{m-1})1) & (\varphi_e \text{ doesn't grow}) \\ & \leq & 2^{y^n_{m-1}+1} & (\forall w(\varphi_e(w) \leq 2^{|w|})) \\ & \leq & \varphi_{f,y^n_m}(z^n_m)\downarrow & \\ & \leq & \varphi_f(y^n_m) & (y^n_m \geq z^n_m \text{ and } \varphi_f \text{ monotonic}) \end{array}$$

Now define $\mathcal{A} = \{A_n : n \text{ is a Schnorr-test}\}$ Then by the remark above there is no Schnorr-test $\langle e, f\rangle$ such that $\mathcal{A} \subseteq S_{\varphi_f}[\varphi_e]$, and hence $\mu_{\text{mod}}(\mathcal{A}) \neq 0$. On the other hand, $\mu_{\text{rec}}(\mathcal{A}) = 0$. Namely, define a rec-martingale d that succeeds on $\mathcal{A}$ as follows. Let $d(\lambda) = 1$, and for $w = 0^n1v$ define $d(w1) = 2d(w)$ if and only if $|w| \in \{y^n_m : m \in \omega\}$, and $d(w1) = d(w)$ otherwise. In any case define $d(w0) = 2d(w) - d(w1)$, so that d is a martingale. Note that the sets $\{y^n_m : m \in \omega\}$ are recursive uniformly in n: to test whether a number k is one of the numbers y^n_m we only have to perform the finite number of computations $\varphi_{f,k}(l)$, where $n = \langle e, f\rangle$ and $l \leq k$. So d is indeed recursive. Also, $\mathcal{A} \subseteq S[d]$ because if $A_n \in \mathcal{A}$ then the set $\{y^n_m : m \in \omega\}$ is infinite. $\square$

Note that by suitably changing the usual definition of "$\varphi_{e,s}(x)\downarrow$" we may assume that the set $\{\langle e, x, s\rangle : \varphi_{e,s}(x)\downarrow\}$ is actually in P, the class of polynomial time computable sets. This implies that the set $\{\langle n, y^n_m\rangle : n, m \in \omega\}$ used in the proof is also in P. Since we can choose the pairing function and its inverses to be polynomial time computable this implies that the martingale d that succeeds on $\mathcal{A}$ is a p-martingale, hence that $\mu_{\text{p}}(\mathcal{A}) = 0$. So we have the following stronger version of Theorem 6.4.2:

6.4.3. THEOREM. *There exists a subset $\mathcal{A}$ of* REC *such that $\mu_{\text{p}}(\mathcal{A}) = 0$ but $\mu_{\text{mod}}(\mathcal{A}) \neq 0$.*

The assertion in this theorem that $\mathcal{A}$ is a subset of REC can not be improved to $\mathcal{A} \subseteq \Delta$ for any of the usual subrecursive complexity classes Δ by the following

6.4.4. PROPOSITION. *For any recursive time bound t, $\mu_{\text{mod}}(\text{DTIME}(t)) = 0$.*

PROOF. Fix a time bound t and let F be a recursive function such that

$$A \in \text{DTIME}(t) \Leftrightarrow \exists i\, \forall x(F(i, x) = A(x)).$$

Since DTIME(t) is dense in 2^ω (every initial segment is a prefix of a set in DTIME(t)) and F is total we can, given an initial segment w, recursively find the least i such that for all $x \leq |w|$, $F(i, x) = w(x)$. Denote this i by i_w. Define a recursive martingale d by $d(\lambda) = 1$ and $d(wb) = d(w) \cdot F(i_w, |w|)$ for $w \in \{0,1\}^*$ and $b \in \{0,1\}$. Then d succeeds on the i-th language in DTIME(t) faster than $2^{|w|-c}$ for some constant c (the number of steps that d was betting on the language using the wrong index i) whence DTIME$(t) \subseteq S_h[d]$ for the the function $h(n) = n$. $\square$

6.5 Partial recursive martingales

Partial recursive (p.r.) martingales were introduced in Fortnow et al. [22] to study the sizes of classes arizing in the context of learning theory. We prove that these martingales do not yield a measure.

6.5.1. Definition. ([22]) A partial function $m : 2^{<\omega} \to \mathbb{Q}^+$ is a *partial martingale* if for all $\sigma \in 2^{<\omega}$, if $m(\sigma 0)\downarrow$ or $m(\sigma 1)\downarrow$ then

- $m(\sigma)\downarrow$, $m(\sigma 0)\downarrow$, and $m(\sigma 1)\downarrow$, and
- $m(\sigma 0) + m(\sigma 1) = 2m(\sigma)$.

A partial martingale m *succeeds on* a set A if for all x, $m(A{\restriction}x)$ is defined and $\limsup_{x\in\omega} m(A{\restriction}x) = \infty$. A class $\mathcal{A}$ has *partial recursive measure zero* (p.r.-measure zero) if there is a partial recursive martingale that succeeds on every $A \in \mathcal{A}$.

Define the class of *self-describing* sets

$$\mathcal{S} = \{A \in \mathrm{REC} : 0^e 1 \sqsubset A \wedge A = \varphi_e\}.$$

That is, a recursive set A is self-describing if its characteristic sequence starts with exactly e zeros where the number e is a characteristic code for A.

6.5.2. Theorem. ([22]) *The class $\mathcal{S}$ does not have rec-measure zero.*

Proof. Let d be any recursive martingale. We construct a recursive set $A \in \mathcal{S}$ on which d does not succeed. By the recursion theorem we know a code e for A in advance of the construction. By induction, define $\sigma_0 = 0^e 1$ and

$$\sigma_{n+1} = \begin{cases} \sigma_n 0 & \text{if } d(\sigma_n 0) \le d(\sigma_n), \\ \sigma_n 1 & \text{if } d(\sigma_n 1) \le d(\sigma_n). \end{cases}$$

Then, if A is defined as $A = \bigcup_n \sigma_n$, it is easily seen that d does not succeed on A, and by definition of σ_0 we have $A \in \mathcal{S}$. □

6.5.3. Theorem. *There exist classes $\mathcal{X}$ and $\mathcal{Y}$, both of partial recursive measure zero, such that $\mathcal{X} \cup \mathcal{Y}$ does not have p.r.-measure zero.*

Proof. Let $\mathcal{Y}$ be any class of p.r.-measure zero that does not have rec-measure zero. For example, we can take $\mathcal{Y} = \mathcal{S}$, the class of self-describing sets. By Theorem 6.5.2 this class does not have rec-measure zero, and it is easy to show that $\mathcal{Y}$ has p.r.-measure zero. Let $\mathcal{X}$ be any dense class of p.r.-measure zero, for example, let $\mathcal{X}$ be the success set of the martingale d defined by $d(\lambda) = 1$,

$d(w0) = 3/2 \cdot d(w)$, $d(w1) = 1/2 \cdot d(w)$. Note that every partial martingale that succeeds on $\mathcal{X}$ must be total. Therefore, $\mathcal{X} \cup \mathcal{Y}$ does not have p.r.-measure zero, since this would imply the existence of a total recursive martingale that succeeds on $\mathcal{Y}$. $\square$

Clearly no partial recursive martingale can succeed on all the recursive sets. Namely, if it is not total, say it is not defined on $w \in 2^{<\omega}$, then it does not succeed on any recursive set with prefix w. Otherwise the martingale is total recursive, and we can diagonalize against it in the usual way. However, as we show in the next theorem, there is not a uniform way of defining for every p.r. martingale a recursive set on which it does not succeed.

6.5.4. THEOREM. *There is no partial recursive function f such that whenever φ_e is a p.r martingale, $f(e)$ is defined and $\varphi_{f(e)}$ is the characteristic function of a recursive set with $\varphi_{f(e)} \notin S[\varphi_e]$.*

PROOF. Suppose $f : \omega \to \omega$ is partial recursive. There is a p.r. function $m : \omega \times \{0,1\}^* \to \mathbb{Q}$ such that for all d, m_d is a p.r. martingale and if $\varphi_{f(d)}$ is total then m_d succeeds on $\varphi_{f(d)}$. Hence there is a recursive function h such that for all d the function $\varphi_{h(d)}$ is a p.r. martingale that succeeds on $\varphi_{f(d)}$ if both $f(d)$ is defined and $\varphi_{f(d)}$ is total. By the recursion theorem choose e with $\varphi_{h(e)} = \varphi_e$. Then φ_e is a p.r. martingale and if $f(e)$ is defined then either $\varphi_{f(e)}$ is not total or $\varphi_{f(e)} \in S[\varphi_e]$. $\square$

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Notation Index

Samenvatting

In dit proefschrift bespreken we een aantal vraagstukken uit de recursietheorie die betrekking hebben op maat en willekeurigheid. Het centrale begrip in de recursietheorie is het begrip 'recursieve verzameling'. Een verzameling is recursief als er een algoritme bestaat om te bepalen of iets een element is van deze verzameling. Bij het bestuderen van deelklassen van de klasse van recursieve verzamelingen spreekt men in het algemeen van complexiteitstheorie.

In hoofdstuk 1 introduceren en bespreken we de centrale begrippen uit dit proefschrift. In het bijzonder bespreken we enige elementaire maattheorie en de presentatie daarvan met behulp van zogenaamde martingalen. Deze functies, die opgevat kunnen worden als gokstrategieën, worden in een groot deel van dit proefschrift gebruikt om constructieve maattheorie te beschrijven. Dit in navolging van het werk van Schnorr en Lutz. In deze theorie worden diverse begrippen uit de klassieke maattheorie constructief gemaakt door de eis op te leggen dat ze *berekenbaar* zijn. De mate van berekenbaarheid fungeert hier als een parameter Δ, waaraan we kunnen denken als een begrenzing op de toegestane methoden. Hierom wordt deze vorm van constructieve maattheorie ook wel 'begrensde maattheorie' genoemd.

Het doel van begrensde maattheorie is tweeledig. Ten eerste wordt het door Δ voldoende strikt te kiezen mogelijk om ideeën uit de klassieke maattheorie toe te passen op de studie van diverse complexiteitsklassen. Dit geeft informatie over hoe de 'meeste' elementen van een complexiteitsklasse zich gedragen. Ten tweede geeft het bestuderen van begrensde maattheorie inzicht in het gedrag van *willekeurige* of *toevals*-verzamelingen. We kunnen hieraan denken als verzamelingen die gegenereerd zijn door een toevalsproces, zoals bijvoorbeeld het opwerpen van een munt. Een verzameling A is Δ-willekeurig als $\{A\}$ niet maat nul heeft in de begrensde maattheorie met parameter Δ. Intuïtief betekent dit dat een algoritme uit de klasse Δ geen regelmaat kan ontdekken in de verzameling A.

Bovenstaande ideeën worden in hoofdstuk 2 toegepast op de complexiteitsklasse E van verzamelingen die berekenbaar zijn in exponentiële tijd. In het eerste deel van dit hoofdstuk worden zogenaamde generische verzamelingen bestudeerd en gebruikt om een generalisatie te bewijzen van een stelling van Juedes en Lutz. In sectie 2.5 worden willekeurigheid en genericiteit met elkaar vergeleken en wordt duidelijk dat in deze context generische verzamelingen opgevat kunnen worden als een zwakke variant van toevalsverzamelingen. In het tweede deel van hoofdstuk 2 worden toevalsverzamelingen gebruikt om een vraag van Lutz te beantwoorden over het bestaan van zwak volledige verzamelingen die niet volledig zijn.

In hoofdstuk 3 wordt het ontwerp van Lutz voor begrensde maattheorie enigszins aangepast om een maat te definiëren die geschikt is voor de studie van recursief opsombare (afgekort r.e., voor 'recursively enumerable') verzamelingen. Dit is een klasse van verzamelingen die een prominente rol speelt in de recursietheorie. We bestuderen zwakke begrippen van volledigheid en verkrijgen een volledig beeld (zie plaatje pagina 57) van de relaties tussen de verschillende begrippen van zwakke en 'gewone' volledigheid. De studie van deze begrippen heeft ook consequenties voor een vraag die het begrip maat niet noemt, namelijk de vraag in hoeverre een onvolledige verzameling op een volledige verzameling kan lijken. Deze vraag wordt behandeld in sectie 3.3.

In hoofdstuk 4 bestuderen we klassen van martingalen corresponderend met de klassen uit de aritmetische hiërarchie. In het bijzonder bestuderen we de begrensde maat gedefinieerd door voor de klasse Δ de verzameling van recursief opsombare functies te nemen. De bijbehorende toevalsverzamelingen zijn precies de verzamelingen die oorspronkelijk geïntroduceerd zijn door Martin-Löf als voorstel voor een algemene definitie van willekeurigheid. Het bestaan van universele recursief opsombare verzamelingen heeft tot gevolg dat de r.e.-willekeurige verzamelingen mooie eigenschappen hebben. We beschrijven de distributie van deze verzamelingen in termen van de bekende reduceerbaarheidsrelaties uit de recursietheorie. We localiseren de klasse $R(\text{r.e.})$ van verzamelingen die geconstrueerd worden door zogenaamde r.e.-constructoren (pagina 65). In tegenstelling tot de klassen $R(\Delta)$ uit Lutz' begrensde maattheorie komt de klasse $R(\text{r.e.})$ niet overeen met een bekende klasse uit de recursietheorie. Tenslotte behandelen we analoge vragen voor de maten behorende bij de niveaus Δ_n van de aritmetische hiërarchie, en bewijzen we dat deze maten samenvallen met de maten behorend bij de niveaus Π_n.

Hoofdstuk 5 behandelt verzamelingen die 'laag' (low) zijn voor twee klassen van toevalsverzamelingen: de klasse $\mathcal{R}$ van Martin-Löf uit hoofdstuk 4 en de klasse $\mathcal{S}$, oorspronkelijk geïntroduceerd door Schnorr als een meer constructieve versie van $\mathcal{R}$. Een verzameling A is laag voor een klasse $\mathcal{C}$ als voor de gerelativeerde versie $\mathcal{C}^A$ van $\mathcal{C}$ geldt dat $\mathcal{C} = \mathcal{C}^A$. Intuïtief: als A relatief $\mathcal{C}$ niet bijdraagt in rekenkracht. Recursieve verzamelingen zijn trivialiter laag voor zowel $\mathcal{R}$ als $\mathcal{S}$.

We bewijzen dat in beide gevallen ook niet-recursieve verzamelingen bestaan die laag zijn. Dit toont aan dat substantiële hoeveelheden informatie op zo'n manier gecodeerd kunnen liggen in een verzameling dat ze niet toegankelijk zijn voor elementen van respectievelijk $\mathcal{R}$ en $\mathcal{S}$. De gevallen $\mathcal{R}$ en $\mathcal{S}$ verschillen aanzienlijk. In het geval van $\mathcal{R}$ construeren we een niet-recursieve, recursief opsombare lage verzameling, en weten we niet of er zulke verzamelingen buiten Δ_2 bestaan. In het geval van $\mathcal{S}$ construeren we $2^{\aleph_0}$ niet-recursieve lage verzamelingen en laten we zien dat deze noodzakelijkerwijs buiten Δ_2 moeten liggen. De resultaten voor de verzamelingen die laag zijn voor $\mathcal{S}$ worden verkregen via een karakterisering van deze verzamelingen in puur recursietheoretische begrippen, dat wil zeggen zonder vermelding van maattheorie. Volgens deze karakterisering zijn de functies die recursief zijn in een lage verzameling recursief *traceerbaar* (sectie 5.4). We laten verder nog zien dat verzamelingen die laag zijn voor $\mathcal{S}$ hyper-immuun zijn, en dat de omkering van deze bewering niet algemeen geldt.

In hoofdstuk 6, tenslotte, bespreken we kort een aantal thema's die betrekking hebben op recursieve martingalen. Aan de orde komen reduceerbaarheid naar toevalsverzamelingen, relaties tussen recursieve willekeurigheid en Kolmogorov-complexiteit, de relatie tussen de maat gedefinieerd door recursieve martingalen en de maat van Schnorr bestudeerd in secties 5.4 en 5.5, partieel recursieve martingalen, en maat in Δ_2.

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