GENERALIZATIONS OF THE WEAK LAW OF THE EXCLUDED MIDDLE

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Abstract. We study a class of formulas generalizing the weak law of the excluded middle, and provide a characterization of these formulas in terms of Kripke frames and Brouwer algebras. We use these formulas to separate logics corresponding to factors of the Medvedev lattice.

1. The weak law of the excluded middle

Let IPC denote the intuitionistic propositional calculus. The weak law of the excluded middle (w.l.e.m. for short) is the principle

$$\neg p \vee \neg\neg p.$$

We view this as an axiom *schema*, in which we can substitute any formula for the variable p. Consider the logic $\mathsf{IPC} + \neg p \vee \neg\neg p$, that is, the closure under deductions and substitutions of IPC and the w.l.e.m. The logic $\mathsf{IPC}+\neg p\vee\neg\neg p$ has been studied extensively, and is known in the literature under various names. It has been called

- *the logic of the weak law of the excluded middle* by Jankov,
- *Jankov logic* by various Russian authors,
- *De Morgan logic* by various American authors,
- *testability logic* by some others, and
- KC by still many others.

The term *principle of testability* for $\neg p \vee \neg\neg p$ goes back to Brouwer himself. In [3, p80] he writes (our comment in brackets):

> "Another corollary of the simple principle of the excluded third [i.e. $\tau \vee \neg\tau$] is the *simple principle of testability*, saying that every assignment τ of a property to a mathematical entity can be *tested*, i.e. proved to be either non-contradictory [$\neg\neg\tau$] or absurd [$\neg\tau$]."

Apparently the name KC comes from Dummett and Lemmon [5], who used LC to denote the "linear calculus", and K alphabetically precedes L, hence the name KC. The name De Morgan logic derives from the fact that the principle $\neg p \vee \neg\neg p$ is equivalent (over IPC) to the validity of the familiar De Morgan laws, or more precisely, to the only intuitionistically problematic implication $\neg(p \wedge q) \to \neg p \vee \neg q$ of these laws.

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In this paper we will study the following sequence $\{\varphi_k\}_{k\geqslant 1}$ of formulas generalizing the w.l.e.m.:

Definition 1.1. Let $\varphi_1 = \neg p \vee \neg\neg p$, and for every $k > 1$ define

$$\varphi_k = \bigvee_{i\neq j} \big(\neg p_i \to \neg p_j\big) \vee \neg(\neg p_1 \wedge \ldots \wedge \neg p_k) \tag{1}$$

(where $1 \leqslant i, j \leqslant k$).

Notice that the formula φ_1 can be seen as a special case of φ_k: indeed, φ_k is equivalent over IPC to

$$\neg p_1 \vee \ldots \vee \neg p_k \vee \bigvee_{i\neq j} \big(\neg p_i \to \neg p_j\big) \vee \neg(\neg p_1 \wedge \ldots \wedge \neg p_k) \tag{2}$$

because $\neg p_i$ implies $\neg p_j \to \neg p_i$ in IPC. Then φ_1 is the special case $k = 1$.

Also note that IPC proves $\varphi_k \to \varphi_{k+1}$ for every $k \geqslant 1$. This follows for example from Theorem 2.4, or from Theorem 3.3 below.

Below, we will study the logics $\mathsf{IPC} + \varphi_k$, which again is the deductive closure of IPC and the axiom schema φ_k. In particular $\mathsf{IPC} + \varphi_k$ proves any substitution instance of φ_k.

2. Kripke semantics

In this section we characterize the formulas φ_k in (1) in terms of Kripke frames, and relate them to a class of formulas introduced by Smorynski [12].

We briefly recall some elementary notions about Kripke semantics. For unexplained terminology about Kripke frames and models we refer the reader to [4] or [7, p67].

A *Kripke frame* $\langle K, R\rangle$ is a nonempty set K, partially ordered by an *accessibility relation* R. Throughout this paper, we will work with Kripke frames that have a *root*, that is, a least element with respect to R, though this is not standardly part of the definition. As usual, we distinguish between models and frames: A *Kripke model* $\langle K, R, V\rangle$ is a Kripke frame together with a *valuation* V, that associates with every variable p a set $V(p) \subseteq K$, such that if $x \in V(p)$ and xRy then $y \in V(p)$ for every x and y. Now the forcing relation $x \Vdash \varphi$, with $x \in K$ and φ a formula, is defined by

- $x \Vdash p$ if $x \in V(p)$;
- $x \Vdash \varphi \wedge \psi$ if and only if $x \Vdash \varphi$ and $x \Vdash \psi$;
- $x \Vdash \varphi \vee \psi$ if and only if $x \Vdash \varphi$ or $x \Vdash \psi$;
- $x \Vdash \varphi \to \psi$ if and only if for every y with $x\,Ry$, if $y \Vdash \varphi$ then $y \Vdash \psi$;
- $x \Vdash \neg\varphi$ if and only if there is no y with $x\,Ry$ and $y \Vdash \varphi$.

A formula φ *holds* in a frame K, denoted by $K \models \varphi$, if $K \Vdash \varphi$ (meaning that $x \Vdash \varphi$ for every $x \in K$), for every valuation V on the frame. A logic L is *complete with respect to*, or *characterizes*, a class of frames $\mathcal{K}$ if a formula is derivable in L if and only if it holds on every frame in $\mathcal{K}$.

Definition 2.1. A Kripke frame with accessibility relation R has *topwidth* k if it has k maximal nodes $x_1, \ldots, x_k$ such that for every $y \in K$ there is an i with yRx_i

Following Jankov [9], Gabbay [7, p67] showed that the logic $\mathsf{IPC} + \neg p \vee \neg\neg p$ is complete with respect to the class of Kripke frames of topwidth 1. Smorynski [12] introduced, for every $k \geqslant 1$, the formula

$$\sigma_k = \bigwedge_{0 \leqslant i < j \leqslant k} \neg\big(\neg p_i \wedge \neg p_j\big) \to \bigvee_{0 \leqslant i \leqslant k} \Big(\neg p_i \to \bigvee_{j \neq i} \neg p_j\Big) \tag{3}$$

and showed that the logic $\mathsf{IPC} + \sigma_k$ characterizes the class of Kripke frames of topwidth at most k (henceforth we refer to this result as Smorynski's Completeness Theorem). In particular, IPC proves that $\sigma_k \to \sigma_{k+1}$ and $\mathsf{IPC}+\sigma_1$ coincides with the logic of the w.l.e.m.. Note that φ_k has k variables and σ_k has $k+1$. The relation between these formulas is sorted out below.

We now turn to a characterization of the formulas φ_k in (1) in terms of Kripke frames. We start with some preliminaries about canonical models. For more on canonical models we refer to [4]. The *canonical model* K of a logic L containing IPC consists of tableaux, that is, pairs (Γ, Δ) of sets of formulas, with the following properties:[1]

(i) (Γ, Δ) is *consistent* with L, meaning that for no $\varphi_1, \ldots \varphi_n \in \Delta$, Γ proves $\varphi_1 \vee \ldots \vee \varphi_n$ over L,
(ii) (Γ, Δ) is *maximal* in the sense that $\Gamma \cup \Delta$ is the set of all formulas.

The accessibility relation R in the canonical model is defined by

$$(\Gamma, \Delta)\, R\, (\Gamma', \Delta') \iff \Gamma \subseteq \Gamma' \iff \Delta \supseteq \Delta'.$$

This defines the canonical frame, and to make it into a model it is defined that every atomic formula in Γ is forced in the node (Γ, Δ). It is a basic property of K that for every node (Γ, Δ) and every formula φ,

$$(\Gamma, \Delta) \Vdash \varphi \iff \varphi \in \Gamma. \tag{4}$$

Note that it follows from properties (i) and (ii) that Γ is closed under L-provability.

Lemma 2.2. *Suppose K is a Kripke frame of topwidth $n+1$ in which φ_k does not hold. Then $\binom{n}{\lfloor n/2 \rfloor} \geqslant k$.*

Proof. Under the assumptions, we prove that the power set $\mathcal{P}(\{1, \ldots, n\})$ has an antichain of size k. The lemma then follows from Sperner's Theorem, ([16]; cf. also [1]) stating that $\binom{n}{\lfloor n/2 \rfloor}$ is the greatest number k for which there is an antichain of k pairwise incomparable subsets of $\{1, \ldots, n\}$.

Since there is a model on the frame K that falsifies φ_k, there must be a maximal node in which $\neg p_1 \wedge \ldots \wedge \neg p_k$ holds. This leaves n nodes to falsify all implications $\neg p_i \to \neg p_j$ with $i \neq j$. Label these nodes by $1, \ldots, n$. Let $S_i \subseteq \{1, \ldots, n\}$ be the set of nodes where p_i holds, with $i = 1, \ldots, k$. Then the sets S_i form an antichain since for every pair $i \neq j$ there is a node that falsifies $\neg p_i \to \neg p_j$, hence in which p_i and $\neg p_j$ hold. $\square$

[1]Gabbay [7] uses saturated sets of formulas to define the canonical model, which is similar but different.

Lemma 2.3. *Suppose* $(\Gamma_1, \Delta_1), \dots, (\Gamma_n, \Delta_n)$ *are distinct maximal nodes in the canonical model of* L. *Then for every* $S \subseteq \{1, \dots, n\}$ *there is a formula* A *such that* $A \in \Gamma_j$ *if and only if* $j \in S$.

Proof. By maximality, the Γ_i are pairwise $\subseteq$-incomparable, hence for every $i \neq j$, there is a formula $A_{i,j} \in \Gamma_i - \Gamma_j$. Hence, taking, $A_i = \bigwedge_{j \neq i} A_{i,j}$, for every i, it is easy to see that $(\Gamma_i, \Delta_i) \Vdash A_i \to \neg A_j$ for every $i \neq j$. Now let $A = \bigvee_{j \in S} A_j$. □

Theorem 2.4. $\mathsf{IPC}{+}\varphi_k$ *is complete with respect to the class of Kripke frames of topwidth at most* n, *where* n *is minimal such that*

$$\binom{n}{\lfloor n/2 \rfloor} \geqslant k.$$

Proof. For the right-to-left implication, suppose K is a frame of topwidth $m + 1 \leqslant n$ in which φ_k does not hold. Then by Lemma 2.2, $\binom{m}{\lfloor m/2 \rfloor} \geqslant k$, hence by minimality of n we have $m \geqslant n$, a contradiction. Hence any frame of topwidth $l \leqslant n$ satisfies φ_k.

For the converse direction, we have to show that if φ is a formula that $\mathsf{IPC} + \varphi_k$ does not prove, then there is a Kripke frame of topwidth at most n, where n and k are related as in the statement of the theorem, in which φ does not hold, i.e. there is a model on this frame on which φ does not hold. We show that a part of the canonical model of $\mathsf{IPC} + \varphi_k$ has this property.

Now if φ is not provable in $\mathsf{IPC} + \varphi_k$, then the tableau $(\{\varphi_k\}, \{\varphi\})$ is consistent, but not maximal. By Lindenbaum's lemma [4, Lemma 5.1] it can be extended to a maximal tableau $t = (\Gamma, \Delta)$, which is thus a node in the canonical model in which φ does not hold by the property (4). Let K^t denote the part of K that is R-reachable from t. We prove that K^t has the required property.

First we note that every node in K is below an R-maximal one: every path in K has an upper bound (by taking unions on the first coordinate and intersections on the second), hence an application of Zorn's lemma gives a maximal element above any node in K.

We now show that K^t has at most n R-maximal nodes. Suppose for a contradiction that there exist at least $n + 1$ distinct maximal nodes

$$(\Gamma_1, \Delta_1), \dots, (\Gamma_{n+1}, \Delta_{n+1}).$$

Since $\binom{n}{\lfloor n/2 \rfloor} \geqslant k$ there is an antichain $S_1, \dots, S_k$ in $\mathcal{P}(\{1, \dots, n\})$ of size k. For every S_i, with the help of Lemma 2.3 choose a formula A_i such that

$$A_i \in \Gamma_j \iff j \in S_i \tag{5}$$

and such that $A_i \notin \Gamma_{n+1}$. Note that by maximality it follows from (5) that

$$\neg A_i \in \Gamma_j \iff A_i \notin \Gamma_j \iff j \notin S_i.$$

But now we can prove that φ_k is not forced in t: First $t \nVdash \neg(\neg A_1 \wedge \dots \wedge \neg A_k)$ because $(\Gamma_{n+1}, \Delta_{n+1}) \Vdash \neg A_1 \wedge \dots \wedge \neg A_k$ by choice of A_i. Also $t \nVdash \neg A_i \to \neg A_{i'}$ for every $i \neq i'$ with $i, i' \leqslant k$. Namely, the elements S_i and $S_{i'}$ of the antichain are incomparable, hence $j \in S_{i'} - S_i$ for some $j \in \{1, \dots, n\}$. Thus, by definition of A_i, we have $A_{i'} \in \Gamma_j$ and $\neg A_i \in \Gamma_j$, and hence $(\Gamma_j, \Delta_j) \Vdash \neg A_i \wedge A_{i'}$. So we see that t does not force the formula $\varphi_k(A_1, \dots, A_k)$ obtained from φ_k by substituting A_i for every variable p_i. But then it follows that $t \nVdash \varphi_k$, for if $t \Vdash \varphi_k$ then t would also force $\varphi_k(A_1, \dots, A_k)$ because we work over the logic $\mathsf{IPC} + \varphi_k$, which by definition proves every substitution instance of φ_k. □

By the property (4), every formula of a logic L holds in the canonical model of L. A logic L is called *canonical* if every formula of L holds in the canonical frame of L. Note that the proof of Theorem 2.4 shows that the logics of φ_k are canonical in this sense.

Following [7, p69], a condition F on a partially ordered set $\langle K, R, 0\rangle$ with least element 0, is *absolute* if it can be formulated in a higher order language (with symbols for $R, 0, =$), and for every $\langle K, R, 0\rangle$ satisfying F, there exists a finite $K_0 \subseteq K$ such that for every K', with $K_0 \subseteq K' \subseteq K$, we have that also $\langle K', R{\upharpoonright}K', 0\rangle$ satisfies F. It is known, see e.g. Gabbay [7, p69], that if L is an intermediate logic which characterizes a class of Kripke frames, consisting of exactly the frames satisfying an absolute condition F, then L also characterizes the class of finite Kripke frames satisfying F. An intermediate logic L is said to have the *finite model property*, if for every φ with $\varphi \notin \mathsf{L}$, there exists a finite Kripke model which does not satisfy φ. By a classical theorem of Harrop ([8]; see also [7, p. 266]), if an intermediate logic L has the finite model property and is finitely axiomatizable, then L is decidable. Therefore we have:

Theorem 2.5. *Each* $\mathsf{IPC} + \varphi_k$ *is complete with respect to the class of* finite *Kripke frames with topwidth at most* n*, where* n *is least such that* $\binom{n}{\lfloor n/2\rfloor} \geqslant k$*. Moreover,* $\mathsf{IPC} + \varphi_k$ *is decidable.*

Proof. The claim follows by the above quoted remark and the fact the condition of being a Kripke frame with topwidth at most n, and n least such that $\binom{n}{\lfloor n/2\rfloor} \geqslant k$, is absolute. □

Finally, we have the following additional characterization of $\mathsf{IPC} + \varphi_k$:

Corollary 2.6. $\mathsf{IPC} + \varphi_k = \mathsf{IPC} + \sigma_n$*, for all* n *and* k *such that* n *is minimal with* $\binom{n}{\lfloor n/2\rfloor} \geqslant k$*.*

Proof. This follows from Theorem 2.4 and Smorynski's Completeness Theorem. □

Notice that the sequence of logics $\mathsf{IPC} + \varphi_k$ is decreasing, but not strictly decreasing, with respect to inclusion. Namely, if $k_1 < k_2$ and n is the least such that $\binom{n}{\lfloor n/2\rfloor} \geqslant k_1$, but n is also the least such that $\binom{n}{\lfloor n/2\rfloor} \geqslant k_2$, then

$$\mathsf{IPC} + \varphi_{k_1} = \mathsf{IPC} + \varphi_{k_2} = \mathsf{IPC} + \sigma_n.$$

3. Algebraic semantics

A *Brouwer algebra* is an algebra $\langle L, +, \times, \rightarrow, \neg, 0, 1\rangle$ where $\langle L, +, \times, 0, 1\rangle$ is a bounded distributive lattice (with $+$ and $\times$ denoting the operations of sup and inf, respectively) and $\to$ is a binary operation satisfying

$$b \leqslant a + c \Leftrightarrow a \to b \leqslant c, \tag{6}$$

or, equivalently,

$$a \to b = \text{least } \{c : b \leqslant a + c\},$$

and $\neg$ is the unary operation, given by $\neg a = a \to 1$. A Brouwer algebra L *satisfies* a propositional formula σ (denoted by $L \models \sigma$; we also say that σ is an *identity* of L) if whatever substitution of elements of L in place of the propositional variables of σ (interpreting the connectives $\vee$, $\wedge$, $\to$, $\neg$ with the operations $\times$, $+$, $\to$, $\neg$, respectively) yields the element 0. (Note that this definition of truth is dual to that in a Heyting algebra; see also the remarks on Heyting algebras below.) Let

$$\mathrm{Th}(L) = \{\sigma : L \models \sigma\}.$$

It is well known that $\mathsf{IPC} \subseteq \mathrm{Th}(L)$, for every Brouwer algebra L. An intermediate logic L is *complete with respect to* a class of Brouwer algebras, if for every formula σ, L derives σ if and only if every algebra in the class satisfies σ.

Recall that in a distributive lattice L, we have that an element $a \in L$ is join-irreducible if and only if $a \leqslant x + y$ implies $a \leqslant x$ or $a \leqslant y$, for every $x, y \in L$. Thus if L is a Brouwer algebra, $b \in L$ with $b = \sum X$, where X is a finite set consisting of join-irreducible elements, then for every $a \in L$,

$$a \to b = \sum\{x \in X : x \nleqslant a\}. \tag{7}$$

This follows from the fact that $b \leqslant a + y$, where $y = \sum\{x \in X : x \nleqslant a\}$, and by join-irreducibility of each element of X, we have that $x \leqslant c$ for every c such that $b \leqslant a + c$ and every $x \in X$ such that $x \nleqslant a$. Thus y is the least such that $b \leqslant a + y$. Finally, if X is an antichain of join-irreducible elements in a distributive lattice, and $I, J \subseteq X$ are finite sets, then

$$\sum I \leqslant \sum J \Leftrightarrow I \subseteq J. \tag{8}$$

Recall the following well-known construction (see [6]) which associates with every Kripke frame a Brouwer algebra, whose identities coincide with the formulas that hold in the frame. Let K be a given Kripke frame, with accessibility relation R: a subset $A \subseteq K$ is *open*, if for every $x, y \in K$ we have that $x \in A$ and xRy then $y \in A$. Let $\mathsf{Op}(A)$ be the collection of open subsets of K.

Lemma 3.1 ([6]). *The distributive lattice* $\mathsf{Alg}(K) = \langle \mathsf{Op}(K), +, \times, \to 0, 1\rangle$ *is a Brouwer algebra, where* $A + B = A \cap B$, $A \times B = A \cup B$, $A \to B = \{x \in K : (\forall y \in K)[xRy \wedge y \in A \Rightarrow y \in B]\}$, $0 = K$, *and* $1 = \emptyset$. *Moreover*

$$\{\varphi : K \models \varphi\} = \{\varphi : \mathsf{Alg}(K) \models \varphi\}.$$

Proof. See [6]. In fact, the theorem in [6] is formulated in terms of Heyting algebras. Recall that L is a Heyting algebra if the dual L^{op} is a Brouwer algebra. If L is a Heyting algebra, we write $L \models^H \sigma$, if $L^{\mathrm{op}} \models \sigma$. In [6] it is shown that the collection of open sets together with the operations $+ = \cup$, $\times = \cap$, $0 = \emptyset$, $1 = K$, and

$$A \to B = \{x \in K : (\forall y \in K)[xRy \wedge y \in A \Rightarrow y \in B]\},$$

is a Heyting algebra which satisfies the same formulas as K. To prove our result, given a frame K, apply Fitting's theorem to get a Heyting algebra, and then take its dual: the claim then follows from the obvious fact that the formulas satisfied (under $\models$) by a Brouwer algebra are the same as the ones satisfied (under $\models^H$) by its dual Heyting algebra. $\square$

Conversely, given a Brouwer algebra L with meet-irreducible 0, let $I(L)$ be the collection of prime ideals of L, which becomes a Kripke frame $\mathsf{Kr}(L) = \langle I(L), \subseteq \rangle$. (Note that $\mathsf{Kr}(L)$ satisfies our assumption that all Kripke frames have a root, since $0 \in L$ is meet-irreducible, so that $\{0\}$ is a prime ideal.)

Lemma 3.2 ([10])**.** *For every Brouwer algebra L, we have*

$$\{\varphi : L \models \varphi\} \subseteq \{\varphi : \mathsf{Kr}(L) \models \varphi\}.$$

Moreover, equality holds if L is finite.

Proof. See [10]. Again, a few words may be spent on the proof, since [10] uses Heyting algebras instead of Brouwer algebras. So, suppose we are given a Brouwer algebra L, take its dual L^{op}, which is a Heyting algebra, and then use [10] to conclude that $\langle F(L^{\mathrm{op}}), \subseteq \rangle$ (where $F(L^{\mathrm{op}})$ is the collection of prime filters of L^{op}) is a Kripke frame K that satisfies $\{\varphi : L^{\mathrm{op}} \models^H \varphi\} \subseteq \{\varphi : K \models \varphi\}$, with equality if L^{op} is finite. The claim then follows from the fact that $\{\varphi : L^{\mathrm{op}} \models^H \varphi\} = \{\varphi : L \models \varphi\}$, and $F(L^{\mathrm{op}})$ is order isomorphic to $I(L)$ under $\subseteq$, as easily follows from recalling that in a distributive lattice L, for every $X \subseteq L$, X is a prime filter if and only if $L - X$ is a prime ideal. $\square$

Theorem 2.5 has the following algebraic counterpart:

Theorem 3.3. $\mathsf{IPC} + \varphi_k$ *is complete with respect to the class of all finite Brouwer algebras L with meet-irreducible* 0 *and at most n coatoms, where n is minimal such that $\binom{n}{\lfloor n/2 \rfloor} \geqslant k$.*

Proof. The proof follows from Theorem 2.4, Lemma 3.1, Lemma 3.2, together with the following observations:

(1) If K has topwidth n, then $\mathsf{Alg}(K)$ has n coatoms: indeed, for every maximal element x in the frame, the singleton $\{x\}$ is open, and this is clearly a coatom in $\mathsf{Alg}(K)$; moreover the coatoms in $\mathsf{Alg}(K)$ are all of this form.
(2) If a finite Brouwer algebra L has n coatoms, then $\mathsf{Kr}(L)$ is of topwidth n: indeed, in a finite Brouwer algebra L, the ideals generated by the coatoms are prime and contain all other prime ideals, generated by meet-irreducible elements. In other words the coatoms correspond exactly to the maximal elements in $\mathsf{Kr}(L)$.

Finally, notice that, for every Kripke frame K, $\mathsf{Alg}(K)$ has meet-irreducible 0, since the Kripke frames in this paper always have a least element. $\square$

For finite Brouwer algebras, we may also describe the completeness property in terms of join-irreducible elements joining to the greatest element 1.

Definition 3.4. For every n, let $\mathfrak{B}_n$ denote the class of Brouwer algebras in which the top element is the join of some antichain of n join-irreducible elements.

Notice that in any distributive lattice, if $\sum X = \sum Y$, where X, Y are finite antichains of join-irreducible elements, then it follows from (8) that $X = Y$. Thus, in a finite distributive lattice L, or more generally in a distributive lattice L having the finite descending chain condition (see e.g. [2, Theorem III.2.2]) each element is the join of a unique antichain of join-irreducibles, and thus L belongs to $\mathfrak{B}_n$, for a unique n.

Lemma 3.5. *If L is a finite Brouwer algebra, then L has exactly n coatoms if and only if $L \in \mathfrak{B}_n$.*

Proof. Suppose that $L \in \mathfrak{B}_n$ is finite, and let $b_1, \dots, b_n$ be the antichain of n join-irreducible elements such that $1 = \sum_{i=1}^n b_i$. For every i, let $\hat{b_i} = \sum_{j \neq i} b_j$. We claim that each $\hat{b_i}$ is a coatom. Indeed $\hat{b_i} < 1$, as $b_i \nleqslant \hat{b_i}$; moreover, assume that $\hat{b_i} \leqslant b$, and let $b = \sum X$ where X is an antichain of join-irreducible elements. (Here we use that L is finite.) By join irreducibility, we have

$$\{b_j : j \neq i\} \subseteq X \subseteq \{b_j : 1 \leqslant j \leqslant n\}$$

thus either $\hat{b_i} = b$ or $b = 1$. It follows that L has at least n coatoms. On the other hand, suppose that L has also a coatom $a \notin \{\hat{b_i} : 1 \leqslant i \leqslant n\}$. Then for every i, $\hat{b_i} + a = 1$, thus $b_i \leqslant \hat{b_i} + a$, hence by join irreducibility, $b_i \leqslant a$. This implies that $\sum_i b_i \leqslant a$, hence $a = 1$, a contradiction.

Conversely, suppose that L is a finite Brouwer algebra that has n coatoms. Since L is finite, there exists m such that $L \in \mathfrak{B}_m$. On the other hand, the above argument shows that $m = n$, so that $L \in \mathfrak{B}_n$. □

Let $\mathfrak{B}_n^\perp$ be the subclass of $\mathfrak{B}_n$, consisting of the algebras with meet-irreducible 0. It follows:

Corollary 3.6. $\mathsf{IPC} + \varphi_k$ *is complete with respect to the class of finite Brouwer algebras $\mathfrak{B}_n^\perp$, where n is minimal such that $\binom{n}{\lfloor n/2 \rfloor} \geqslant k$.*

Proof. Immediate from Theorem 3.3, and Lemma 3.5. □

Finally, we prove Theorem 3.8 below, which holds also of Brouwer algebras that are not necessarily finite. We need a preliminary lemma, which illustrates the range of $\neg$ in a Brouwer algebra from $\mathfrak{B}_n$.

Lemma 3.7. *Let $L \in \mathfrak{B}_n$, and let $b_1, \dots, b_n$ be an antichain of join-irreducible elements such that $1 = b_1 + \cdots + b_n$. Then every negation $\neg a$ in L is of the form $\neg a = \sum_{i \in I} b_i$ for some subset $I \subseteq \{1, \dots, n\}$ (where, of course, $\neg a = 0$ if $I = \emptyset$). In particular, $\neg b_i = \sum_{j \neq i} b_j$.*

Proof. By (7) we have $\neg a = \sum_{i \in I} b_i$, where $I = \{i : b_i \nleqslant a\}$. □

Theorem 3.8. *Let $\binom{n}{\lfloor n/2 \rfloor} = k$. Then the following hold:*

(i) *If $L \in \mathfrak{B}_m$ and $m \leqslant n$, then $L \models \varphi_k$;*
(ii) *if $L \in \mathfrak{B}_m^{\perp}$ and $m > n$ then $L \not\models \varphi_k$.*

Proof. (i) Let k and n be as in the statement of the theorem. Let $L \in \mathfrak{B}_m$, $m \leqslant n$, with $b_1, \ldots, b_m$ join-irreducible elements that join to 1. In order to show that φ_k holds in L, we take any sequence a_i of k elements in L and show that φ_k evaluates to 0 for $p_i = a_i$. If there are $i \neq j$ such that $\neg a_i$ and $\neg a_j$ are comparable then the first clause of φ_k is satisfied. So suppose that all $\neg a_i$ are pairwise incomparable. We have to show that then the last clause of φ_k is satisfied, i.e. that $\neg(\neg a_1 + \ldots + \neg a_k) = 0$, or equivalently, $\sum_{i=1}^{k} \neg a_i = 1$. By Lemma 3.7 every $\neg a$ is of the form $\neg a = \sum_{i \in I} b_i$. Note that $\sum_{i \in I} b_i \leqslant \sum_{j \in J} b_j$ if and only if $I \subseteq J$, as follows from (8). So to the k incomparable negations $\neg a_i$ corresponds a collection of k pairwise $\subseteq$-incomparable subsets of $\{1, \ldots, m\}$. Sperner's Theorem says that $\binom{m}{\lfloor m/2 \rfloor}$ is the maximum number k for which there is such an antichain of k pairwise incomparable subsets of $\{1, \ldots, m\}$. Hence because $\binom{m}{\lfloor m/2 \rfloor} \leqslant k$, the collection corresponding to the $\neg a_i$ covers all of $\{1, \ldots, m\}$, and in particular

$$\sum_{i=1}^{k} \neg a_i = \sum_{i=1}^{m} b_i = 1,$$

which is what we had to prove.

(ii) Suppose that $L \in \mathfrak{B}_m^{\perp}$, with $m > n$: let

$$I = \{b_1, \ldots, b_n, b_{n+1}, \ldots, b_m\}$$

be an antichain of join-irreducible elements such that in L we have $1 = \sum_{1 \leqslant i \leqslant m} b_i$. By Sperner's Theorem take a collection of k incomparable subsets $\{I_i : 1 \leqslant i \leqslant k\}$ of $\{1, \ldots, n\}$. For every $i = 1, \ldots, k$ choose a_i so that $\neg a_i = \sum_{j \in I_i} b_j$. (The proof of Lemma 3.7 shows how to achieve this: take $a_i = \sum_{j \notin I_i} b_j$.) Then the negations $\neg a_i$ are incomparable because the sets I_i form an antichain, and hence the first clause of φ_k is nonzero (as 0 is meet-irreducible in L). We also have

$$\sum_{i=1}^{k} \neg a_i = \sum_{\substack{1 \leqslant i \leqslant k \\ j \in I_i}} b_j \neq 1$$

(because no b_j, with $j > n$, is included), hence $\neg(\sum_{i=1}^{k} \neg a_i) \neq 0$ and the second clause of φ_k is also nonzero. So φ_k does not evaluate to 0 in L, since in this algebra, 0 is meet-irreducible. □

4. An application to the Medvedev lattice

This section is an addendum to [15]. We thank Paul Shafer [11] for pointing out some inaccuracies in that paper. In [15] logics of the form $\mathrm{Th}(\mathfrak{M}/\mathbf{A})$ are studied, where $\mathfrak{M}$ is the Medvedev lattice, $\mathbf{A} \in \mathfrak{M}$, and $\mathfrak{M}/\mathbf{A}$ is the initial segment of $\mathfrak{M}$ consisting of all $\mathbf{B} \in \mathfrak{M}$ such that $\mathbf{B} \leqslant \mathbf{A}$. The

Medvedev lattice arises from the following reducibility on subsets of ω^ω (also called *mass problems*): if $\mathcal{A}, \mathcal{B}$ are mass problems, then $\mathcal{A} \leqslant \mathcal{B}$, if there is an oracle Turing machine which, when given as oracle any function $g \in \mathcal{B}$, computes a function $f \in \mathcal{A}$. The *Medvedev degrees*, or simply, *M-degrees*, are the equivalence classes of mass problems under the equivalence relation generated by $\leqslant$. The collection of all M-degrees constitutes a bounded distributive lattice, called the *Medvedev lattice*, which turns out to be in fact a Brouwer algebra, i.e. it is equipped with a suitable operation $\to$, satisfying (6). Hence every factor of the form $\mathfrak{M}/\mathbf{A}$ is itself a Brouwer algebra, being closed under $\to$, with $\neg$ given by $\neg\mathbf{B} = \mathbf{B} \to \mathbf{A}$. In the following we use the notation from [15], to which the reader is also referred for more details and information about the Medvedev lattice and intermediate propositional logics.

In order to show that there are infinitely many logics of the form $\mathrm{Th}(\mathfrak{M}/\mathbf{A})$, in [15] a sequence of M-degrees $\mathbf{B}_n$, $n \in \omega$, is introduced. In Corollary 5.8 of [15] it is claimed that the logics $\mathrm{Th}(\mathfrak{M}/\mathbf{B}_n)$ are all different but no detailed proof of this is given. Below we prove that indeed these logics are all different from each other. In particular for any $f \in \omega^\omega$ consider the mass problem

$$\mathcal{B}_f = \left\{ g \in \omega^\omega : g \not\leqslant_T f \right\} :$$

then the Medvedev degree $\mathbf{B}_f$ of $\mathcal{B}_f$ is join-irreducible, [13]. Recall that the top element 1 of $\mathfrak{M}/\mathbf{B}_n$ is the join

$$\mathbf{B}_n = \mathbf{B}_{f_1} + \ldots + \mathbf{B}_{f_n}$$

where $\left\{ f_i : i \in \omega \right\}$ is a collection of functions whose Turing degrees are pairwise incomparable. In particular, the top element of $\mathfrak{M}/\mathbf{B}_1$ is join-irreducible and the top elements of all other factors $\mathfrak{M}/\mathbf{B}_n$ are not. Hence $\mathrm{Th}(\mathfrak{M}/\mathbf{B}_1)$ can be distinguished from all the other theories by the formula $\neg p \vee \neg\neg p$. Namely, the w.l.e.m. holds in a factor $\mathfrak{M}/\mathbf{A}$ if and only if $\mathbf{A}$ is join-irreducible, cf. [14]. We recall that the least element of $\mathfrak{M}$, and thus of every factor $\mathfrak{M}/\mathbf{A}$, is meet-irreducible. Hence $\mathfrak{M}/\mathbf{B}_n \in \mathfrak{B}_n^\perp$. (This is in fact enough for the proof below.)

Corollary 4.1. *If* $m \neq n$ *then* $\mathrm{Th}(\mathfrak{M}/\mathbf{B}_m) \neq \mathrm{Th}(\mathfrak{M}/\mathbf{B}_n)$.

Proof. Assume $n < m$, and let $k = \binom{n}{\lfloor n/2 \rfloor}$. Since $\mathfrak{M}/\mathbf{B}_n \in \mathfrak{B}_n^\perp$, by Theorem 3.8, we have that $\varphi_k \in \mathrm{Th}(\mathfrak{M}/\mathbf{B}_n)$, but $\varphi_k \notin \mathrm{Th}(\mathfrak{M}/\mathbf{B}_m)$. Notice also that by Corollary 2.6, we can now also conclude that $\sigma_n \in \mathrm{Th}(\mathfrak{M}/\mathbf{B}_n)$, but $\sigma_n \notin \mathrm{Th}(\mathfrak{M}/\mathbf{B}_m)$ □

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