10.2.2. *Null geodesics.* The preceding discussion was for a congruence of timelike geodesics. We can consider the same questions for a congruence of null geodesics. If k^μ the tangent vector field for a congruence of (affinely parametrized) null geodesics, then this satisfies $k^\mu k_\mu = 0$ and $\nabla_k k = 0$. We can define $\Theta^\mu_{\ \nu} := k^\mu_{\ |\nu}$. The derivation of eq. (10.6) remains the same, but the significance of Θ is more subtle.

Any neighboring geodesic of a timelike geodesic is also timelike. However, not all neighbors of a null geodesic are null. This means that the deviation between null geodesics is more special than the deviation between timelike geodesics.

The geodesic deviation equation shows that a deviation parallel to the tangent vector evolves trivially, and can be ignored. It also shows that we can consistently require the deviation to be orthogonal to the tangent vector.

For timelike or spacelike geodesics, this means that we can get rid of the trivial (parallel) part by restricting attention to deviation that is orthogonal to the geodesics, but for null geodesics, the tangent vector k is orthogonal to itself. This means that restricting to orthogonal deviation does not remove the trivial part. Effectively, the space of interesting deviations between null geodesics is 2-dimensonal — corresponding to 2 spatial directions. Geodesic deviation tells us how the shape of a light beam evolves.

As for a timelike congruence, Θ is orthogonal to the tangent vector, $k^\nu \Theta^\mu_{\ \nu} = 0 = k_\mu \Theta^\mu_{\ \nu}$. For a beam of light, Θ will be geometrically equivalent if we add multiples of k,

$$\Theta^\mu_{\ \nu} \mapsto \Theta^\mu_{\ \nu} + k^\mu u_\nu + v^\mu k_\nu \tag{10.8}$$

(for any vectors u and v orthogonal to k).

The expansion for a null congruence is denoted $\hat\theta := \Theta^\mu_{\ \mu} = k^\mu_{\ |\mu}$. The derivative of the expansion is

$$\dot{\hat\theta} = -R_{\mu\nu}k^\mu k^\nu - \Theta^\mu_{\ \nu}\Theta^\nu_{\ \mu}.$$

Exercise 10.3. Check that the transformation (10.8) does not change $\Theta^\mu_{\ \nu}\Theta^\nu_{\ \mu}$.

To write a Raychaudhuri equation for a null congruence, we need to decompose Θ into expansion, shear, and vorticity parts. First consider what Θ should be for a congruence with no shear or vorticity. This is ambiguous because of the general ambiguity in Θ. In the timelike case, $\delta^\mu_{\ \nu} - v^\mu v_\nu$ is a rank 3 projection orthogonal to v. In the null case, the simplest form for Θ is proportional to $P^\mu_{\ \nu}$, where $P_{\mu\nu} = P_{\nu\mu}$, $P^\mu_{\ \lambda}P^\lambda_{\ \nu} = P^\mu_{\ \nu}$ and $0 = P^\mu_{\ \nu}k^\nu$, and P has rank 2. This is unique up to adding a multiple of $k^\mu k_\nu$, so any such P will give a geometrically equivalent Θ. The trace of P is $P^\mu_{\ \mu} = 2$, so the shear and vorticity free form for Θ is $\frac{1}{2}\hat\theta P^\mu_{\ \nu}$. This suggests the general decomposition

$$\Theta_{\mu\nu} = \tfrac{1}{2}\hat\theta P_{\mu\nu} + \hat\sigma_{\mu\nu} + \hat\omega_{\mu\nu}$$

where $\hat\sigma_{\mu\nu} = \hat\sigma_{\nu\mu}$ and $\hat\omega_{\mu\nu} = -\hat\omega_{\nu\mu}$. The shear $\hat\sigma$ has 2 geometric degrees of freedom and the vorticity $\hat\omega$ has only 1.

This leads to the Raychaudhuri equation for an infinitesimal congruence of null geodesics,

$$\dot{\hat\theta} = -R_{\mu\nu}k^\mu k^\nu - \tfrac{1}{2}\hat\theta^2 - 2\hat\sigma^2 + 2\hat\omega^2. \tag{10.9}$$

The scalars $\hat{\sigma}^2$ and $\hat{\omega}^2$ are defined as in the timelike case and only depend upon the geometric parts of the shear and vorticity. Note that this is formally the same as eq. (10.7), except for the factor of $\frac{1}{2}$ instead of $\frac{1}{3}$; this corresponds to the 2 rather than 3 degrees of freedom for deviation.

10.3. **Schwarzschild.** Instead of the standard Cartesian coordinates (t, x, y, z) on Minkowski space, we can use a system of polar coordinates (t, r, θ, ϕ). These are related by

$$
\begin{aligned}
t &= t \\
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta.
\end{aligned}
$$

In this coordinate system, the metric is

$$ds^2 = dt^2 - dr^2 - r^2 d\sigma^2$$

where $d\sigma^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ is the metric for a unit sphere.

We can construct these coordinates in a way that generalizes to any spacetime that is static, spherically symmetric, and asymptotically flat. Let M be such a spacetime.

Spherically symmetry means that there is a group of symmetries ($SO(3)$) which sweeps out 2-spheres. Each of these spheres has constant intrinsic curvature. So, we can define a function $r \in C^\infty(M)$ such that the sphere through any given point has area $4\pi r^2$ and circumference $2\pi r$.

Static means that there exists a timelike Killing vector ξ which is hypersurface orthogonal. Asymptotic flatness means that we can normalize so that $\|\xi\|^2 \xrightarrow[r\to\infty]{} 1$. There exists a function $t \in C^\infty(M)$ which is constant on the hypersurfaces orthogonal to ξ, and such that $\xi(t) = 1$. This is unique of to adding a constant.

Now, choose one of the 2-spheres of symmetry at $t = 0$ and parametrize it with spherical polar coordinates θ and ϕ. We can extend these coordinates over the $t = 0$ hypersurface by requiring θ and ϕ to be constant along the curves perpendicular to the 2-spheres (i.e., the radial lines). Finally, extend these coordinates over M by requiring $\xi(\theta) = \xi(\phi) = 0$.

With these choices, $\xi = \partial_t$. The assumption that ξ is orthogonal to the constant t hypersurfaces means that $g_{tr} = g_{t\theta} = g_{t\phi} = 0$. The assumption that θ and ϕ are constant along the curves orthogonal to the 2-spheres of symmetry means that $g_{t\theta} = g_{r\theta} = g_{t\phi} = g_{r\phi} = 0$.

So, the metric must be of the form

$$ds^2 = \alpha\, dt^2 - \beta\, dr^2 - r^2 d\sigma^2$$

where α and β are functions of r alone. Asymptotic flatness and the normalization of t mean that

$$\alpha, \beta \xrightarrow[r\to\infty]{} 1. \tag{10.10}$$

Consider the congruence of outgoing radial (constant θ and ϕ) null curves. Because of spherical symmetry, these are automatically geodesics. The expansion $\hat{\theta}$ is by definition the rate at which a tube of these geodesics expands. Using a dot to indicate the derivative along the geodesics, the expansion can be computed as the derivative of the logarithm of the area $4\pi r^2$,

$$\hat{\theta} = (4\pi r^2)^{-1}(4\pi r^2)^{\cdot} = 2r^{-1}\dot{r}.$$

Now assume that $T_{\mu\nu} = 0$, and thus $R_{\mu\nu} = 0$. By spherical symmetry, the spin and shear of this family of null geodesics must vanish. So, the Raychaudhuri equation simplifies considerably to

$$0 = \dot{\hat{\theta}} + \tfrac{1}{2}\hat{\theta}^2 = 2r^{-1}\ddot{r}.$$

This means that $\dot{r}$ is constant. (Equivalently, r is an affine parameter for these null geodesics.)

Because $\xi = \partial_t$ is a Killing vector, $\xi_\mu \dot{x}^\mu = \alpha\dot{t}$ is constant. Because these are null geodesics, $0 = \alpha\dot{t}^2 - \beta\dot{r}^2$. Putting these facts together,

$$\alpha\beta = \left(\frac{\alpha\dot{t}}{\dot{r}}\right)^2$$

must be constant. By the normalization of t, this must be $\alpha\beta = 1$, and the metric simplifies to

$$ds^2 = \alpha\, dt^2 - \alpha^{-1}dr^2 - r^2 d\sigma^2.$$

The volume form is just $\epsilon = r^2\, dt \wedge dr \wedge \Omega$, where

$$\Omega := \sin\theta\, d\theta \wedge d\phi.$$

is the volume form for a unit sphere. This is closed, and the integral of Ω over a 2-surface is the total solid angle subtended.

If we lower the index, then the Killing vector ξ becomes the 1-form $\xi = \alpha\, dt$. Its exterior derivative is $d\xi = -\alpha'\, dt \wedge dr$. If we raise the indices, then the nonvanishing component is $(d\xi)^{tr} = \alpha'$. The Hodge dual is

$$*d\xi = \alpha' r^2 \Omega$$

The Komar mass integral gives the mass by integrating over any of the 2-spheres of symmetry,

$$M = \frac{1}{8\pi}\int_{S^2} *d\xi = \frac{1}{8\pi}\int_{S^2} \alpha' r^2 \Omega = \tfrac{1}{2}\alpha' r^2.$$

Integrating α' gives $\alpha = C - \frac{2M}{r}$, but (10.10) implies $C = 1$. We thus obtain the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\sigma^2. \tag{10.11}$$

Note that this metric is singular at $r = 2M$. This does not mean that the geometry is singular there, just that the coordinate system goes bad. This coordinate system is only good for $r > 2M$, $0 < \theta < \pi$, and $-\pi < \phi < \pi$.

This metric approximately describes the geometry outside a star or any reasonably round astronomical object. Of course the geometry inside a star is different, since $T^{\mu\nu}$ is far from 0 there.

10.3.1. *Reissner-Nordström.* We can generalize the Schwarzschild solution a little by adding an electromagnetic field. So, consider a spherically symmetric, static spacetime with a rotationally invariant, static electromagnetic field and no other matter.

Let k be the tangent vector to a rotationally invariant congruence of outgoing null geodesics. (Rotational invariance, implies that these are radial.) The component of the Ricci tensor contributing to the Raychaudhuri equation is

$$\begin{aligned} R_{\mu\nu}k^\mu k^\nu &= 8\pi T_{\mu\nu}k^\mu k^\nu - 4\pi T^\mu_\mu k^\nu k_\nu = 8\pi T_{\mu\nu}k^\mu k^\nu \\ &= -8\pi F^{\mu\nu}k_\nu F_{\mu\lambda}k^\lambda + 2\pi F^{\alpha\beta}F_{\alpha\beta}k^\mu k_\mu = -8\pi F^{\mu\nu}k_\nu F_{\mu\lambda}k^\lambda. \end{aligned}$$

However, the vector $F^{\mu\nu}k_\nu$ is rotationally invariant (and thus radial) and orthogonal to k. By "radial", I mean some linear combination of ∂_t and ∂_r. A radial vector orthogonal to k must be proportional to k — and thus a null vector. Therefore $R_{\mu\nu}k^\mu k^\nu = 0$.

So, by the same argument as before, the metric is of the form,

$$ds^2 = \alpha\, dt^2 - \alpha^{-1}dr^2 - r^2 d\sigma^2.$$

Suppose that the electromagnetic field is due to an electric charge Q, and no magnetic monopole charge. Then for any sphere at constant r and t,

$$\oint_{S^2} F = 0 \qquad \text{and} \qquad \oint_{S^2} *F = Q.$$

By symmetry, this implies that $*F = \frac{Q}{4\pi}\Omega$ or equivalently, $F = -\frac{Q}{4\pi r^2}dt \wedge dr$.

Using Einstein's equation, the t-t-component of the Ricci tensor is

$$R_{tt} = \frac{Q^2}{4\pi r^4}\alpha.$$

Using the timelike Killing vector $\xi := \partial_t$ again,

$$*(R_{\mu\nu}\xi^\mu dx^\nu) = \frac{Q^2}{4\pi r^2}dr \wedge \Omega.$$

If we convert ξ into a 1-form ξ_μ, then

$$d{*}d\xi = (\alpha' r^2)' dr \wedge \Omega.$$

Killing's second equation (see Sec. 9.1) gives

$$(\alpha' r^2)' = \frac{Q^2}{2\pi r^2}.$$

This integrates to

$$\alpha' r^2 = C - \frac{Q^2}{2\pi r}.$$

The Komar mass integral shows that the mass inside radius r is $\frac{1}{2}C - \frac{Q^2}{2\pi r}$, so $\frac{1}{2}C$ is just the total mass M. So, $\alpha' = \frac{2M}{r^2} - \frac{Q^2}{2\pi r^3}$. Integrating this with the normalizing condition $\alpha(\infty) = 1$, gives

$$\alpha = 1 - \frac{2M}{r} + \frac{Q^2}{4\pi r^2}.$$

So, we find the Reissner-Nordström metric,

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{4\pi r^2}\right)^{-1} dr^2 - r^2 d\sigma^2.$$

This coordinate system is only valid for $r > M + \sqrt{M^2 - \frac{Q^2}{4\pi}}$.

This is usually written without the 4π. That corresponds to a different choice of units in which Coulomb's law is simplified, but Maxwell's equations contain a factor of 4π. With that choice $\alpha = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$.

On the other hand, the most rational system of units would set Newton's and Coulomb's constants equal to $\frac{1}{4\pi}$. With that choice $\alpha = 1 - \frac{M}{2\pi r} + \frac{Q^2}{(4\pi r)^2}$.

If there is a magnetic monopole charge as well, then the electromagnetic field is,

$$F = \frac{Q_M}{4\pi}\Omega - \frac{Q_E}{4\pi r^2} dt \wedge dr$$

and the metric is the same, but with Q^2 replaced by $Q_E^2 + Q_M^2$.