The structure of perturbative quantum gauge theories

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What Feynman graphs are...

Graphs built from a fixed set \( \{v_1, \ldots, v_k\} \) of types of vertices and a fixed set \( \{e_1, \ldots, e_N\} \) of types of edges.

Examples:

- **Scalar \( \phi^3 \)-theory:**
  
  \[ \text{vertex: } \quad \text{edge: } \]

  and one constructs graphs such as

- **Quantum electrodynamics:**

  \[ \text{vertex: } \quad \text{edges: } \]

  and one constructs graphs such as
Quantum chromodynamics:

vertices: , , ,

edges: , ,

and one constructs graphs such as , , ,
Perturbative quantum gauge theory

Idea: probability amplitudes for physical processes are given by expansions in Feynman graphs.

**Example**: interaction of photon with electron (QED)

\[ G \sim = \quad + \quad + \quad + \quad \cdots \]

A physicist is interested in numbers, and the **Feynman rules** associate a complex number to a Feynman graph \( \Gamma \)

\[ \Gamma \mapsto U(\Gamma) \in \mathbb{C} \]

However, these numbers are typically infinite... \( \sim \) need to **renormalize**
Idea of renormalization

1. **Regularization**: introduce a parameter $z \in \mathbb{C}$ and define new Feynman rules $U_z$:

$$\Gamma \mapsto U_z(\Gamma) \in \mathbb{C}$$

The previous infinity becomes a **pole at** $z = 0$ of the Laurent series expansion in $z$.

2. **Subtraction**: get rid of the whole pole part of the Laurent series expansion: this gives the renormalized amplitude

$$\Gamma \mapsto R_z(\Gamma) \in \mathbb{C}$$

This applies to any Feynman graph, and in particular to subgraphs of Feynman graphs:

For a generic graph $\Gamma$: $R_z(\Gamma)$ defined by a recursive procedure

Example: renormalization of involves and
Quantum gauge symmetries

- Gauge field theories possess a gauge symmetry: this forms the infinite dimensional \textit{gauge group}.
- Mathematically, the gauge group can be understood as sections of the bundle of groups $P \times_G G$ associated to a principal $G$-bundle $P$ (on which the gauge field is a connection).
- After a successful (perturbative) quantization of gauge field theories, the gauge symmetry is lost, but \textit{quantum gauge symmetries} appear:

\begin{center}
\textbf{There are certain identities between Feynman graphs}
\end{center}

Example: in quantum electrodynamics we have linear Ward identities, eg.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0.5) -- (0.5,1) -- (1,0.5);
\end{tikzpicture} = \begin{tikzpicture}
\draw (0,0) -- (1,0);
\end{tikzpicture}
\end{center}

Instead, in quantum chromodynamics, the (Slavnov–Taylor) identities are \textit{quadratic} in the Feynman graphs.
Mathematical structure of renormalization

Group of ‘Feynman rules’

It turns out that the collection of all Feynman rules constitute a group.

We start by considering the Feynman rules $\Gamma \mapsto U(\Gamma) \in \mathbb{C}$ as characters on the **free commutative algebra** $H$ generated by all $1PI$ Feynman graphs **with residue in** $\{v_1, \ldots, v_k\} \cup \{e_1, \ldots, e_N\}$:

- **One-particle irreducible graphs:**
  
  \[ 1PI: \quad \begin{array}{c}
  \quad \text{and not } 1PI \ (1PR): \quad \end{array} \]

- **Residue of a graph:**

\[
\text{res} \left( \begin{array}{c}
  \quad \end{array} \right) = \begin{array}{c}
  \quad \end{array} \quad \text{and} \quad \text{res} \left( \begin{array}{c}
  \quad \end{array} \right) = \begin{array}{c}
  \quad \end{array} \]

Example of a graph not allowed:

\[
\quad \text{since } 1PR \text{ and residue } \not\equiv v_i
\]
Group structure on characters of $H$

- **Unit** $\epsilon \in G := \text{Hom}_C(H, \mathbb{C})$ is understood as a counit $\epsilon : H \rightarrow \mathbb{C}$.
- **Multiplication** $\ast : G \times G \rightarrow G$ induced by a coproduct $\Delta : H \rightarrow H \otimes H$.
- **Inverse** induced by the antipode $S : H \rightarrow H$.

**Theorem (Connes–Kreimer, 2000)**

There exists a counit, coproduct and antipode on the algebra $H$ of Feynman graphs, turning $H$ into a Hopf algebra (and $G$ a group). The counit is

$$\epsilon(\Gamma) = \begin{cases} 1 & \text{if } \Gamma = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and the coproduct is defined by

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subseteq \Gamma} \gamma \otimes \Gamma / \gamma,$$

where the sum is over (disjoint unions of) 1PI subgraphs with residue $v_i$ or $e_j$. 
Examples of the coproduct with $\nu = \leftarrow$ and $e = \rightarrow$

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subseteq \Gamma} \gamma \otimes \Gamma / \gamma$$

\[\Delta \begin{pmatrix} \text{Diagram 1} \end{pmatrix} = \begin{pmatrix} \text{Diagram 2} \end{pmatrix} \otimes 1 + 1 \otimes \begin{pmatrix} \text{Diagram 3} \end{pmatrix} + \begin{pmatrix} \text{Diagram 4} \end{pmatrix} \otimes \begin{pmatrix} \text{Diagram 5} \end{pmatrix} + \begin{pmatrix} \text{Diagram 6} \end{pmatrix} \otimes \begin{pmatrix} \text{Diagram 7} \end{pmatrix} + 2 \begin{pmatrix} \text{Diagram 8} \end{pmatrix} \otimes \begin{pmatrix} \text{Diagram 9} \end{pmatrix} + 2 \begin{pmatrix} \text{Diagram 10} \end{pmatrix} \otimes \begin{pmatrix} \text{Diagram 11} \end{pmatrix} + \begin{pmatrix} \text{Diagram 12} \end{pmatrix} \otimes \begin{pmatrix} \text{Diagram 13} \end{pmatrix} + \begin{pmatrix} \text{Diagram 14} \end{pmatrix} \otimes \begin{pmatrix} \text{Diagram 15} \end{pmatrix} + \begin{pmatrix} \text{Diagram 16} \end{pmatrix} \otimes \begin{pmatrix} \text{Diagram 17} \end{pmatrix} + \begin{pmatrix} \text{Diagram 18} \end{pmatrix} \otimes \begin{pmatrix} \text{Diagram 19} \end{pmatrix} \]
Renormalization as a decomposition in $G$

- The above Hopf algebra $H$ is the algebraic structure underlying the recursive procedure of renormalization.
- In fact, for a character $U_z : H \rightarrow \mathbb{C}$, there exists a character $C_z : H \rightarrow \mathbb{C}$ (‘counterterm’) defined for $z \neq 0$, such that
  $$R_z = C_z * U_z$$
  is finite at $z = 0$ [Connes and Kreimer, 2000].
- This decomposition is unique if one requires $C_{z=\infty} = \epsilon$ and can be interpreted as a Birkhoff decomposition in the group $G = \text{Hom}_\mathbb{C}(H, \mathbb{C})$:
  $$U_z = C_z^{-1} * R_z$$

![Diagram showing the decomposition process with points 0 and \(\infty\).]
Structure of the Hopf algebra of Feynman graphs
Gradings on $H$

• Grading by loop number $L(\Gamma) = h^1(\Gamma)$:

$H = \bigoplus_{l \in \mathbb{Z}_{\geq 0}} H^l$, \quad q_l : H \to H^l$

• Multigrading by number of vertices:

$d_i(\Gamma) = \# \text{vertices } v_i \text{ in } \Gamma - \delta_{v_i, \text{res}(\Gamma)}$

with

$H = \bigoplus_{n_1, \ldots, n_k \in \mathbb{Z}^k} H^{n_1, \ldots, n_k}$, \quad p_{n_1, \ldots, n_k} : H \to H^{n_1, \ldots, n_k}$
Physical interest: Green’s functions

For each vertex and edge we define Green’s functions

\[ G^{v_i} = 1 + \sum_{\text{res}(\Gamma) = v_i} \frac{\Gamma}{|\text{Aut}(\Gamma)|}, \quad G^{e_j} = 1 - \sum_{\text{res}(\Gamma) = e_j} \frac{\Gamma}{|\text{Aut}(\Gamma)|}, \]

corresponding to the physical probability amplitudes.

Problem: Write the coproduct on these Green’s functions

For example, for scalar $\phi^3$-theory (with one type of vertex $v = \bigtriangleup$ and one type of edge $e = \longrightarrow$) we have

Proposition

The elements $X = G^v(G^e)^{-3/2}$ and $G^e$ generate a Hopf subalgebra in $H$:

\[
\Delta(X) = \sum_{l=0}^{\infty} X^{2l+1} \otimes q_l(X), \quad \Delta(G^e) = \sum_{l=0}^{\infty} G^e X^{2l} \otimes q_l(G^e)
\]
Hopf subalgebras and ideals

In general (vertices \{v_1, \ldots, v_k\} and edges \{e_1, \ldots, e_N\}), we define

\[ X_{v_i} := \left( \frac{G^{v_i}}{\prod_j (G^{e_j})^{\text{val}_j(v_i)/2}} \right)^{1/\text{val}(v_i) - 2} \]

Theorem (vS, 2008)

1. The elements \(X_{v_i}\) and \(G^{e_j}\) generate a Hopf subalgebra \(H'\) in \(H\) with dual group

\[ G := \text{Hom}_\mathbb{C}(H', \mathbb{C}) \subset (\mathbb{C}[[\lambda_1, \ldots, \lambda_k]])^N \rtimes \text{Diff}(\mathbb{C}^k) \]

2. The ideal \(J := \langle X_{v_i} - X_{v_j} \rangle\) in \(H'\) is a Hopf ideal, i.e. \(H'/J\) is a Hopf algebra with dual group

\[ \text{Hom}_\mathbb{C}(H'/J, \mathbb{C}) \subset (\mathbb{C}[[\lambda]])^N \rtimes \text{Diff}(\mathbb{C}) \]

Can we explain the existence of these Hopf ideals from the classical gauge symmetry?
We now establish a connection between the Hopf algebra of renormalization and a Gerstenhaber structure in the context of gauge theories.

- In general, we assign to each vertex $v_i$ a parameter $\lambda_i$ for $i = 1, \ldots, k$.
- To each edge $e_j$ we assign a field $\phi_j$ and a corresponding antifield $\phi_j^\dagger$ for $j = 1, \ldots, N$ (with certain degrees); the anti-bracket is defined by

$$
(\phi_i(x), \phi_j^\dagger(y)) = \delta_{ij} \delta(x - y).
$$

and zero otherwise.

- This makes the following a **Gerstenhaber algebra**:

$$
A := \mathcal{F}([\phi_1, \phi_1^\dagger, \ldots, \phi_N, \phi_N^\dagger]) \otimes \mathbb{C}[\lambda_1, \cdots, \lambda_k]
$$

i.e. a graded algebra with a Lie bracket of degree 1.
The algebra $A = \mathcal{F}([\phi_1, \phi_1^\dagger, \ldots, \phi_N, \phi_N^\dagger]) \otimes \mathbb{C}[[\lambda_1, \ldots, \lambda_k]]$ consists of $\mathbb{C}[[\lambda_1, \ldots, \lambda_k]]$-linear functionals in the fields.

**Proposition (vS, 2008)**

The algebra $A$ is a Gerstenhaber comodule algebra over $H'$. In other words, there exists a map $\rho : A \to A \otimes H'$ compatible with the coproduct on $H'$ and respecting the bracket in $A$.

Consequently, there is an action of $G \subset (\mathbb{C}[[\lambda_1, \ldots, \lambda_k]]^\times)^N \rtimes \text{Diff}(\mathbb{C}^k)$ on $A$.

For instance, we have

$$
\rho : \phi_j \mapsto \sum_{n_1 \cdots n_k} \phi_j \lambda_1^{n_1} \cdots \lambda_k^{n_k} \otimes p_{n_1 \cdots n_k} \left( (G^{e_j})^{1/2} \right) \quad \text{(invertible series)}
$$

$$
\rho : \lambda_i \mapsto \sum_{n_1 \cdots n_k} \lambda_i \lambda_1^{n_1} \cdots \lambda_k^{n_k} \otimes p_{n_1 \cdots n_k} \left( (X_v)^{\text{val}(v_i)-2} \right) \quad \text{(formal diffeos)},
$$

where we recall $X_{v_i} = \left( \frac{G^{v_i}}{\prod_j (G^{e_j})^{\text{val}(v_i)/2}} \right)^{1/\text{val}(v_i)-2} \in H'$.
Master equation

- Next, one considers an element \( S \in A \) (the action) of the following form

\[
S = \sum_{j=1}^{N} \int dx\ m(e_j)(x) + \sum_{i=1}^{k} \lambda_i \int dx\ m(v_i)(x)
\]

with \( m(e_j), m(v_i) \) monomials in the fields that interact/propagate at \( e_j, v_i \), resp..

- The ideal \( I = \langle (S, S) \rangle \) implements the ‘master equation’ \((S, S) = 0\) and

\[
I = \langle p_\alpha(\lambda_1, \ldots, \lambda_k) \rangle_\alpha, \quad p_\alpha \text{ polynomials}
\]

- A theory (defined by \( S \)) is called simple if \( I = \langle \lambda_i - \lambda^{\text{val}(v_i) - 2} \rangle_i \) with \( \lambda := \lambda_j \) corresponding to some fixed valence 3 vertex \( v_j \).

Proposition (vS, 2008)

If \( S \) defines a simple theory, then the subgroup \( G^I \subset G \) leaving \( I \) invariant is dual to \( H'/J \) (with \( J = \langle X_{v_i} - X_{v_j} \rangle_{i,j} \)), i.e.

\[
G^I \simeq \text{Hom}_\mathbb{C}(H'/J, \mathbb{C}) \subset (\mathbb{C}[\lambda]^\times)^N \times \overline{\text{Diff}}(\mathbb{C}).
\]
Application to Yang–Mills theory

The action is (essentially) the Yang–Mills action for a connection one-form $\omega$ (with simple gauge group):

$$S = \| F(\omega) \|^2 = \int F^a_{\mu\nu} F^a_{\mu\nu} , \quad \text{with} \quad F = d\omega + \frac{1}{2} \omega \wedge \omega.$$ 

Feynman graphs are built from $\nu_3 = \text{graph}$ and $\nu_4 = \text{graph}$

The corresponding Hopf algebra $H$ coacts on the Gerstenhaber algebra $A$ of $\mathbb{C}[[\lambda_3, \lambda_4]]$-linear functionals in $\omega, \omega^\perp, \ldots$, and

$$\text{Hom}_\mathbb{C}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_3, \lambda_4]]^\times \rtimes \text{Diff}(\mathbb{C}^2)$$
• Gauge symmetry $\rightsquigarrow$ master equation $(S, S) = 0 \iff \lambda_4 - \lambda_3^2 = 0$

• The subgroup $G'$ of $\text{Hom}_\mathbb{C}(H', \mathbb{C}) \subset \mathbb{C}[[\lambda_3, \lambda_4]]^\times \rtimes \text{Diff}(\mathbb{C}^2)$ that leaves this equation invariant is dual to the Hopf algebra $H'/J$ with

$$J = \langle X_{\lambda_4} - X_{\lambda_3} \rangle$$

so that $G' \subset \mathbb{C}[[\lambda]]^\times \rtimes \text{Diff}(\mathbb{C})$ identifying $\lambda_4 = \lambda_3^2 \equiv \lambda^2$.

• Thus, in $H'/J$ the identities $X_{\lambda_4} = X_{\lambda_3}$ hold, or, explicitly

$$G_{\lambda_4} = (G_{\lambda_3})^2$$

These “quantum gauge symmetries” are known in physics as the Slavnov–Taylor identities for the coupling constants.

• Their appearance as generators of a Hopf ideal proves that the ST-identities are compatible with renormalization.

• In fact, if $U_z$ satisfies ST-identities, it follows that $R_z, C_z$ do so as well:

the renormalized and counterterm maps satisfy ST-identities


