On the geometry of noncommutative gauge fields

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Our goal is to find a proper formulation of noncommutative gauge field theories; we do this by studying an example of a noncommutative 4-dimensional sphere $S_\theta^4$; it is the base space of a noncommutative principal bundle:

$$SU(2) \to S_\theta^7 \to S_\theta^4$$

This is a quantization of the Hopf fibration on the commutative 4-sphere, that plays a central role in the construction of (charge 1) instantons on $S^4$.

Recall: moduli space $\mathcal{M}_1$ of charge 1 instantons on $S^4$ is isomorphic to the homogeneous space:

$$\mathcal{M}_1 \simeq SL(2, \mathbb{H}) / Sp(2, \mathbb{H})$$

conformal transformations gauge transformations

Question: What does the moduli space of (charge 1) instantons on $S_\theta^4$ look like?

To answer this question, we first need a notion of gauge fields on $S_\theta^4$. 
Toric noncommutative manifolds

Let $M$ a compact Riemannian spin manifold ($\dim M = m$) with a smooth isometrical action of the $n$-torus $\mathbb{T}^n$.

- action $\sigma_s$ of $\mathbb{T}^n$ on $C^\infty(M)$ by automorphisms:
  \[ \sigma_s(f)(x) = f(s^{-1} \cdot x). \]

- decompose $f = \sum f_r$ in homogeneous elements of degree $r$:
  \[ \sigma_s(f_r) = e^{2\pi i r \cdot s} f_r. \]

- $C^\infty(M)$ is represented on Hilbert space $\mathcal{H} = L^2(M, S)$ of spinors by pointwise multiplication: $\pi : C^\infty(M) \to B(\mathcal{H})$.

- There is a representation $U$ of $\mathbb{T}^n$ on $\mathcal{H}$ such that
  \[ U(s)DU(s)^{-1} = D, \]
  \[ U(s)\pi(f)U(s)^{-1} = \pi(\sigma_s(f)). \]
Given any real $n \times n$ anti-symmetric matrix $\theta = (\theta_{\mu\nu})$ a twisted representation $L_\theta$ of $C^\infty(M)$ is defined by

$$L_\theta(f) = \sum_r f_r U(r_\mu \theta_{\mu 1}, \ldots, r_\mu \theta_{\mu n}).$$

and set $C^\infty(M_\theta) := L_\theta(C^\infty(M))$;

- quantization map $L_\theta : C^\infty(M) \rightarrow C^\infty(M_\theta)$ satisfying $L_\theta(f \times_\theta g) = L_\theta(f)L_\theta(g)$ with on homogeneous elements:

$$f_r \times_\theta g_{r'} = f_r \sigma_{r,\theta}(g_{r'}) = e^{2\pi i r \cdot \theta \cdot r'} f_r g_{r'}.$$

- the triple $(C^\infty(M_\theta), \mathcal{H}, D)$ satisfies all properties of Connes’ noncommutative spin geometry of dim $m$

  (i.e. an $m$-summable spectral triple)
Noncommutative integral given by Dixmier trace:
\[
\int L_\theta(f) = \text{Tr}_\omega L_\theta(f) |D|^{-m}.
\]

Lemma (GIV05)

If \( f \in C^\infty(M) \) then
\[
\int L_\theta(f) = \int_M f \, d\nu
\]

Also, Connes-Moscovici local index formula takes a simple form.

Theorem

For a projection \( p \in M_N(C^\infty(M_\theta)) \), the index of the twisted Dirac operator \( D_p = pDp \) is given by:

\[
\text{Index } D_p = \text{Res}_{z=0} z^{-1} \text{tr} \left( \gamma p |D|^{-2z} \right) \\
+ \sum_{k \geq 1} c_k \text{Res}_{z=0} \text{tr} \left( \gamma \left( p - \frac{1}{2} \right) [D, p]^{2k} |D|^{-2(k+z)} \right)
\]
Differential calculus on $M_\theta$

Let $(\Omega(M), d)$ be the usual differential calculus on $M$.

- Extend the map $L_\theta : C^\infty(M) \rightarrow C^\infty(M_\theta)$ to $\Omega(M)$ by imposing it to commute with $d$. The image $L_\theta(\Omega(M))$ will be denoted by $\Omega(M_\theta)$.

- Similarly, there is a Hodge star operator on $\Omega(M_\theta)$ defined by $\star_\theta L_\theta(\omega) = L_\theta(\star \omega)$.

with $\star : \Omega^p(M) \rightarrow \Omega^{m-p}(M)$ the classical Hodge star operator.

- inner product on $\Omega(M_\theta)$:

$$ (\alpha, \beta)_2 = \int \star_\theta (\alpha^* \star_\theta \beta) $$

since $\star_\theta (\alpha^* \star_\theta \beta)$ is an element in $C^\infty(M_\theta)$. 

Yang-Mills theory on $M_\theta$

A connection on a right $C^\infty(M_\theta)$-module $E$ is a map

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^1(M_\theta)$$

obeying Leibniz rule, curvature $F = \nabla^2 : \mathcal{E} \rightarrow \mathcal{E} \otimes_{C^\infty(M_\theta)} \Omega^2(M_\theta)$.

- Yang-Mills action for a connection $\nabla$ on a finite projective $C^\infty(M_\theta)$-module $\mathcal{E}$ with curvature $F$ is defined by

$$S(\nabla) = \int \text{tr}^*_{\theta}(F^*_{\theta}F)$$

- Gauge invariance: $S(u^*\nabla u) = S(\nabla)$ for $u \in U(C^\infty(M_\theta))$

  Infinitesimally: $S(\nabla + [\nabla, X]) = S(\nabla)$ with $X \in \text{End}_{C^\infty(M_\theta)}^s(\mathcal{E})$.

- Equations of motion: nc Yang-Mills equations

$$[\nabla, *_{\theta}F] = 0$$

- Bianchi identity $[\nabla, F] = 0 \implies$ connections with (anti)selfdual curvature $*_{\theta}F = \pm F$ (instantons) are solutions of the YM equations; absolute minima of YM-action.
The (Connes-Landi) sphere $S^4_\theta$

With $\theta$ a real parameter, the algebra $\mathcal{A}(S^4_\theta)$ of polynomial functions on the sphere $S^4_\theta$ is generated by elements $z_0 = z_0^*$, $z_j$, $z_j^*$, $j = 1, 2$, subject to

$$z_\mu z_\nu = \lambda_{\mu\nu} z_\nu z_\mu, \quad z_\mu z_\nu^* = \lambda_{\nu\mu} z_\nu z_\mu^*, \quad \mu, \nu = 0, 1, 2,$$

with deformation parameters given by

$$(\lambda_{\mu\nu}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \lambda \\ 1 & \lambda & 1 \end{pmatrix}; \quad \lambda := e^{2\pi i \theta}$$

and with the spherical relation $\sum_\mu z_\mu^* z_\mu = 1$.

- **Isospectral deformation**: nc spin geometry $(\mathcal{A}(S^4_\theta), \mathcal{H}, D)$ of dim 4;
- **Differential calculus** $(\Omega(S^4_\theta), \text{d})$ as before, with $*_\theta : \Omega^p(S^4_\theta) \to \Omega^{4-p}(S^4_\theta)$. 

The sphere $S^{7}_{\theta'}$

With $\lambda'_{ab} = e^{2\pi i \theta'_{ab}}$ and $(\theta'_{ab})$ a real antisymmetric matrix, the algebra $\mathcal{A}(S^{7}_{\theta'})$ is generated by elements $\psi_a, \psi^*_a$, $a = 1, \ldots, 4$, subject to

$$\psi_a \psi_b = \lambda'_{ab} \psi_b \psi_a, \quad \psi_a \psi^*_b = \lambda'_{ba} \psi^*_b \psi_a,$$

and the spherical relation:

$$\sum_a \psi^*_a \psi_a = 1.$$

- Differential calculus $(\Omega(S^{7}_{\theta'}), d)$ as before.
Noncommutative Hopf fibration

A minimal requirement for \( A(S^4_\theta) \) and \( A(S^7_{\theta'}) \) to constitute a noncommutative \( SU(2) \)-principal bundle is that there is an action \( \alpha \) of \( SU(2) \) on \( A(S^7_{\theta'}) \) such that \( A(S^4_\theta) \) can be identified with the subalgebra of invariant elements under this action.

These conditions express \( \theta' \) in terms of \( \theta \) and we identify:

\[
\begin{align*}
z_0 &= \psi_1^* \psi_1 + \psi_2^* \psi_2 - \psi_3^* \psi_3 - \psi_4^* \psi_4 \\
z_1 &= 2(\mu \psi_3^* \psi_1 + \psi_2^* \psi_4) \\
z_2 &= 2(-\mu \psi_4 \psi_1^* + \psi_2 \psi_3^*)
\end{align*}
\]

\[
\theta'_{ab} = \frac{\theta}{2} \left( \begin{array}{cccc} 0 & 0 & -1 & 1 \\
0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \end{array} \right)
\]

where \( \mu = \sqrt{\lambda} = e^{\pi i \theta} \), giving the inclusion \( A(S^4_\theta) \hookrightarrow A(S^7_{\theta'}) \).

The \( * \)-action \( \alpha \) of \( SU(2) \) on \( A(S^7_{\theta'}) \) is given by

\[
\alpha_w : (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \mapsto (\psi_1, -\psi_2^*, \psi_3, -\psi_4^*) \cdot \left( \begin{array}{c} w \\
0 \\
w \end{array} \right)
\]

with \( w = \left( \begin{array}{c} w_1 \\
-w_2 \\
-w_1 \end{array} \right) \in SU(2) \).
Quantization of $S^7 \to S^4$ by an action of $\mathbb{T}^2$

The noncommutative Hopf fibration $S^7_{\theta'} \to S^4_\theta$ is a deformation of $S^7 \to S^4$ by an action of a 2-torus:

- Action of $\mathbb{T}^2 \subset \text{SO}(5)$ on $S^4$ gives deformed algebra $\mathcal{A}(S^4_\theta)$
- Action of double cover $\tilde{\mathbb{T}}^2 \subset \text{Spin}(5)$ of $\mathbb{T}^2$ on $S^7$ gives deformed algebra $\mathcal{A}(S^7_{\theta'})$
- This lifted action commutes with the action of the structure group $\text{SU}(2)$ which guarantees that the inclusion $\mathcal{A}(S^4_\theta) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ is a noncommutative $\text{SU}(2)$ principal bundle
Associated vector bundles $S^7_{\theta'} \times_{SU(2)} V$

We associate $\mathcal{A}(S^4_{\theta})$-modules to the noncommutative principal bundle $\mathcal{A}(S^4_{\theta}) \hookrightarrow \mathcal{A}(S^7_{\theta'})$ by all finite-dimensional representations of $SU(2)$.

- Let $\rho$ be a representation of $SU(2)$ on $V^{(n)} = \mathbb{C}^{n+1}$:

$$\mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho} V^{(n)} = \left\{ f \in \mathcal{A}(S^7_{\theta'}) \otimes V^{(n)} : (\alpha_w \otimes \text{id})(f) = (\text{id} \otimes \rho(w)^{-1})(f) \right\}$$

- There are projections $p(n) \in M_{4n}(\mathcal{A}(S^4_{\theta}))$ s.t. finite projective

$$\mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho} V^{(n)} \simeq p(n)\mathcal{A}(S^4_{\theta})^{4n}.$$ 

- Twisted Dirac operator $D_{(n)} = p(n)Dp(n)$ with coefficients in the module $\mathcal{A}(S^7_{\theta'}) \boxtimes_{\rho} V^{(n)}$.

Proposition

The index of the Dirac operator $D_{(n)}$ on $S^4_{\theta}$ is given by

$$\text{Index } D_{(n)} = \frac{1}{6}n(n+1)(n+2)$$
**Basic (charge 1) instanton on $S^4_{\theta}$**

A generic element in the module $\mathcal{E} = \mathcal{A}(S^7_{\theta'}) \boxtimes \rho \mathbb{C}^2$ can be written as $\Psi^* f$, $f \in \mathcal{A}(S^4_{\theta})^4$ with

$$
\Psi = \begin{pmatrix}
\psi_1 & -\psi_2^*\\
\psi_2 & \psi_1^* \\
\psi_3 & -\psi_4^* \\
\psi_4 & \psi_3^*
\end{pmatrix} ; \quad \text{satisfying } \Psi^* \Psi = \mathbb{I}_2.
$$

recall that: $\alpha_w : \Psi \mapsto \Psi \cdot \left( \begin{array}{cc}
w_0 & 0 \\
0 & w_0 \end{array} \right)$

Thus, $p = \Psi \Psi^*$ is a projection in $M_4(\mathcal{A}(S^4_{\theta}))$ and in fact $\mathcal{E} \simeq p \mathcal{A}(S^4_{\theta})^4$.

Explicitly:

$$
p = \frac{1}{2} \begin{pmatrix}
1 + z_0 & 0 & z_1 & -\mu z^*_2 \\
0 & 1 + z_0 & z_2 & \mu z^*_1 \\
z^*_1 & z^*_2 & 1 - z_0 & 0 \\
-\mu z_2 & \bar{\mu} z_1 & 0 & 1 - z_0
\end{pmatrix}
$$
Grassmann connection $\nabla_0 = p \, d \rightarrow$ curvature satisfies

\[ *_{\theta} F_0 = F_0 \]

For this reason, $\nabla_0$ is called the **basic instanton** on $S^{4}_{\theta}$, of charge 1, since

\[ \left\langle [S^{4}_{\theta}], \text{ch}_2(p) \right\rangle = 1 \]

In terms of $f \in A(S^{7}_{\theta'}) \boxtimes \rho \, \mathbb{C}^2$ we have

\[(\nabla_0 f)_i = df_i + \omega_{ij} f_j.\]

where $\omega = \Psi^* d \Psi$ is the **basic instanton gauge potential**, a $2 \times 2$-matrix with entries in $\Omega^1(S^{7}_{\theta'})$ satisfying:

\[ \overline{\omega}_{ij} = -\omega_{ji} ; \quad \sum_i \omega_{ii} = 0 . \]

**noncommutative** $su(2)$ gauge field
Moduli space of (charge 1) instantons

[AH78] Starting with the basic instanton $\nabla_0$ on $\mathcal{E}$, any other $(su(2))$ connection on $\mathcal{E}$ is given by $\nabla_0 + t\alpha$, with

$$\alpha \in \Omega^1(\text{ad}(S_\theta^7)) := \Omega^1(S_\theta^4) \otimes_{C^\infty(S_\theta^4)} \Gamma(\text{ad}(S_\theta^7))$$

where $\Gamma(\text{ad}(S_\theta^7))$ is the associated module to the adjoint representation of $SU(2)$ on $su(2)$.

- Linearized selfdual equation: $P_-[\nabla_0, \alpha] = 0$; $P_- = \frac{1}{2}(1 - *_\theta)$.
- If $\alpha$ were obtained from an infinitesimal gauge transformation, then $\alpha = [\nabla_0, X]$ with $X \in \Gamma(\text{ad}(S_\theta^7))$.
- Since $P_-[\nabla_0, [\nabla_0, X]] = [P_- F_0, X] = 0$, we have the selfdual complex

$$0 \rightarrow \Omega^0(\text{ad}(S_\theta^7)) \xrightarrow{[\nabla_0, \cdot]} \Omega^1(\text{ad}(S_\theta^7)) \xrightarrow{P_-[\nabla_0, \cdot]} \Omega^2(\text{ad}(S_\theta^7)) \rightarrow 0$$

and look for an element in the first cohomology group $H^1$.
We compute the alternating sum $h^0 - h^1 + h^2$ of the dimensions of the cohomology groups as the index of a twisted Dirac operator, with coefficients in the noncommutative vector bundle $S^- \otimes \text{ad}(S^7_{\theta'})$ on $S^4_{\theta}$ (N.B. $S^-$ is isomorphic to the charge $-1$ instanton bundle)

- Local index formula:

$$\text{Index} \ (\mathcal{D}) = \left\langle [S^4_{\theta}], \text{ch}(S^- \otimes \text{ad}(S^7_{\theta'})) \right\rangle$$

$$= 2 \cdot \left\langle [S^4_{\theta}], \text{ch}_2(\text{ad}(S^7_{\theta'})) \right\rangle - 3 \cdot \left\langle [S^4_{\theta}], \text{ch}_2(S^-) \right\rangle$$

$$= 2 \cdot 4 - 3 \cdot 1 = 5$$

- Using a vanishing argument for $h^0$ and $h^2$, we find that $h^1 = 5$.

The tangent space of the moduli space at $\nabla_0$ has dimension 5

Problem: How can we generate a 5-parameter family of instantons?
Twisted infinitesimal symmetries

Consider the Lie algebra $\mathfrak{so}(5)$ with generators $H_1, H_2, E_r$ for the eight roots $r = (\pm 1, \pm 1), (0, \pm 1), (\pm 1, 0)$;

- Action on generators $z_0, z_1, z_1^*, z_2, z_2^*$ of $\mathcal{A}(S_4^\theta)$:

\[
H_1 = z_1 \partial_1 - z_1^* \partial_1^*, \quad H_2 = z_2 \partial_2 - z_2^* \partial_2^*,
\]

\[
E_{+1,+1} = z_2 \partial_1^* - z_1 \partial_2^*, \quad \text{et cetera}
\]

- Extended to the whole of $\mathcal{A}(S_4^\theta)$ as twisted derivations [Sit01]:

\[
E_r(ab) = E_r(a)\lambda^{-r_1}H_2(b) + \lambda^{-r_2}H_1(a)E_r(b),
\]

\[
H_j(ab) = H_j(a)b + aH_j(b),
\]

- The action of $\mathfrak{so}(5)$ by twisted derivations can be lifted to an action on $\mathcal{A}(S_7^\theta')$ of the same twisted type and extended to $\Omega^1(S_7^\theta')$.

Proposition

The instanton gauge potential $\omega$ is invariant under the twisted action of $\mathfrak{so}(5)$; in other words, $H_j(\omega) = E_r(\omega) = 0$. 
(Twisted) Conformal Lie algebra

Consider the Lie algebra $so(5, 1)$: it consists of the generators of $so(5)$ and generators $H_0, G_r$ with $r = (±1, 0), (0, ±1)$.

- Action of $so(5, 1)$ on the generators of $A(S^4_\theta)$:
  \[ H_0 = \partial_0 - z_0(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 + z_2^*\partial_2^*), \]
  \[ G_{1,0} = 2\partial_1^* - z_1(z_0\partial_0 + z_1\partial_1 + z_1^*\partial_1^* + \overline{\lambda}z_2\partial_2 + \lambda z_2^*\partial_2^*), \]
  \[ et cetera \]

extended to $A(S^4_\theta)$ as twisted derivations; lift to $A(S^7_\theta')$ and $\Omega(S^7_\theta')$.

Proposition

1. The instanton gauge potential $\omega = \Psi^* d\Psi$ transforms to $\omega + t\delta\omega_i$, $i = 0, \ldots, 4$ under $tH_0, tG_r$ ($t \in \mathbb{R}$)

2. The curvature $F_0$ of basic instanton transforms to $F_0 + t\delta F_i + O(t^2)$ with $\delta F_0 = -2z_0 F_0$,
   \[ \delta F_1 = -2z_1 \lambda H^2 F_0; \quad \delta F_2 = -2z_2 \lambda H^1 F_0; \]

and $\delta F_3 = \delta F_1^*$, $\delta F_4 = \delta F_2^*$, which are still selfdual.
Drinfel’d twists

This twisting of \( so(5) \) and \( so(5, 1) \) can be understood as a Drinfel’d twist:

- Universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) is a Hopf algebra with coproduct \( \Delta(X) = X \otimes 1 + 1 \otimes X \), counit \( \epsilon(X) = 0 \) and \( S(X) = -X \) for \( X \in \mathfrak{g} \).
- This structure is twisted by an invertible element \( \mathcal{F} \in H \otimes H \):
  \[
  \Delta_{\mathcal{F}}(h) = \mathcal{F} \Delta(h) \mathcal{F}^{-1}.
  \]
- The twist of \( \mathcal{U}(so(5)) \) and \( \mathcal{U}(so(5, 1)) \) is given by the element
  \[
  \mathcal{F} = \lambda \frac{1}{2} (-H_1 \otimes H_2 + H_2 \otimes H_1), \quad \lambda = e^{2\pi i \theta}
  \]
yielding the Hopf algebras \( \mathcal{U}_\theta(so(5)) \) and \( \mathcal{U}_\theta(so(5, 1)) \).
We have a 5-parameter family of *infinitesimal* instantons.

Since the tangent space at $\nabla_0$ of the moduli space of charge 1 instantons on $S^4_\theta$ is 5-dimensional, this is a complete family.

How to integrate this infinitesimal family? Two possibilities:

1. Imitate [AHS78] to integrate tangent spaces to obtain local charts on moduli space $\mathcal{M}_1$, then patch these together.

2. Dualize Drinfel’d twist of $so(5,1)$ and $so(5)$ and quantize [Rie93] the corresponding Lie groups $SL(2, \mathbb{H})$ and $Sp(2, \mathbb{H})$; then generate instantons via a coaction of these quantum groups on the noncommutative Hopf fibration.

Problem with the first option is that we need a global version of (noncommutative) $SU(2)$ gauge transformations, corresponding to the infinitesimal gauge transformations given by $ad(S^7_{\theta'})$. This is an known problem in noncommutative geometry: determinant not central.

We take the second approach and quantize these Lie groups...
Quantum groups $\text{SL}_\theta(2, \mathbb{H})$ and $\text{Sp}_\theta(2, \mathbb{H})$

- We have $\mathbb{T}^2 \subset \text{Sp}(2, \mathbb{H}) \subset \text{SL}(2, \mathbb{H})$ by setting:

$$ (t_1, t_2) \mapsto \begin{pmatrix} e^{2\pi i t_1} & 0 \\ 0 & e^{2\pi i t_2} \end{pmatrix}. $$

- Adjoint action of $\mathbb{T}^2$ on $\text{Sp}(2, \mathbb{H}) \subset \text{SL}(2, \mathbb{H})$ give deformed products $\times_\theta$ on the algebras of (polynomial) functions $\mathcal{A}(\text{Sp}(2, \mathbb{H}))$ and $\mathcal{A}(\text{SL}(2, \mathbb{H}))$.

$$ \mathcal{A}(\text{Sp}_\theta(2, \mathbb{H})) = (\mathcal{A}(\text{Sp}(2, \mathbb{H})), \times_\theta) $$
$$ \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) = (\mathcal{A}(\text{SL}(2, \mathbb{H})), \times_\theta) $$

- Since $\text{Sp}(2, \mathbb{H})$ and $\text{SL}(2, \mathbb{H})$ are Lie groups, the algebras $\mathcal{A}(\text{Sp}(2, \mathbb{H}))$ and $\mathcal{A}(\text{SL}(2, \mathbb{H}))$ are Hopf algebras, with coproduct $\Delta$, counit $\epsilon$ and antipode $S$.

- It turns out that $(\mathcal{A}(\text{Sp}_\theta(2, \mathbb{H})), \Delta, \epsilon, S)$ and $(\mathcal{A}(\text{SL}_\theta(2, \mathbb{H})), \Delta, \epsilon, S)$ are still Hopf algebras.
Noncommutative family of instantons

- We can ‘embed’ $S^7_{\theta'}$ in $\mathbb{H}^2_{\theta}$ to obtain a coaction:

$$\Delta_L : \mathcal{A}(S^7_{\theta'}) \to \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^7_{\theta'})$$

and similarly for $\mathcal{A}(\text{Sp}_\theta(2, \mathbb{H}))$ which is a quotient of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))$.

- This induces a coaction of $\mathcal{A}(\text{SL}_\theta(2, \mathbb{H}))^{\mathbb{Z}_2} =: \mathcal{A}(\text{SO}_\theta(5, 1))$ on $\mathcal{A}(S^4_\theta)$.

- The basic instanton projection $p \in M_4(\mathcal{A}(S^4_\theta))$ is mapped to $P := \Delta_L(p)$ which is an element in $M_4(\mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$.

- The projection $P$ is understood as a noncommutative family of projections parametrized by the noncommutative space $\text{SL}_\theta(2, \mathbb{H})$.

Proposition

1. The family of connection $\widetilde{\nabla}_0 = P \circ (\text{id} \otimes d)$ is self-dual:

$$(\text{id} \otimes \star_\theta) P( (\text{id} \otimes d) P)^2 = P( (\text{id} \otimes d) P)^2.$$

2. The projection $P$ is (Murray-von Neumann) equivalent to the projection $1 \otimes p$ in the algebra $M_4(\mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$. 
Since $P \sim 1 \otimes p$, it follows that $\text{ch}_n(P) \in \text{HC}_{2n}(\mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$ coincides with the pushforward $\phi_*\text{ch}_n(p)$ of $\text{ch}_n(p) \in \text{HC}_{2n}(\mathcal{A}(S^4_\theta))$ under the algebra map

$$\phi : \mathcal{A}(S^4_\theta) \rightarrow \mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta), \quad a \mapsto 1 \otimes a.$$ 

As a consequence, both $\text{ch}_0(P)$ and $\text{ch}_1(P)$ are zero since $\text{ch}_0(p)$ and $\text{ch}_1(p)$ vanish \cite{CL01}.

Then, one uses the map $\phi$ to pull back the fundamental class $[S^4_\theta] \in \text{HC}^4(\mathcal{A}(S^4_\theta))$ to a class $\phi^*[S^4_\theta]$ in $\text{HC}^4(\mathcal{A}(\text{SL}_\theta(2, \mathbb{H})) \otimes \mathcal{A}(S^4_\theta))$. When paired with $\text{ch}_2(P)$ it gives

$$\left\langle \phi^*[S^4_\theta], \text{ch}_2(P) \right\rangle = \left\langle \phi^*[S^4_\theta], \phi_*\text{ch}_2(p) \right\rangle = \left\langle [S^4_\theta], \text{ch}_2(p) \right\rangle = 1,$$

In other words, the **instanton charge of the family $P$ is 1**.
The family of connections

We can express $\widetilde{\nabla}_0$ in terms of a family of connection one-forms:

$$\widetilde{\omega}_{ij} = \Delta_L(\omega_{ij}) \in A(S\text{L}_\theta(2, \mathbb{H}) \otimes A(S^7_{\theta'}))$$

Proposition

The instanton connection 1-form $\omega$ is invariant under the coaction of the quantum group $S\text{p}_\theta(2, \mathbb{H})$, that is for this quantum group one has

$$\Delta_L(\omega_{ab}) = 1 \otimes \omega_{ab}.$$ 

In other words, the family of connections is parametrized by the quantum quotient $M_\theta := S\text{L}_\theta(2, \mathbb{H})/S\text{p}_\theta(2, \mathbb{H})$, defined in terms of its function algebra:

$$A(M_\theta) := \{ a \in A(S\text{L}_\theta(2, \mathbb{H})) | \Delta_R(a) = 1 \otimes a \}.$$ 

where $\Delta_R = (\pi \otimes \text{id}) \circ \Delta$ is the (right) coaction of $A(S\text{p}_\theta(2, \mathbb{H}))$ on $A(S\text{L}_\theta(2, \mathbb{H}))$ via the quotient map $\pi$. 
Structure of the noncommutative parameter space $\mathcal{M}_\theta$

The algebra $\mathcal{A}(\mathcal{M}_\theta)$ is generated by $m, n, g_1, g_2$ with relations:

\[ g_1 g_2 = \lambda g_2 g_1, \quad g_1 g_2^* = \bar{\lambda} g_2^* g_1, \]
\[ mn - (g_1^* g_1 + g_2^* g_2) = 1, \]

Thus, $\mathcal{M}_\theta$ is a deformation of a hyperboloid in 6 dimensions.

In fact, if we write $w = \frac{1}{2}(m + n)$, $Y = \frac{1}{2}w^{-1}(m - n)$ and $G_1 = w^{-1} g_1, G_2 = w^{-1} g_2$:

\[ Y^2 + G_1^* G_1 + G_2^* G_2 = 1 - w^{-2}. \]

and in the ‘limit $w \to \infty$’ (the ‘boundary’ of $\mathcal{M}_\theta$), we find noncommutative 4-spheres $S^4_\theta$. 


