1

THE STANDARD MODEL

These notes contain more details and some references about the main topics of the classical Standard Model that I have discussed in the seminar ”Standard Model in Noncommutative Geometry”. It is possible that the different expressions for Lagrangians in the text below are not completely consistent with each other: it may, for instance, occur that some scalar factors are missing. This is due to the fact that I have used multiple sources (with different conventions) while writing these notes. I have not taken the effort to fix these inconsistencies since I focussed on the underlying concepts.

I share these notes since I think they may be useful for studying the different topics. Probably these notes are not error-free. If you find some errors, please let me know. Furthermore, it could be that some of you know a lot more about some of these topics than I do. If you like to add some useful or interesting information, I will surely appreciate that.

1.1 INTRODUCTION

The Standard Model describes all interactions (except for interactions due to gravity) between the elementary particles. The Standard Model consists of fermions (six quarks and six leptons, all spin $\frac{1}{2}$) and bosons (g, $W^{\pm}$, Z^0, γ, H, all spin-1 except for H which is spin-0). All these particles, except for the Higgs boson, have indeed been measured. Moreover, the Standard Model, especially, gives various relations between different parameters. These relations have been verified up to a very high precision. Actually, it is quite remarkable that the predictions of the Standard Model agree so well with the experimental observations.

The bosons in the Standard Model (except for the Higgs-boson) are gauge bosons and the three interactions are mediated through these gauge bosons. The boson g (gluon) accounts for the strong force and is a spin-1 particle, the $W^{\pm}$, Z^0 mediate the weak force and the photon is responsible for the electromagnetic force. The Higgs boson enters the Standard Model after symmetry breaking. The graviton (supposed to be a spin-2 particle) is not included in the Standard Model, so the Standard Model does not account for gravity. As you probably know, the unification of a theory of gravity (e.g. the theory of general relativity) and the Standard Model is still a major problem.

The physics in the (quantised) Standard Model is contained in the (Standard Model) Lagrangian and theoretical predictions of experiments are obtained by doing perturbation calculations. Carrying out these calculations is a business on its own and requires knowledge about renormalisation and dimension regularisation etc., topics that will not be treated here.

1.1.1 *What will come?*

First I will tell something about the Lorentz group and its finite-dimensional irreducible representations together with the representations of the groups $SO^+(1,3)$ and $SL(2,\mathbb{C})$, the universal covering group of $SO^+(1,3)$. Particles in the Standard Model transform under a certain transformation of one of these groups. I will use these representations to introduce the fermionic particles in the Standard Model. Since all fermions in the Standard Model are spin-$\frac{1}{2}$ I will not discuss fermions that have higher spin.

Table 1.1: Particles in the Standard Model

Fermions						Bosons		
Quarks			Leptons					
up	charm	top	e^-	μ^-	τ^-	photon	$W^\pm$,Z^0	gluon
down	strange	bottom	ν_e	ν_μ	ν_τ	EM	weak	strong

The next topic I will discuss are the different gauge theories in the Standard Model. As sort of an introduction I will shortly explain what gauge theories are. Having done that we have gathered all the necessary equipment to describe the Standard Model. First of all, I will discuss the $SU(3) \times SU(2)_L \times U(1)_Y$ gauge symmetries of the Standard Model and write down the gauge- and Lorentz-invariant Lagrangian belonging to this theory. We will see that is it not possible to put in, by hand, a gauge-invariant mass term for the gauge fields in the Lagrangian. Therefore in the unbroken Standard Model the gauge bosons are massless. This problem is solved by breaking the $SU(2)_L \times U(1)_Y$ symmetry to a $U(1)_{EM}$ symmetry (the $SU(3)$-symmetry is untouched in this procedure). I will tell how this symmetry breaking is done and how one gets relations between different parameters in the Standard Model. Moreover, in this procedure the Higgs particle is introduced in the Standard Model.

The Glashow-Weiberg-Salam Model is used for deriving mass terms for the fermionic fields in the Standard Model. The most general form of this mass term eventually leads to quark mixing. That is, quarks of different generations may interact with each other through the $W^\pm$-bosons.

At the end I will discuss some miscellaneous topics such as Majorana spinors and the difference between helicity, chirality and spin. The main topic is the occurence of massive neutrinos. When the minimal Standard Model was written down it was assumed that all neutrinos were massless. Nowadays, it is known that not all neutrinos have the same mass so that at least one of them is not mass-less. I will shortly describe the implications of having massive neutrino fields.

In table (1.1.1) I have made a list of the particles in the Standard Model.

1.2 REPRESENTATIONS OF THE LORENTZ GROUP

Definition 1.2.1. *Let $(\mathbb{R}^{1,3}, \eta)$ be Minkowsky space-time, where $\eta = diag(1,-1,-1,-1)$ is the Minkowsky metric. The full Lorentz-group, denoted by $O(1,3)$ consists of all linear maps $\Lambda : \mathbb{R}^{1,3} \to \mathbb{R}^{1,3}$ that preserve the metric: $\Lambda \in O(1,3)$ if and only if $x^T \Lambda^T \eta \Lambda y = x^T \eta y$ for all $x, y \in \mathbb{R}^{1,3}$ (i.e. $\Lambda^T \eta \Lambda = \eta$).*

The full Lorentz group is a Lie group that consists of four connected components:

$$\begin{aligned} L_+^\uparrow &= \{\Lambda \in O(1,3) | \det \Lambda = 1, \Lambda_{00} \geq 1\}, \\ L_-^\uparrow &= \{\Lambda \in O(1,3) | \det \Lambda = -1, \Lambda_{00} \geq 1\}, \\ L_+^\downarrow &= \{\Lambda \in O(1,3) | \det \Lambda = 1, \Lambda_{00} \leq -1\}, \\ L_-^\downarrow &= \{\Lambda \in O(1,3) | \det \Lambda = -1, \Lambda_{00} \leq -1\}, \end{aligned}$$

where the first component is obviously the connected component of the identity. The subgroup $L_+^\uparrow \cup L_+^\downarrow$ are called the proper Lorentz transformations and elements in $L_+^\uparrow \cup L_-^\uparrow$ are called the orthochronous transformations. The component $L_+^\uparrow$ is the connected component of the identity and is therefore a subgroup of $O(1,3)$. We will also write $SO^+(1,3)$ for this group (which has the same Lie-algebra as $O(1,3)$). Its elements are called the proper orthochronous transformations.

Remark 1.2.2. One can show that $L_+^\uparrow$ is homeomorphic to $\mathbb{R}^3 \times SO(3)$ and is therefore connected. The union of the four components are indeed the full Lorentz-group since $-1 = \det \eta = \det(\Lambda^T)\det(\eta)\det(\Lambda) = (\det \Lambda)^2$ and $1 = \langle(1,0,0,0),(1,0,0,0)\rangle = \langle\Lambda(1,0,0,0),\Lambda(1,0,0,0)\rangle = \Lambda_{00}^2 - \Lambda_{11}^2 - \Lambda_{22}^2 - \Lambda_{33}^2$. Furthermore, if we define $P = \text{diag}\,(1,-1,-1,-1)$, $T = \text{diag}\,(-1,1,1,1) \in O(1,3)$, then $P \cdot L_+^\uparrow = L_-^\uparrow$, $T \cdot L_+^\uparrow = L_-^\downarrow$ and $PT \cdot L_+^\uparrow = L_+^\downarrow$, so that all four components are connected.

Let us take a look at the transformation of a real scalar field ϕ (in the classical case this is a real-valued function). Under a Lorentz transformation Λ the field ϕ transforms as:

$$\phi'(x) = \phi(\Lambda^{-1}x). \tag{1.1}$$

If we allow ϕ to be a multiplet of scalar fields, then under a Lorentz transformation the most general transformation of the field ϕ is of the form

$$\Lambda \cdot \phi^a(x) = D_b^a(\Lambda)\phi^b(\Lambda^{-1}x). \tag{1.2}$$

We require this to be a representation. In that case, under the composition of Lorentz transformation Λ and Λ' the D_b^a obey the rule:

$$D_c^a(\Lambda\Lambda') = D_b^a(\Lambda)D_c^b(\Lambda'), \tag{1.3}$$

so that D is a matrix representation of the Lorentz group. The conclusion is that, in order to determine the different transformation properties we have to determine the (finite-dimensional) representations of the Lorentz group. For a general Lie group G, every irreducible representation of G descends to an irreducible representation of the Lie algebra $\mathfrak{g}$ of G. Since it is easier to determine the irreducible representations of the Lie algebra we will first determine those.

The Lie algebras of $O(1,3)$ and $SO^+(1,3)$ are the same, namely $so(1,3)$ which consists of all 4×4-matrices A satisfying $A\eta + A^T\eta = 0$. To get representations of $SO^+(1,3)$ we first study the representation theory of $so(1,3)$.

The Lie-algebra $so(1,3)$ has six generators, three rotation and three boost generators:

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

We can now establish an isomorphism $S : \mathbb{C}\otimes so(1,3) \to \mathbb{C}\otimes su(2) \oplus \mathbb{C}\otimes su(2)$ of complex Lie-algebras by setting $T_r = \frac{1}{2}(X_r + iB_r)$ and $\bar{T}_r = \frac{1}{2}(X_r - iB_r)$. The finite-dimensional (unitary) irreducible representations of $su(2)$ are indexed by positive half-integers s and the dimension of the representation is equal to $2s+1$.

Define the following representations of $\mathbb{C} \otimes so(1,3)$:

$$D_{s,t}(X) := (D_s \otimes D_t)(SX), \quad (X \in so(1,3)). \tag{1.4}$$

where $D_s \otimes D_t$ is the representation $D_s \otimes D_t(X,Y) = D_s(X) \otimes 1 + 1 \otimes D_t(Y)$ of the Lie-algebra $\mathbb{C} \otimes su(2) \oplus \mathbb{C} \otimes su(2)$. This representation has dimension $(2s+1)(2t+1)$. We can restrict this representation to $so(1,3)$ and from now on we will denote $D_{s,t}$ for this representation of $so(1,3)$.

Theorem 1.2.3. *The representation $D_{s,t}$ is irreducible and any (finite-dimensional) irreducible representation of $so(1,3)$ is isomorphic to $D_{s,t}$ for some s, t.*

Proof. We will not prove this theorem here, but instead refer to [5], Chapter 6, for more details on this subject. With this theorem we now know all finite-dimensional irreducible representations of $so(1,3)$. These representations are indexed by two half-integers. □

Proposition 1.2.4. *If we restrict the representation $D_s \otimes D_t$ to the subalgebra generated by the rotation generators X_i in $(so(1,3)$ (this subalgebra is isomorphic to $su(2)$), then we get the splitting*

$$D_{s,t}|_{rotations} = \oplus_{r=|s-t|}^{s+t} D_r, \tag{1.5}$$

of $su(2)$-representations and the number $s+t$ is the spin of the particle. [1]

Remark 1.2.5. We can take the complex conjugate representation of $D_{r,s}$. That is, we consider the representation $D^*_{r,s}$. One can show that $D^*_{r,s} \cong D_{s,r}$. We call a representation ρ real if $\rho^* = \rho$.

There is one 1-dimensional representation of the Lorentz-algebra, namely the trivial one, and the scalar particles transform according to this representation. There are two 2-dimensional representations, namely $D_{\frac{1}{2},0}$ and $D_{0,\frac{1}{2}}$ which correspond to the left-handed and right-handed fields respectively. (both 2-dimensional representations are real Lie-algebra isomorphisms from $so(1,3)$ to $sl(2)$ and $D_{\frac{1}{2},0} = -D^\dagger_{0,\frac{1}{2}}$)) Finally, there are three 4-dimensional representations. Since the vector representation of $so(1,3)$ is real and irreducible this representation is equivalent to $D_{\frac{1}{2},\frac{1}{2}}$.

Conclusion: the scalar particles transform according to the trivial representation, the Weyl-spinors according to the $D_{\frac{1}{2},0}$ and $D_{0,\frac{1}{2}}$ representations and the spin-1 bosons sit in the vector representation $D_{\frac{1}{2},\frac{1}{2}}$. The Dirac-spinors transform as $D_{\frac{1}{2},0} \oplus D_{0,\frac{1}{2}}$ (left-handed and right-handed part) under the Lorentz-algebra.

Remark 1.2.6. Not all representation of the Lie-algebra $so(1,3)$ lift to representations of the group $SO(1,3)$. They do if and only if the number $s+t$ in $D_{s,t}$ is an integer (see [5], Chapter 6 and 7). As you might know for every finite-dimensional real Lie-algebra $\mathfrak{g}$ there is a unique connected and simply connected (real) Lie group G such that $\mathfrak{g}$ is the Lie algebra of G ([3], Chapter 1.10). Moreover, every representation of $\mathfrak{g}$ lifts to a representation of the group G. In our case this Lie group would be the universal covering group of $SO(1,3)$ which is the group $SL(2,\mathbb{C})$. The group $SL(2,\mathbb{C})$ double-covers $SO(1,3)$. We denote the covering homomorphism by Λ. To construct it, we first note that there is an isomorphism $\mathbb{R}^{1,3} \to H(2,\mathbb{C})$, where $H(2,\mathbb{C})$ is the space of 2×2 hermitian matrices given by $x \mapsto \tilde{x} := x_0 I + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$. Note that $\langle x, x\rangle = \det \tilde{x}$. Now $\Lambda : SL(2,\mathbb{C}) \to SO(1,3)$ is given by $(\Lambda(\tilde{A})x) = A\tilde{x}A^\dagger$, with $(x \in \mathbb{R}^{1,3}$. The kernel of the spin homomorphism Λ is $\pm I$. Thus, the important thing to note here is that for fermionic particles the irreducible finite-dimensional Lie-algebra representations of $so(1,3)$ lift to irreducible representations of $SL(\mathbb{C},2)$ and not to representations of $SO(1,3)$.

An important consequence is the following. The group $SL(2)$ acts on Dirac-spinors as:

$$\rho(A) = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^\dagger \end{pmatrix} \tag{1.6}$$

1 Klaas pointed out that this definition of spin is only correct for massive particles. For massless particles there is a notion of helicity. A more precise definition of spin can be written down when we consider the full group of symmetries: the Poincare-group, which is a semi-direct product of the abelian group of all translations and the Lorent-group. The finite-dimensional irreducible representations of the Poincare Lie-algebra are determined by two numbers, m and s, namely the mass and the spin of the particle.

Take $A_\theta = \text{diag}\,(e^{-i\theta}, e^{i\theta})$. Then $\Lambda(A)$ is rotation about the x_3-axis by 2θ (leaving x_0 fixed). If $\theta = \pi$, then $\Lambda(A_\pi)$ is a rotation about 2π while the spinors ψ are transformed into a negative. This describes the effect that spinors are transformed into its negative when space undergoes a complete rotation.

1.3 GAUGE THEORIES

I will now briefly explain the main ideas of gauge theories in physics, starting with the simplest one: the $U(1)$-gauge theory of Electrodynamics. (This part is based on different chapters in [2].) First of all, the Lagrangian density of a spin-$\frac{1}{2}$ particle is given by

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \tag{1.7}$$

We observe that this Lagrangian has a global $U(1)$ symmetry: if we replace $\psi(x)$ by $e^{-i\alpha}\psi(x)$, where $\alpha \in \mathbb{R}$ is a constant, then the Lagrangian is invariant under this transformation. This symmetry is still present if we let the field ψ interact with an electromagnetic field determined by the potential A_μ. The Lagrangian of a charged spin-$\frac{1}{2}$-field is easily written down using minimal coupling:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - m)\psi, \tag{1.8}$$

and the variational principle gives the Dirac-equation:

$$(\gamma^\mu(i\partial_\mu - qA^\mu) - m)\psi = 0, \tag{1.9}$$

where q is the electric charge of the field ψ. We know that the electromagnetic field is invariant if we replace the potential A_μ by $A_\mu + \partial_\mu \chi$ where χ is a function of space-time. The Dirac equation also stays invariant if we, in addition, replace $\psi(x)$ by $e^{-iq\chi(x)}\psi(x)$, as is easily verified. The transformation

$$\psi \mapsto e^{-iq\chi}\psi, \quad A_\mu \mapsto A_\mu + \partial_\mu \chi \tag{1.10}$$

is called a local gauge transformation.

We can look at this transformation from a different point of view. The Lagrangian (1.7) is invariant under the global $U(1)$ transformation $\psi \mapsto e^{-i\alpha}\psi$ where α is constant. If we now ask for the Lagrangian to be invariant under a local transformation $\psi(x) \mapsto e^{-i\alpha(x)}\psi(x)$, where $\alpha(x)$ is an arbitrary smooth function of space and time, we are forced to introduce the gauge field A_μ with transformation property $A_\mu \mapsto A_\mu + \partial_\mu \chi$, in order to cancel the additional terms which arise. From this point of view, the electromagnetic field appears as a consequence of the requirement of invariance of the Lagrangian under a local symmetry transformation. It is this last viewpoint that is generalised to other gauge theories with other gauge groups.

Definition 1.3.1. *Let ψ be an n-tuple of spinor fields and assume that the Lagrangian of the free spinor-fields is given by (1.7), where ψ now has n components. Assume that we are also given an n-dimensional unitary representation of a matrix Lie-group G. The Lagrangian (1.7) is then invariant under the global transformation:*

$$\psi \to g\psi, \quad (g \in G). \tag{1.11}$$

If we require local G symmetry of the Lagrangian density with respect to this representation, that is the Lagrangian is invariant under the transformation of the particle field $C^\infty(\mathbb{R}^{1,3}, V) \ni \psi \mapsto g \cdot \psi$, where $g \in C^\infty(\mathbb{R}^{1,3}, G)$ acts pointwise on ψ, then we need to

- *introduce a gauge field $A_\mu \in C^\infty(\mathbb{R}^{1,3}, \mathfrak{g})$ transforming as*

$$A_\mu(x) \mapsto g(x)A_\mu(x)g^{-1}(x) - (\partial_\mu g)(x)g^{-1}(x) \tag{1.12}$$

when $\psi(x) \mapsto g(x)\psi(x)$. Here A_μ acts on V in the representation induced by the one of G.

- *define the Lagrangian to be the expression obtained from (1.7) by replacing ∂_μ by $D_\mu := \partial_\mu + A_\mu$.*

The new Lagrangian is now invariant under the local gauge transformation:

$$\psi(x) \mapsto g(x)\psi(x), \quad A_\mu(x) \mapsto g(x)A_\mu(x)g^{-1}(x) - (\partial_\mu g)(x)g^{-1}(x). \tag{1.13}$$

Remark 1.3.2. More generally this should be viewed as follows. Let M be a space-time manifold and G a linear Lie-group. To obtain a gauge theory with gauge group G one considers a principal G-bundle P with an associated vector bundle (fibre V) obtained by the representation of G on V. The spinors ψ are then sections of this associated bundle. The gauge field A_μ above is equivalent to a ($\mathfrak{g}$-valued) connection 1-form on the bundle P. The field strength of the gauge potential is given by the curvature tensor $F = dA + A \wedge A$. In case the gauge group G is an abelian group the wegde product $A \wedge A = 0$, which in physics means that the gauge field A does not couple to itself. This is different in the nonabelian case where $A \wedge A$ is in general unequal to zero and gives three/four couplings of A to itself.[2]

We remark here that the Lagrangian of the free gauge fields is given by

$$\mathcal{L}_{gauge} \sim \text{Tr } F_{\mu\nu}F^{\mu\nu}, \tag{1.14}$$

where the trace is taken in the Lie-algebra part. It is not difficult to check that this Lagrangian is invariant under gauge transformations. The full Lagrangian is now of the form:

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{fermions} = -\alpha \text{Tr } F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu + A_\mu)\psi \tag{1.15}$$

with $\alpha \in \mathbb{R}$ and A_μ a 1-form with values in the Lie-algebra $\mathfrak{g}$. It is not possible to introduce a mass term for the gauge fields by hand. Such a mass term is of the form $m^2 A_\mu A^\mu$ but as an easy calculcation shows it is not gauge-invariant. As a consequence all gauge bosons in gauge theories are massless. This does not match with the fact that the weak vector bosons $W^\pm$ and Z^0 are massive particles. This problem is fixed by the Higgs mechanism which we will discuss below.

1.4 THE STANDARD MODEL

We have now discussed all the tools necessary to give a formulation of the (classical) Standard Model. In the above text I already stretched that the spin of the particle is determined by the representations of the Lorentz-algebra (or the Poincare algebra). Other properties of fermionic particles are captured in the representations of the appropiate fields with respect to the gauge groups. We therefore first discuss the $SU(3) \times SU(2)_L \times U(1)_Y$-symmetry in the unbroken model and the representations according to which the different particle fields transform. As mentioned before all gauge bosons remain mass-less. To get massive gauge boson we break the symmetry by the Higgs-mechanism. Finally, I will introduce mass terms for the different fermionic field and how this is related to quark-mixing. Most of this section is based on Chapters 11 and 12 in [2].

1.4.1 *The unbroken model*

In the unbroken Standard Model the local gauge symmetries are $SU(3) \times SU(2)_L \times U(1)_Y$. Below the symmetry breaking scale the symmetry group reduces to $SU(3) \times U(1)_{EM}$, where $U(1)_{EM}$ is a

2 For more details on the description of gauge theories in terms of principal bundles you should take a look in [1]. In Chapter 1 you can find three equivalent definition of a connection 1-form on a principal bundle P. From the third definition you can easily see that the transformation of A_μ in (1.12) determines a connection 1-form on P. In Chapter 2 you can find more information about the curvature of a connection 1-form on a principal bundle.

subgroup of the group $SU(2)_L \times U(1)_Y$. It is interesting to note that above the symmetry scale we also loose symmetries, namely C, T, P since the weak interactions are not invariant under these discrete symmetries.

The $SU(3)$-symmetry is an exact symmetry. It is established by assigning an extra quantum number to quarks, called colour. The idea of colour comes from the observation of the Δ^{++} particle which consists of three u-quarks, each of them with spin up. However, the Pauli exclusion principle forbids this state to occur (two fermions can never be in the same state). In order not to violate this exclusion principle a new quantum number was introduced: colour. Every quark was assigned a colour: red, green or blue (or whatever, as long as there are three colours: quarks sit in the standard representation of $SU(3)$ (this repsentation is denoted by 3, anti-quarks sit in the dual representation of the standard representation ($\bar{3}$)). We remark here that isolated quarks have never been observed. Theoretically, this problem is solved by acquiring colour confinement. Quarks can only combine into colourless combinations, baryons and mesons. Baryons consists of three quarks and mesons consist of a quark and an anti-quark. Being colourless they transform trivially under an $SU(3)$ transformation. This implies that the colour part is totally anti-symmetric. In turn, this implies that the spatial and spin part are symmetric. The gluon is the force mediator of the stronge force and couples to itself via three or four-point interactions (this is because $SU(3)$ is non-abelian).

The $SU(2)$-symmetry is called the weak isospin symmetry. It says that the Lagrangian is invariant under a $SU(2)$-transformation:

$$\begin{pmatrix} (\nu_{eL} \\ e_L \end{pmatrix}. \tag{1.16}$$

It is experimentally determined that the weak isospin gauge bosons only couple to the left-handed spinor part of a Dirac-spinor. Therefore, the $SU(2)$-transformations act trivially on the right-handed fields and the right-handed fields form isospin singlets such as e_R^-; in the Standard Model there are no right-handed neutrinos. Another isospin doublet is

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \tag{1.17}$$

and the same doublets exist in the other two generations. Following [2] I will write $\begin{pmatrix} \nu_i \\ e_i \end{pmatrix}$ and $\begin{pmatrix} u_i \\ d_i \end{pmatrix}$ for the $SU(2)$-doublets where i denotes the generation ($i = 1, 2, 3$).

Recalling the above discussion the gluon only couples to quarks and to itself since these are the particles that carry colour. The weak force only couples to the left-handed doublets. We can summarise this information by giving the representations according to which the particle fields transform, as in the following way:[3]

$$(3, 2, \frac{1}{6})_L, \quad (3, 1, \frac{2}{3})_R, \quad (3, 1, \frac{1}{3})_R, \quad (1, 2, \frac{1}{2})_L, \quad (1, 1, 1)_R, \tag{1.18}$$

which correspond to ψ_L^q (quark), ψ_R^u (right-handed u), ψ_R^d (right-handed d), ψ_L^l (lepton), ψ_R^e, (right-handed electron) respectively. The last number denotes the eigenvalue of the $u(1)$-generator Y. The second number corresponds to the dimension of the irreducible representation of $SU(2)$ according to which the field transforms under a gauge transformation. Since left-handed fields are $SU(2)$-doublets that transform according to the $SU(2)$-standard representation, we have written a 2. Since right-handed fields are isospin singlets we have written a 1, this is the trivial representation. In the

3 this list is taken from [4], Chapter 2

trivial representation of the Lie-algebra $su(2)$ every element is mapped to zero. This precisely means that the $su(2)$-gauge potential does not couple to fields in this representation.

The first number is the dimension of the irreducible $SU(3)$-representation. Here 1 denotes the only 1-dimensional irreducible representation of $SU(3)$, namely the trivial representation, and 3 stands for the standard representation of $SU(3)$. The group $SU(3)$ has another irreducible representation of dimension 3, the dual representation of the standard representation. This representation is denoted by $\overline{3}$ and anti-quarks live in here. Once again, particle fields transforming according to 3 carry colour and couple to gluons whereas particle fields transforming according to 1 do not couple to the gauge fields. Therefore we have denoted a 3 for the quark fields and a 1 for the others.

Note that the right-handed neutrino is not included in the list. If we did, it would have been $(1, 1, 0)$ since it does not couple to anything. Actually, equation (1.18), together with the Lagrangian (1.15), contain all the information of the unbroken Standard Model.

1.4.2 *Symmetry breaking and the Salam-Weinberg model*

Since symmetry breaking does not affect the $SU(3)$-gauge symmetry, we will forget about this symmetry in this section. We will break the $SU(2)_L \times U(1)_Y$ symmetry to the $U(1)_{EM}$-symmetry. This is also known as the Higgs mechanism. In this procedure a Lagrangian density for a real scalar field arises: this will be the Higgs field.

To break the symmetry we add a $SU(2)$-doublet consisting of two complex scalar fields $\Phi = (\Phi_A, \Phi_B)$ to the Lagrangian which transforms as $(1, 2, \frac{1}{2})$. If we require its Lagrangian to be invariant and the theory to be renormalisable, the most general form of $\mathcal{L}_\phi$ is:

$$\mathcal{L}_\phi = (D_\mu\Phi)^\dagger D_\mu\Phi - \mu^2\Phi^\dagger\Phi - \frac{1}{4}\lambda(\Phi^\dagger\Phi)^2, \tag{1.19}$$

where $D_\mu\Phi = \partial_\mu + \frac{1}{2}ig_1B_\mu Y + \frac{1}{2}ig_2T^a_\mu\tau^a$. Here $T^a = \frac{1}{2}\sigma^a$, $(a = 1, 2, 3)$ are the generators of the Lie-algebra $isu(2)$, and Y denotes the $iu(1)$-generator. Now, assume that μ^2 is negative, then the potential no longer has a minimum at the field $\Phi = 0$, but at some non-trivial value which, using $SU(2)$ and $U(1)$ symmetry can be brought to the form $(0, \phi_0)$ for some real value phi_0. All excited states are of the form $\phi = (0, \phi_0 + \frac{1}{\sqrt{2}}h(x))$ where h is a real scalar field. This field is interpreted as the Higgs field. We have now fixed the gauge and therefore we lost the gauge symmetry $SU(2) \times U(1)$. What remains is the $U(1)$-symmetry $e^{-i\theta/2}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix}$ with generator $T^3 + Y$, which will turn out to be the $U(1)$-symmetry belonging to electromagnetism.

Remark 1.4.1. Since the $SU(2) \times U(1)$ symmetry breaks down to a $U(1)_{EM} \subset SU(2)_L \times U(1)_Y$-symmetry, the doublet representations ψ^q, ψ^l are no longer irreducible representations and break into two 1-dimensional irreducible representations of $U(1)$. We denote the generator of $iu(1)$ by $T_3 + Y$. This yields the following table:

$$\begin{aligned}
(3, 2, \frac{1}{6})_L &\rightarrow (3, \frac{2}{3})_L + (3, -\frac{1}{3})_L, \\
(3, 1, \frac{2}{3})_R &\rightarrow (3, \frac{2}{3})_R, \\
(3, 1, -\frac{1}{3})_R &\rightarrow (3, -\frac{1}{3})_R, \\
(1, 2, -\frac{1}{2})_L &\rightarrow (1, -1)_L + (1, 0)_L, \\
(1, 1, -1)_R &\rightarrow (1, -1)_R,
\end{aligned} \tag{1.20}$$

and of course we get three copies of each fields (three generations).

The total Lagrangian is now of the form

$$\mathcal{L} = \mathcal{L}_\Phi + \mathcal{L}_{gauge} + \mathcal{L}_{fermions}, \tag{1.21}$$

where $\mathcal{L}_{free}$ is the Lagrangian density of the free gauge fields and $\mathcal{L}_{fermions}$ is the Lagrangian density of the fermion fields $i\overline{\psi}\gamma^\mu D_\mu \psi$ where D_μ is the covariant derivative with respect to all gauge fields. Later on we will add another Lagrangian density which, among other things, contains the mass terms of the fermion fields.

If we substitute $\phi = (0, \phi_0)$ into the Lagrangian (1.19) (for a moment we forget the Higgs field h), and write $W^\pm = \frac{1}{\sqrt{2}}(W^1 \mp i W^2)$, then we get the Lagrangian (see also [2], Chapter 11 and [4], Chapter 2):

$$\mathcal{L}_\phi = \frac{1}{2} g_2^2 \phi_0^2 W_\mu^- W^{+\mu} + (\frac{g_2^2}{4} W_\mu^3 W^{3\mu} - \frac{g_1 g_2}{2} W_\mu^3 B^\mu + \frac{g_1^2}{4} B_\mu B^\mu)\phi_0^2. \tag{1.22}$$

Notice that the mass matrix of the vector $(A_\mu^1, A_\mu^2, A_\mu^3, B_\mu)$ mixes A_μ^3 and B_μ. To get a diagonal mass matrix (eigen vectors of the mass matrix are interpreted as the 'true' particles) we pass to the following basis:

$$W_\mu^+, \quad W_\mu^-, \quad A_\mu = W_\mu^3 \sin\theta_w + B_\mu \cos\theta_w, \quad Z_\mu = W_\mu^3 \cos\theta_w - B_\mu \sin\theta_w, \tag{1.23}$$

where

$$\cos\theta_w = \frac{g_2}{(g_1^2 + g_2^2)^{\frac{1}{2}}}, \quad \sin\theta_w = \frac{g_1}{(g_1^2 + g_2^2)^{\frac{1}{2}}}. \tag{1.24}$$

θ_w is called the Weinberg angle. In the new basis the Lagrangian becomes

$$\mathcal{L}_\phi = \frac{1}{2} g_2^2 \phi_0^2 W_\mu^- W^{+\mu} + \frac{1}{4}\phi_0^2 (g_1^2 + g_2^2) Z_\mu Z^\mu. \tag{1.25}$$

and we see that A_μ is massless and Z_μ has a mass of $\frac{1}{4}\sqrt{g_1^2 + g_2^2}\phi_0$. The Lagrangian density of the particle fields has the following form in terms of $W_\mu^\pm$, Z^0, A_μ:

$$\mathcal{L}_{fermions} = i\overline{\psi}\gamma^\mu(\partial_\mu - i g_2 \frac{1}{\sqrt{2}}(W_\mu^+ T^- + W_\mu^- T+) - i\frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} A_\mu (T_3 + Y) - i\frac{g_2^2 T^3 - g_1^2 Y}{\sqrt{g_1^2 + g_2^2}} Z_\mu)\psi, \tag{1.26}$$

where $T^\pm = T^1 \pm i T^2$. We see therefore that A_μ is the gauge field belonging to the $U(1)$-symmetry with generator $T_3 + Y$. When we set

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} = g_2 \sin\theta_w = g_1 \cos\theta_w, \quad Q = T_3 + Y, \tag{1.27}$$

we see that the $U(1)$-symmetry is precisely $U(1)_{EM}$ and A_μ is the photon field.

The remaining terms of the Lagrangian, namely L_{free} and the extra terms we get if we replace ϕ_0 by $\phi_0 + \frac{1}{\sqrt{2}}h$, where h is the Higgs field, will give all the remaining couplings between all the boson particles etc. Since this expression is very large we will not carry out this calculation. One result is that the W^+ particle couples to the photon field A_μ with coupling strength $g_2 \sin\theta_w = e$. Another relation we can read off is that

$$M_W / M_Z = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} = \cos\theta_w. \tag{1.28}$$

It remains to discuss the mass terms for the fermionic fields. We distinguish here between the lepton sector and the quark sector. First, let us discuss the fermions. To get massive charged leptons and massless neutrinos after symmetry breaking we start with the following general gauge- and Lorentz-invariant Lagrangian

$$\mathcal{L}_{mass} = -\sum \left[G_{ij}(\mathbf{L}_i^\dagger \Phi)e_{Rj} + G_{ij}^* e_{Rj}^\dagger(\Phi^\dagger \mathbf{L}_i) \right], \tag{1.29}$$

where the sums are over the three lepton families and G_{ij} is a 3×3 complex matric. It is a familiar result that any 3×3-complex matrix can be put into a real diagonal form C with the help of two unitary matrices, U_L and U_R (see [2], Appendix A, Exercise A.4.):

$$G = U_L^\dagger C U_R. \tag{1.30}$$

The U_L and U_R are in general unique except that both may be multiplied with the same 'phase factor'

$$\begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} \end{pmatrix}. \tag{1.31}$$

If we define $e'_{Ri} = U_{Rij}e_{Rj}$, $L'_i = U_{Lij}L_j$ we get a diagonal mass matrix:

$$\mathcal{L}_{mass} = -\phi_0 \sum_i c_i[\mathbf{L'}_i^\dagger \Phi)e'_{Ri} + e_{Ri}^{'\dagger}(\Phi^\dagger L_i)]. \tag{1.32}$$

After symmetry breaking this becomes (we drop the primes):

$$\mathcal{L}_{mass} = -\phi_0 \sum_i c_i[e_{Li}^\dagger e_{Ri} + e_{Ri}^\dagger e_{Li}]. \tag{1.33}$$

The dynamical part of the fermion Lagrangian is unaffected by this transformation since all neutrinos are eigenstates belonging to the same eigenvalue (namely 0). This is different in the quark sector where all six quarks have different masses.

In the unbroken Standard Model the most general form we might consider to give the d_j-quarks masses is:

$$\mathcal{L}(d)_{mass} = -\sum \left[G_{ij}(\mathbf{L}_i^\dagger \Phi)d_{Rj} + G_{ij}^* d_{Rj}^\dagger(\Phi^\dagger \mathbf{L}_i) \right], \tag{1.34}$$

Breaking the symmetry gives:

$$\mathcal{L}(d)_{mass} = -\phi_0 \sum \left[G_{ij}^d d_{Li}^\dagger d_{Rj} + G_{ij}^{d*} d_{Rj}^\dagger d_{Li} \right], \tag{1.35}$$

In the same way as before we can find unitary matrices D_L, D_R such that

$$\phi_0 G^d = D_L^\dagger m^d D_R, \tag{1.36}$$

where m^d is a real diagonal matrix. Once again, if the diagonal elements are distinct then D_L and D_R are again unique. Since the u-type quarks are not massless, we need a mass term for them as well. A suitable invariant Lagrangian is then

$$\mathcal{L}(u)_{mass} = -\sum \left[G_{ij}^u(\mathbf{L}_i^\dagger \varepsilon \Phi^*)u_{Rj} + G_{ij}^{u*} u_{Rj}^\dagger(\Phi^T \varepsilon \mathbf{L}_i) \right], \tag{1.37}$$

where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. That $\Phi^T \varepsilon L^i$ is gauge-invariant can be checked by showing $U^T \varepsilon U = \det U \cdot \varepsilon$. Again we can turn G^u in a real diagonal form m^u:

$$\phi_0 G^u = U_L^\dagger m^u U_R \tag{1.38}$$

where $U_{L,R}$ are unitary matrices. U_L and U_R may both be multiplied on the left by a phase factor matrix, say

$$\begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix} \tag{1.39}$$

Replacing:

$$\begin{aligned} d'_{Li} &= D_{Lij} d_{Lj}, \quad d'_{Ri} = D_{Rij} d_{rj} \\ u'_{Li} &= U_{Lij} u_{Lj}, \quad u'_{Ri} = U_{Rij} u_{rj}, \end{aligned} \tag{1.40}$$

the quark mass contribution becomes:

$$\mathcal{L}_{mass} = -\sum_{i=1}^{3} \left[m_i^d (d'^\dagger_{Li} d'_{Ri} + d'^\dagger_{Ri} d'_{Li}) + m_i^u (u'^\dagger_{Li} u'_{Ri} + u'^\dagger_{Ri} u'_{Li}) \right]. \tag{1.41}$$

Since all quarks are massive, the kinetic part of the quark Lagrangian does change when we make the substitution (1.40). To see this, let's us take a look at the neutral current (the same argument holds for the interaction with the photon): it is of the form

$$u^\dagger_{Li} \tilde{\sigma}^\mu u_{Li}, \tag{1.42}$$

and the same for the right-handed particles and the d-particles. Transforming gives:

$$u'^\dagger_{Li} U_{Lij} \tilde{\sigma}^\mu U^\dagger_{ji} u'_{Li} = u'^\dagger_{Li} \tilde{\sigma}^\mu u'_{Li}. \tag{1.43}$$

Thus, in the coupling of Z^0 there is no mixing of generations (GIM-mechanism). However, in the charged current

$$u^\dagger_L \tilde{\sigma}^\mu d_{Li} W^+_\mu + h.c. \tag{1.44}$$

the u- and d-quarks get mixed and the interaction in terms of the "true" quark fields becomes (we denote u, c, t, d, s, b for these fields):

$$(u^\dagger_L, c^\dagger_L, t^\dagger_L) V \begin{pmatrix} \tilde{\sigma}^\mu d_L \\ \tilde{\sigma}^\mu s_L \\ \tilde{\sigma}^\mu b_L \end{pmatrix} W^+_\mu + h.c. \tag{1.45}$$

where $V = U_L D^\dagger_L$. The matrix V is known as the Cabbibo-Kobayashi-Maskawa matrix. It tells us that the $W^\pm$-interactions can mix the generations of quarks. What are the degrees of freedom of this matrix? We do this for an arbitrary number of generations N. Well, a unitary matrix $N \times N$-matrix is determined by N^2 real parameters ($U = e^{iA}$ for some hermitian matrix A). The $2N$ phases can be used to fix $2N$ of these parameters. However, not all $2N$ parameters can be used. If we multiply the CKM-matrix from the left with $e^{i\phi}\mathbf{1}_N$ and from the right with its inverse, it is invariant. So, when there are N generations we have $N^2 - (2N - 1)$ parameters for the CKM-matrix. If $N = 3$ we get $9 - 5 = 4$. These four parameters need to be determined by experiment.

1.5 MISCELLANEOUS TOPICS

1.5.1 *Massive neutrinos*

Remember that the mass term of a Dirac-spinor in the Lagrangian is of the form $m\bar{\psi}\psi$ where $\psi \in D(\frac{1}{2}, 0) \oplus D(0, \frac{1}{2})$. For the mass term to be Lorentz-invariant it must be of the form $m\psi_L^\dagger \psi_R + m\psi_R + m\psi_R^\dagger \psi_L$. Expressions like $\psi_L^\dagger \psi_L$ are not Lorentz-invariant.

Nowadays, one has determined that neutrinos from different generations have different masses. This conclusion is drawn because of the observation of neutrino oscillations which can only occur if neutrinos have different masses. The previous paragraph then implies that right-handed neutrinos must exist in order to get a mass term for the neutrinos. Since neutrinos have different masses the charged weak currents, just as in the case of quarks, mix different generations of leptons. In the Standard Model the number of leptons of a given generation is preserved under all interactions since the neutrinos are massless. If neutrinos do have mass, which is most likely to be the case, interactions between different generations of leptons is possible and only the total number of leptons is a conserved quantity. However, it is not clear whether neutrinos are Dirac or Majorana fermions. In the latter case, also the total number of leptons is no longer conserved.[4]

1.5.2 *Majorana spinors*

Majorana spinors are their own antiparticles. More concretely, if we use the charge conjugation operator to transform the particle into its anti-particle, then a Majorana spinor satisfies:

$$\psi = \psi^c := i\gamma^2\psi^* \tag{1.46}$$

In the representation of a Majorana spinor as a Dirac-spinor the left- and right-handed components are not independent. One can check that

$$\nu_R = i\sigma^2\nu_L^* \tag{1.47}$$

1.5.3 *Chirality and helicity*

A Dirac spinor consists of a left-handed Weyl spinor and a right-handed Weyl-spinor. Of course, the left-handedness and right-handedness is a Lorentz-invariant property (when time- and space orientation is preserved). The $W^\pm$ only interact with the left-handed particles. There is also a concept known as helicity. A spin-$\frac{1}{2}$-particle is said to have positive helicity if the direction of the spin is the same as the direction of the momentum. It has negative helicity if the directions are opposite. For massless particles helicity is a Lorentz-invariant and helicity is the same as chirality. However, for massive particles helicity is not a Lorentz-invariant property. Its helicity can be reversed by performing a large enough boost in the opposite direction of the momentum. Therefore, for massive particles helicity and chirality are separate properties.

4 Actually, the conservation of lepton number etc. can also be derived from the Lagrangian using Noether's theorem. For more information on this topic see also [2].

[1] D. Bleecker. *Gauge theories and variational principles*. Addison-Wesley Publishing Company, 1981.

[2] W.N. Cottingham and D.A. Greenwood. *An Introduction to the Standard Model of Particle Physics*. Cambridge University Press, 2007.

[3] J.J. Duistermaat and J.A.C. Kolk. *Lie Groups*. Springer-Verlag Berlin Heidelberg, 2000.

[4] A.N. Schellekens. Beyond the standard model. Lecture Notes, 2009.

[5] R. Ticciati. *Quantum Field Theory for Mathematicians*. Cambridge University Press, 1999.