Noncommutative geometry and particle physics

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Noncommutative manifolds

- Basic device: a spectral triple $(A, \mathcal{H}, D)$:
  - algebra $A$ of bounded operators on
  - a Hilbert space $\mathcal{H}$,
  - a self-adjoint operator $D$ with compact resolvent
    such that the commutator $[D, a]$ is bounded for all $a \in A$. 
Noncommutative manifolds

- Basic device: a spectral triple \((\mathcal{A}, \mathcal{H}, D)\):
  - algebra \(\mathcal{A}\) of bounded operators on
  - a Hilbert space \(\mathcal{H}\),
  - a self-adjoint operator \(D\) with compact resolvent such that the commutator \([D, a]\) is bounded for all \(a \in \mathcal{A}\).

- Grading \(\gamma : \mathcal{H} \to \mathcal{H}\) such that

\[
\gamma^2 = \text{id}, \quad D\gamma + \gamma D = 0, \quad \gamma a = a\gamma \quad (a \in \mathcal{A})
\]
Noncommutative manifolds

- Basic device: a spectral triple \((A, \mathcal{H}, D)\):
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  \]

- Real structure \(J : \mathcal{H} \to \mathcal{H}\), anti-unitary operator such that
  \[
  JD = \pm JD, \quad J\gamma = \pm \gamma J.
  \]
  defining an \(A\)-bimodule structure on \(\mathcal{H}\) via
  \[
  (a, b) \cdot \psi = aJb^*J^{-1}\psi \quad (\psi \in \mathcal{H})
  \]
  and we require (first order):
  \[
  [[D, a], JbJ^{-1}] = 0
  \]
Example: Riemannian spin geometry

Let $M$ be a compact $m$-dimensional Riemannian spin manifold.

- $\mathcal{A} = C^\infty(M)$
- $\mathcal{H} = L^2(S)$, square integrable spinors
- $D = \slashed{\partial}$, Dirac operator
- $\gamma = \gamma_{m+1}$ if $m$ even (chirality)
- $J = C$ (charge conjugation)

Then $D$ has compact resolvent because $\slashed{\partial}$ elliptic self-adjoint. Also $[D, f]$ bounded for $f \in C^\infty(M)$. 
Morita equivalence

Suppose $\mathcal{A} \sim_M \mathcal{B}$.

Can we construct a spectral triple on $\mathcal{B}$ from $(\mathcal{A}, \mathcal{H}, D)$?

- Let $\mathcal{B} \simeq \text{End}_\mathcal{A}(\mathcal{E})$ with $\mathcal{E}$ finitely generated projective. Define

  $$\mathcal{H}' = \mathcal{E} \otimes_\mathcal{A} \mathcal{H}$$

  Then $\mathcal{B}$ acts as bounded operators on $\mathcal{H}'$. 

Definition of operator $D'$ $(\eta, \psi)$ := $\nabla(\eta) \psi + \eta \otimes D \psi$ requires a (compatible) connection on $\mathcal{E}$:

$\nabla : \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega^1_D$ with respect to the derivation $d := [D, \cdot]$ and the Connes' differential one-forms are

$$\Omega^1_D(A) = \begin{cases} \sum_j a_j [D, b_j] : a_j, b_j \in \mathcal{A} \end{cases}$$

Then $(\mathcal{B}, \mathcal{H}', D')$ is a spectral triple [Connes, 1996].
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- Then $(\mathcal{B}, \mathcal{H}', D')$ is a spectral triple [Connes, 1996].
Morita equivalence
with real structure

Again, suppose $\mathcal{A} \sim_\mathcal{M} \mathcal{B}$.

- If there is a real structure $J$ on $(\mathcal{A}, \mathcal{H}, D)$, then instead

$$\mathcal{H}' := \mathcal{E} \otimes_\mathcal{A} \mathcal{H} \otimes_\mathcal{A} \overline{\mathcal{E}}$$

in terms of the conjugate (left $\mathcal{A}$-) module $\overline{\mathcal{E}}$ and define

$$D'(\eta \otimes \psi \otimes \overline{\rho}) = \nabla(\eta)\psi \otimes \overline{\rho} + \eta \otimes D\psi \otimes \overline{\rho} + \eta \otimes \psi \overline{\nabla}(\overline{\rho})$$

where

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_\mathcal{A} \Omega^1_D(\mathcal{A})$$

$$\overline{\nabla} : \overline{\mathcal{E}} \rightarrow \Omega^1_D(\mathcal{A}) \otimes_\mathcal{A} \overline{\mathcal{E}},$$

and

$$J' : \mathcal{H}' \rightarrow \mathcal{H}', \quad \eta \otimes \psi \otimes \overline{\rho} \mapsto \rho \otimes J\psi \otimes \overline{\eta}$$

complete the definition of a real spectral triple $(\mathcal{B}, \mathcal{H}', D', J')$. 
In the case $B = A$ (i.e. $\mathcal{E} = A$) we have of course $\mathcal{H}' \simeq \mathcal{H}$ and $J' \equiv J$. 

$D' \mapsto UD'U^*$ implies $A \mapsto uAu^* + u[D, u^*]$. The element $A$ is the gauge field and it acts as $A \pm JAJ^{-1}$ on $H$, that is, in the adjoint representation.
In the case $B = A$ (i.e. $E = A$) we have of course $\mathcal{H}' \simeq \mathcal{H}$ and $J' \equiv J$.

However, the operator $D$ is perturbed to $D' = D_A \equiv D + A \pm JAJ^{-1}$ with $A^* = A \in \Omega^1_D(A)$ the connection one-form (gauge potential) in $\nabla = d + A$. These are the so-called inner fluctuations.
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The (gauge) group $U(A)$ of unitary elements in $A$ acts on $\mathcal{H}$ in the adjoint, i.e. via the unitary $U = uJu^{-1}$ for $u \in U(A)$. 

Morita self-equivalence
Morita self-equivalence

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- The (gauge) group $U(\mathcal{A})$ of unitary elements in $\mathcal{A}$ acts on $\mathcal{H}$ in the adjoint, i.e. via the unitary $U = uJuJ^{-1}$ for $u \in U(\mathcal{A})$.
- This induces an action of $U(\mathcal{A})$ on the connection one-form $A$, since $D' \mapsto UD'U^*$ implies

$$A \mapsto uAu^* + u[D, u^*]$$
Morita self-equivalence

- In the case $B = A$ (i.e. $E = A$) we have of course $H' \cong H$ and $J' \equiv J$.

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  with $A^* = A \in \Omega^1_D(A)$ the connection one-form (gauge potential) in
  $\nabla = d + A$. These are the so-called inner fluctuations.

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  adjoint, i.e. via the unitary $U = uJuJ^{-1}$ for $u \in U(A)$.

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- The element $A$ is the gauge field and it acts as $A \pm JAJ^{-1}$ on $H$, that
  is, in the adjoint representation.
Spectral action principle

Given a (real) spectral triple \((A, \mathcal{H}, D)\), we define an action functional on \(A \in \Omega^1_D(A)\) and \(\psi \in \mathcal{H}\) as

\[
S_\Lambda[A, \psi] := \text{Tr} (f(D_A/\Lambda)) - \text{Tr} (f(D/\Lambda)) + \langle \psi, D_A \psi \rangle
\]

with \(f\) a function on \(\mathbb{R}\) \((...)\) and \(\Lambda \in \mathbb{R}\) a cut-off.

- Gauge invariance: \(S_\Lambda[u^* A u + u^* [D, u], u \psi] = S_\Lambda[A, \psi]\).
Spectral action principle

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S_{\Lambda}[A, \psi] := \text{Tr} \left( f\left( \frac{D A}{\Lambda} \right) \right) - \text{Tr} \left( f\left( \frac{D}{\Lambda} \right) \right) + \langle \psi, D_A \psi \rangle
\]

with \(f\) a function on \(\mathbb{R}\) \((\ldots)\) and \(\Lambda \in \mathbb{R}\) a cut-off.

- **Gauge invariance**: \(S_{\Lambda}[u^* Au + u^* [D, u], u\psi] = S_{\Lambda}[A, \psi]\).
- The part \(\text{Tr} \left( f\left( \frac{D}{\Lambda} \right) \right)\) is purely ‘gravitational’ (this terminology is justified by applying it to the commutative spectral triple associated to \(M\)).
Heat kernel expansion

One obtains an explicit expression for

$$\text{Tr } (f(D_A/\Lambda))$$

in terms of the heat expansion for the operator $e^{-t(D_A/\Lambda)^2}$.

- Assume simple dimension spectrum for $D$ and a heat expansion

$$\text{Tr } e^{-tD_A^2} \sim \sum_{\alpha} t^\alpha a_\alpha(D_A) \quad (t \to 0)$$
Heat kernel expansion

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- Assume simple dimension spectrum for \( D \) and a heat expansion

\[ \text{Tr} \ e^{-tD_A^2} \sim \sum_{\alpha} t^\alpha a_\alpha(D_A) \quad (t \to 0) \]

- Then write \( f \) as a Laplace transform of \( \phi \)

\[ \text{Tr} \left( f\left( D_A/\Lambda \right) \right) = \int_{t>0} \phi(t) e^{-t(D_A/\Lambda)^2} \, dt = \sum_{\alpha} f_{-\alpha} \Lambda^{-\alpha} a_\alpha(D_A) \]
Example: Yang–Mills theory

Given a compact 4-dimensional Riemannian spin manifold $M$, consider:

- $\mathcal{A} = \mathcal{C}^\infty(M) \otimes M_n(\mathbb{C})$
- $\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C})$
- $D = \partial \otimes 1$
- $\gamma = \gamma_5 \otimes 1$, $J = C \otimes (.)^*$

<table>
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Proposition (Chamseddine-Connes)

- The self-adjoint operator $A + JAJ^{-1}$ with $A = A^* \in \Omega^1_D(A)$ describes an $\mathfrak{su}(n)$-valued one-form on $M$.
- The gauge group $\mathcal{U}(A) \simeq C^\infty(M, U(n))$ acts on $\mathcal{H}$ in the (usual) adjoint representation.
- The spectral action is given by

\[
S_\Lambda[A, \psi] = \frac{f(0)}{24\pi^2} \int_M \text{Tr} F_A \wedge F_A + \langle \psi, (\partial + \text{ad}A)\psi \rangle + O(\Lambda^{-1})
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with $F_A$ the curvature of the connection one-form corresponding to $A$. 
We make two observations.

1. The $\mathfrak{su}(n)$-valued one-form defines a connection one-form on the trivial principal bundle $M \times SU(n)$.

Is there a spectral triple that describes Yang–Mills theory on topologically non-trivial principal bundles?
We make two observations.

1. The $\mathfrak{su}(n)$-valued one-form defines a connection one-form on the trivial principal bundle $M \times SU(n)$.

Is there a spectral triple that describes Yang–Mills theory on topologically non-trivial principal bundles?

2. With the fermions in the adjoint representation of $U(A)$, the above action is a candidate for defining a supersymmetric theory.

How does supersymmetry appear, and can we extend it to physically realistic models? (eg. MSSM)
Geometry of Yang–Mills fields

Let $P \to M$ be a $G$-principal bundle. A convenient way to define connections on $P$ is through covariant derivatives on the associated bundle(s).

- A covariant derivative (or, connection) on $E = P \times_G V$ is a map

$$\nabla : \Gamma^\infty(E) \to \Gamma^\infty(E) \otimes_{C^\infty(M)} \Omega^1(M))$$

satisfying the Leibniz rule $\nabla(sf) = \nabla(s)f + s \otimes df$. This implies that $\nabla = \nabla_0 + A$ with $A \in \Gamma^\infty(adP) \otimes_{C^\infty(M)} \Omega^1(M)$ for any two $\nabla, \nabla_0$. 
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- The **curvature** of $\nabla$ is $F_\nabla := \nabla^2 \in \Gamma^\infty(\text{ad}P) \otimes_{C^\infty(M)} \Omega^2(M)$. 


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- The curvature of $\nabla$ is $F_\nabla := \nabla^2 \in \Gamma^\infty(\text{ad}P) \otimes_{C^\infty(M)} \Omega^2(M)$.

- The gauge group $\text{Aut}_1(P) \simeq \Gamma^\infty(\text{Ad}P)$ acts on $\nabla$

$$\nabla \mapsto g \nabla g^{-1}$$

and, consequently, $F_\nabla \mapsto g F_\nabla g^{-1}$.
Given the above, we may define the Yang–Mills action functional (for simplicity, assume $G = U(n)$ or $SU(n)$)

$$S_{YM}[A] = \int_M \text{Tr} \ F_\nabla \wedge * F_\nabla$$

writing $\nabla = \nabla_0 + A$ for some fixed connection $\nabla_0$
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• This describes the dynamics and self-interactions of a single gauge boson (eg. photon, W-boson, gluon, ...)

Example: QCD has $G = SU(3)$. Gluons are $su(3)$-valued one-forms on $M$; quarks are sections of $E = P \times SU(3) C_3$. Their dynamics and interaction are described by $S_{YM} + S_M$. 

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- Physical matter (fermions) can be included (on a spin manifold) as sections of the tensor product of the spinor bundle $S$ the associated bundles $E$ to $P$:

$$S_M[A, \psi] = \langle \psi, \gamma \circ \nabla \psi \rangle$$

(eg. electrons, quarks, ...)
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Yang–Mills theory (non-trivial)

Algebra

Let $\mathcal{A}$ be a finitely generated, projective $C^\infty(M)$-module $*$-algebra. Thus, the module structure is compatible with the $*$-algebra structure:

$$f(ab) = (fa)b = a(fb), \quad (fa)^* = \bar{f}a^*, \quad \text{et cetera.}$$

Recall that an algebra bundle $B \rightarrow M$ is a vector bundle with an algebra structure on the fibers; also, the local trivializations are algebra maps.
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Algebra

Let $\mathcal{A}$ be a finitely generated, projective $C^\infty(M)$-module $\ast$-algebra. Thus, the module structure is compatible with the $\ast$-algebra structure:

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Recall that an algebra bundle $B \to M$ is a vector bundle with an algebra structure on the fibers; also, the local trivializations are algebra maps.

Proposition (Serre-Swan for algebra bundles)

There is a one-to-one correspondence between finite rank (involutive) algebra bundles on $M$ and finitely generated projective $C^\infty(M)$-module $(\ast)$-algebras.

The correspondence being $\mathcal{A} \simeq \Gamma^\infty(M, B)$ for an algebra bundle $B \to M$. 
Yang–Mills theory (non-trivial)
Hilbert space and Dirac operator

We define a Hilbert space \( \mathcal{H} := \mathcal{A} \otimes C^\infty(M) L^2(M, S) \). Let \( \nabla_0 \) be a (compatible) connection on the finitely generated projective module \( \mathcal{A} \):

\[
\nabla_0 : \mathcal{A} \rightarrow \mathcal{A} \otimes C^\infty(M) \Omega^1_\partial(C^\infty(M))
\]

A self-adjoint operator \( D \) on \( \mathcal{H} \) is defined as \( D = \nabla_0 \otimes 1 + 1 \otimes \partial \).
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Hilbert space and Dirac operator

We define a Hilbert space $\mathcal{H} := \mathcal{A} \otimes_{C^\infty(M)} L^2(M, S)$. Let $\nabla_0$ be a (compatible) connection on the finitely generated projective module $\mathcal{A}$:

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A self-adjoint operator $D$ on $\mathcal{H}$ is defined as $D = \nabla_0 \otimes 1 + 1 \otimes \partial$.

**Theorem**

- *The set of data $(\mathcal{A}_{C^\infty(M)}, \mathcal{H}, D)$ defines a spectral triple.*

Also, there exists a grading $\gamma = 1 \otimes \gamma_5$ (assuming $M$ even dimensional) and a real structure $J = (\cdot)^* \otimes C$.

Next, we study the inner fluctuations of this spectral triple.
Yang–Mills theory (non-trivial)

Principal bundles

From the transition functions \( t_{\alpha\beta} \) of the algebra bundle \( B \) (for which \( \mathcal{A} \simeq \Gamma^\infty(M, B) \)) we build a \( SU(n) \)-principal bundle:

- Assume for simplicity that the fiber of \( B \) is isomorphic to \( M_n(\mathbb{C}) \).
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- The resulting $SU(n)$-principal bundle $P$ has as an associated bundle:

$$B = P \times_{SU(n)} M_n(\mathbb{C})$$
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- The connection $\nabla_0$ defines a covariant derivative $\nabla_0$ on the associated bundle $B$. 
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- The connection $\nabla_0$ defines a covariant derivative $\nabla_0$ on the associated bundle $B$.
- The inner fluctuations $D \mapsto D' = D + A + JAJ^{-1}$ give rise to connections $\nabla$ on $B$, such that $D' = \gamma \circ \nabla$. They are parametrized by elements in $\Omega^1(\text{ad}P)$. 
Yang–Mills theory (non-trivial)

Spectral action

We collect this in a

Theorem

- \((A_{\infty}(M), A \otimes C_{\infty}(M)) L^2(S), D = \nabla_0 \otimes 1 + 1 \otimes \partial, \gamma = 1 \otimes \gamma_5, J = (.)^* \otimes C)\)
  is an even real spectral triple.

- The self-adjoint operator \(A + JAJ^{-1}\) with \(A = A^* \in \Omega^1_D(A)\) describes a section of \(\text{ad}P \times \Lambda^1(M)\).

- The gauge group \(U(A) \simeq \Gamma_{\infty}(\text{Ad}P)\), and acts on \(H\) in the adjoint representation.

- The spectral action is given by

\[
S_{\Lambda}[A, \psi] = \frac{f(0)}{24\pi^2} \int_M \text{Tr} \ F_A \wedge \ast F_A + \langle \psi, (\partial + \text{ad}A)\psi \rangle + O(\Lambda^{-1})
\]

with \(F_A\) the curvature of the connection \(\nabla\) corresponding to \(D + A + JAJ^{-1}\).
The noncommutative torus for rational $\theta$ is of the above type.

More generally, one can construct from a spectral triple $(\mathcal{A}_0, \mathcal{H}_0, D_0)$ and a (fin.gen.proj.) $\mathcal{A}_0$-module algebra $\mathcal{A}$, equipped with a $D_0$-connection $\nabla$ another spectral triple

$$(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}_0 \mathcal{H}, \nabla \otimes 1 + 1 \otimes D_0)$$

(similar to Morita equivalence)

Relation to the work of Ćaćić (MPIM, Caltech)?

Include topological terms through addition of $\text{Tr} \left( \gamma f(D_A/\Lambda) \right)$.

Supersymmetric Yang–Mills theory

Again, consider the spectral triple $(\mathcal{C}^\infty(M) \otimes M_n(\mathbb{C}), L^2(S) \otimes M_n(\mathbb{C}), \mathcal{D} \otimes 1)$ and the spectral action

$$S_\Lambda[A, \psi] = \frac{f(0)}{24\pi^2} \int_M \text{Tr} \ F \wedge \ast F + \langle \psi, D_A \psi \rangle + \mathcal{O}(\Lambda^{-1})$$

With the fermions $\psi \in \mathcal{H}$ in the adjoint representation of the gauge group $\mathcal{U}(\mathcal{A})$, it might be possible to exchange $\psi \leftrightarrow A$ (in some way), while leaving the spectral action invariant.
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- With the fermions \(\psi \in \mathcal{H}\) in the adjoint representation of the gauge group \(\mathcal{U}(A)\), it might be possible to exchange \(\psi \leftrightarrow A\) (in some way), while leaving the spectral action invariant.
- First, we need to obtain the correct degrees of freedom:

\[
\text{H} = L^2(S) \otimes R_u(n)\]
Supersymmetric Yang–Mills theory

Again, consider the spectral triple \((C^\infty(M) \otimes M_n(\mathbb{C}), L^2(S) \otimes M_n(\mathbb{C}), \emptyset \otimes 1)\)
and the spectral action

\[ S_\Lambda[A, \psi] = \frac{f(0)}{24\pi^2} \int_M \text{Tr} \ F \wedge *F + \langle \psi, D_A \psi \rangle + \mathcal{O}(\Lambda^{-1}) \]

- With the fermions \(\psi \in \mathcal{H}\) in the adjoint representation of the gauge group \(U(A)\), it might be possible to exchange \(\psi \leftrightarrow A\) (in some way), while leaving the spectral action invariant.
- First, we need to obtain the correct degrees of freedom:
  - Instead of \(\langle \psi, D_A \psi \rangle\) we consider
    \[ \langle \widetilde{\chi}, D_A \widetilde{\psi} \rangle \]
    in terms of a anti-chiral \(\widetilde{\chi}\) and chiral \(\widetilde{\psi}\) (this is in accordance with the usual independent variables \(\overline{\psi}\) and \(\psi\) in the Lorentzian case [vNW]).
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- Write \(\mathcal{H} = L^2(S) \otimes M_n(\mathbb{C}) \cong L^2(S) \otimes_{\mathbb{R}} u(n)\) and
  \[ \langle \tilde{\chi}, D_A \tilde{\psi} \rangle = \langle \text{Tr} \tilde{\chi}, D \text{Tr} \tilde{\psi} \rangle + \langle \chi, D_A \psi \rangle \]
  where \(\tilde{\psi} = \text{Tr} \tilde{\psi} + \psi\), \(\tilde{\chi} = idem\) is the decomposition according to \(u(n) = \mathbb{R} \oplus su(n)\). Thus, the spinors \(\text{Tr} \tilde{\chi}\) and \(\text{Tr} \tilde{\psi}\) decouple.
We restrict the inner product to $\chi$ and $\psi$ in $L^2(S) \otimes_{\mathbb{R}} \mathfrak{su}(n)$ and consider

$$\text{SYM}[A, \chi, \psi] = \frac{f(0)}{24\pi^2} \int_M \text{Tr} \ F_A \wedge *F_A + \langle \chi, D_A \psi \rangle$$
We restrict the inner product to $\chi$ and $\psi$ in $L^2(S) \otimes_{\mathbb{R}} \mathfrak{su}(n)$ and consider

$$S_{SYM}[A, \chi, \psi] = \frac{f(0)}{24\pi^2} \int_M \text{Tr } F_A \wedge *F_A + \langle \chi, D_A \psi \rangle$$

Consider the following supersymmetry transformations

$$\delta A := c_1 \gamma^\mu \otimes (\epsilon_-, \gamma_\mu \psi) + c_2 \gamma^\mu \otimes (\chi, \gamma_\mu \epsilon_+)$$

$$\delta \psi := c_3 F \epsilon_+$$

$$\delta \chi := c_4 F \epsilon_-.$$  

with $\epsilon_+ \in L^2(S)$ constant spinors such that $\gamma_5 \epsilon_\pm = \pm \epsilon_\pm$. 

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\delta \chi := c_4 F \epsilon_-
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with $\epsilon_\pm \in L^2(S)$ constant spinors such that $\gamma_5 \epsilon_\pm = \pm \epsilon_\pm$.

**Proposition**

For certain constants $c_i$ the action functional $S_{\text{SYM}}$ is invariant under the supersymmetry transformations:

$$\frac{d}{dt} S_{\text{SYM}}[A + t\delta A, \chi + t\delta \chi, \psi + t\delta \psi] \bigg|_{t=0} = 0$$
Guided by physics: super-QCD

- The $SU(3)$-gauge field $A$ describes what is called a **gluon**, its supersymmetric partner $\psi$ (together with $\chi$) a **gluino**.
Guided by physics: super-QCD

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- We would like to include also quarks, as well as their superpartners: squarks, but keep the gauge group to be $SU(3)$.
- Quarks are fermions in the defining representation of $SU(3)$ rather then in the adjoint representation. We therefore extend our finite-dimensional Hilbert space $M_3(\mathbb{C})$ to
  \[ V := \mathbb{C}^3 \oplus M_3(\mathbb{C}) \oplus \overline{\mathbb{C}^3} \]
  and let $M_3(\mathbb{C})$ act on both $\mathbb{C}^3$ and $M_3(\mathbb{C})$ by left matrix multiplication, and as the identity on $\overline{\mathbb{C}^3}$. 
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- The real structure is now given on $V$ by the map

$$J_V : (q_1, T, \overline{q_2}) \mapsto (q_2, T^*, \overline{q_1})$$

(eventually combined with the real structure on $M$).
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(eventually combined with the real structure on $M$).
- Thus, the algebra $\mathcal{A} = C^\infty(M) \otimes M_3(\mathbb{C})$ acts on $\mathcal{H} = L^2(S) \otimes V$ and $J = C \otimes J_V$ defines an anti-unitary operator on $\mathcal{H}$. 
Deriving the squarks

As said, we do not want to change the gauge group $SU(3)$ so the algebra should remain $C^\infty(M) \otimes M_3(\mathbb{C})$. Squarks, being superpartners of quarks, are bosons. We want to obtain them as inner fluctuations. This motivates to replace the operator $\partial / \otimes 1$ on $H$ by $D = \partial / \otimes 1 + \gamma_5 \otimes D_V$ with $D_V: V \to V$ given by

$$
\begin{pmatrix}
0 \\
0 \\
e_0 \\
d_0 \\
0 \\
e_0
\end{pmatrix}
$$

with $d: M_3(\mathbb{C}) \to \mathbb{C}^3$, $g \mapsto g \cdot v$ and $e: M_3(\mathbb{C}) \to \mathbb{C}^3$, $g \mapsto g^t \cdot v$ for some vector $v \in \mathbb{C}^3$. Proposition $(C^\infty(M) \otimes M_3(\mathbb{C}), L^2(S) \otimes V, D, \gamma_5 \otimes 1, J)$ defines a real, even spectral triple.
**Deriving the squarks**

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---

**Proposition** $(C^\infty(M) \otimes M_3(\mathbb{C}), L^2(S) \otimes V, D, \gamma_5 \otimes 1, J)$ defines a real, even spectral triple.
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- This motivates to replace the operator $\partial \otimes 1$ on $\mathcal{H}$ by

$$D = \partial \otimes 1 + \gamma_5 \otimes D_V$$

with $D_V : V \rightarrow V$ given by

$$D_V := \begin{pmatrix}
0 & d & 0 \\
\text{d}^* & 0 & \text{e}^* \\
0 & \text{e} & 0
\end{pmatrix}$$

with $d : M_3(\mathbb{C}) \rightarrow \mathbb{C}^3, g \mapsto g \cdot \nu$ and $e : M_3(\mathbb{C}) \rightarrow \overline{\mathbb{C}^3}, g \mapsto g^t \cdot \overline{\nu}$ for some vector $\nu \in \mathbb{C}^3$. 
Deriving the squarks

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- Squarks, being superpartners of quarks, are bosons. We want to obtain them as inner fluctuations.
- This motivates to replace the operator $\frac{\partial}{\partial} \otimes 1$ on $\mathcal{H}$ by
  
  $$D = \frac{\partial}{\partial} \otimes 1 + \gamma_5 \otimes D_V$$

  with $D_V : V \to V$ given by

  $$D_V := \begin{pmatrix} 0 & d & 0 \\ d^* & 0 & e^* \\ 0 & e & 0 \end{pmatrix}$$

  with $d : M_3(\mathbb{C}) \to \mathbb{C}^3, g \mapsto g \cdot v$ and $e : M_3(\mathbb{C}) \to \overline{\mathbb{C}}^3, g \mapsto g^t \cdot \overline{v}$ for some vector $v \in \mathbb{C}^3$.

Proposition

$(C^\infty(M) \otimes M_3(\mathbb{C}), L^2(S) \otimes V, D, \gamma_5 \otimes 1, J)$ defines a real, even spectral triple
Deriving the squarks

Inner fluctuations

Again, consider \((C^\infty(M) \otimes M_3(\mathbb{C}), L^2(S) \otimes V, D, \gamma_5 \otimes 1, J)\).

The inner fluctuations \(D_A = D + A + JAJ^{-1}\) of \(D\) can be written as

\[
D + A + A\tilde{q}
\]

where \(A\) is parametrized by an \(u(3)\)-valued one-form and \(A\tilde{q}\) by an element \(\tilde{q} \in C^\infty(M) \otimes \mathbb{C}^3\). In fact, we can write

\[
A\tilde{q}(q_1, g, \bar{q}_2) = (g\tilde{q}, \bar{q}_1\tilde{q}^t + \tilde{q}\bar{q}_2^t, g^t\tilde{q})
\]
Deriving the squarks

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  \[
  \tilde{A}_\tilde{q}(q_1, g, \bar{q}_2) = (g\tilde{q}, \bar{q}_1\tilde{q}^t + \tilde{q}\bar{q}_2^t, g^t\tilde{q})
  \]

Proposition

- The gauge group \(U(A) \simeq C^\infty(M, U(3))\) acts on the Hilbert space as:
  \[
  (q_1, g, \bar{q}_2) \mapsto (uq_1, ugu^*, \bar{u}\bar{q}_2)
  \]

- The gauge transformation \(D_A \rightarrow UD_AU^*\) transforms the gauge fields as
  \[
  A \mapsto uAu^* + u[D, u^*]; \quad \tilde{A}_\tilde{q} \mapsto \tilde{A}_u\tilde{q}
  \]
The spectral action

Interestingly, \[ [\partial + A, A_\tilde{q}] = \gamma^\mu A (\partial_\mu + A_\mu) \tilde{q}. \]

Proposition

In addition to the Yang–Mills action, we have in the (bosonic) spectral action:

\[
\int_M \left[ - \left( \frac{6f_2}{\pi^2 \Lambda^2} + 3R \right) \Lambda^2 |\tilde{q}(x)|^2 + \frac{f(0)}{4 \pi^2} \left( 8 |\tilde{q}(x)|^4 + 6 |(\partial_\mu + A_\mu) \tilde{q}(x)|^2 \right) \right] d\mu g(x) 
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Proposition

The fermionic action $\langle \psi, D_A \psi \rangle$ contains in addition

$$\langle \psi_q, (\partial + A)\psi_q \rangle + \langle \chi_g, (\partial + \text{ad}A)\psi_g \rangle + \langle \bar{\psi}_q, (\partial + A)\bar{\psi}_q \rangle +$$

$$\langle \psi_q, \psi_g \bar{q} \rangle + \langle \chi_g \bar{q}, \chi_q \rangle + \langle \chi_g \bar{q}, \bar{\psi}_q \rangle + \langle \bar{\psi}_q, \psi_g^t \bar{q} \rangle$$

where $\psi = \psi_q \oplus (\psi_g \oplus \chi_g) \oplus \bar{\psi}_q$
Interpretation/comparison with the MSSM

So, in addition to the previous SYM terms, we have

\[
\int_M \left[ - \left( \frac{6f_2}{\pi^2 \Lambda^2} + 3R \right) \Lambda^2 |\tilde{q}(x)|^2 + \frac{f(0)}{4\pi^2} \left( 8|\tilde{q}(x)|^4 + 6| (\partial_\mu + A_\mu)\tilde{q}(x)|^2 \right) \right] \, d\mu_g(x)
\]

\[
\langle \psi_q, (\not{\partial} + A)\psi_q \rangle + \langle \chi_g, (\not{\partial} + \text{ad}A)\psi_g \rangle + \langle \overline{\psi}_q, (\not{\partial} + \overline{A})\overline{\psi}_q \rangle + \\
\langle \psi_q, \psi_g \tilde{q} \rangle + \langle \chi_g \tilde{q}, \chi_q \rangle + \langle \chi_g^t \tilde{q}, \overline{\psi}_q \rangle + \langle \overline{\psi}_q, \psi_g^t \tilde{q} \rangle
\]

We recognize from the MSSM [Kramml]:

- **squark kinetic term** \( \propto |\partial_\mu \tilde{q}|^2 \).
- **squark mass term** \( \propto |\tilde{q}|^2 \).
- **squark quartic self-interaction** \( \propto |\tilde{q}|^4 \).
- **squark-gluon interactions** \( \propto |(\partial_\mu + A_\mu)\tilde{q}|^2 \).
- **squark-quark-gluino interaction** \( \propto \langle \chi_g \tilde{q}, \psi_q \rangle \).
Outlook (Part 2)

- Unimodularity to reduce $U(n)$ to $SU(n)$. Fermion doubling. [CCM].
- An essential further step is to identify the coefficients of the terms just considered. However, the literature is on the MSSM, whereas we considered only part of that, namely super-QCD.
- Future plan is to include the electro-weak sector as well, exploiting the same ideas. This could lead to a noncommutative geometrical description of the MSSM, whose Lagrangian is highly non-trivial to write down. We hope to derive it as the spectral action of some noncommutative manifold.