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# The Dirac operator on quantum $SU(2)$

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# Introduction

- Noncommutative Geometry vs. Quantum Groups
- Construct $q$-version of spin geometry on $SU(2)$:
  - Homogeneous space:

$$SU(2) = \frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)} = \frac{SU(2) \times SU(2)}{SU(2)} \tag{1}$$

    with $\mathrm{Spin}(3)$ the diagonal $SU(2)$ subgroup of $\mathrm{Spin}(4)$.
    Quotient map: $(p, q) \mapsto pq^{-1}$
  - Action of $\mathrm{Spin}(4) = SU(2) \times SU(2)$ on $SU(2)$:

$$(p, q) \cdot x = pxq^{-1} \tag{2}$$

## Algebraic preliminaries

Let $q$ be a positive real number, $q \neq 1$.

**Definition.** *Define the algebra* $\mathcal{A} := \mathcal{A}(SU_q(2))$ *of polynomials on* $SU_q(2)$ *to be the* $*$*-algebra generated by* $a$ *and* $b$*, subject to the following relations:*

$$\boxed{\begin{gathered} ba = qab, \qquad b^*a = qab^*, \qquad bb^* = b^*b \\ a^*a + q^2b^*b = 1, \qquad aa^* + bb^* = 1. \end{gathered}}$$

*As a consequence,* $a^*b = qba^*$ *and* $a^*b^* = qb^*a^*$*.*

*Correspondence with [Kl-Schm],[Chakr-Pal],[Con]:* $a \leftrightarrow a^*, b \leftrightarrow -b$.

*This becomes a Hopf $*$-algebra with*

- *the coproduct* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ *defined by*

$$\Delta a := a \otimes a - q\, b \otimes b^*,$$
$$\Delta b := b \otimes a^* + a \otimes b,$$

- *the counit* $\varepsilon : \mathcal{A} \to \mathbb{C}$ *defined by* $\varepsilon(a) = 1$ *and* $\varepsilon(b) = 0$*,*

- *the antipode* $S : \mathcal{A} \to \mathcal{A}$ *defined as an antilinear map by*

$$Sa = a^*, \qquad Sb = -qb,$$
$$Sb^* = -q^{-1}b^*, \qquad Sa^* = a.$$

**Definition.** *The $*$-algebra $\mathcal{U} := \mathcal{U}_q(su(2))$ is generated by elements $e, f, k$, with $k$ invertible, satisfying the relations*

$$\boxed{\quad ek = qke, \qquad kf = qfk, \qquad k^2 - k^{-2} = (q - q^{-1})(fe - ef) \quad}$$

*Correspondence with [Kl-Schm]: $q \leftrightarrow q^{-1}$, or, equivalently: $e \leftrightarrow f$.*

*Hopf $*$-algebra structure given by: coproduct $\Delta$:*

$$\Delta k = k \otimes k, \qquad \Delta e = e \otimes k + k^{-1} \otimes e, \qquad \Delta f = f \otimes k + k^{-1} \otimes f,$$

*counit $\epsilon(k) = 1, \epsilon(f) = \epsilon(e) = 0$, antipode $S$,*

$$Sk = k^{-1}, \qquad Sf = -qf, \qquad Se = -q^{-1}e,$$

*and star structure: $k^* = k, f^* = e$.*

## Representation theory of $\mathcal{U}_q(su(2))$

The irreducible finite dimensional representations $\sigma_l$ of $\mathcal{U}_q(su(2))$ are labelled by nonnegative half-integers $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and given by

$$\boxed{\begin{aligned}\sigma_l(k)\,|lm\rangle &= q^m\,|lm\rangle,\\ \sigma_l(f)\,|lm\rangle &= \sqrt{[l-m][l+m+1]}\,|l, m+1\rangle,\\ \sigma_l(e)\,|lm\rangle &= \sqrt{[l-m+1][l+m]}\,|l, m-1\rangle,\end{aligned}}$$

on the irreducible $\mathcal{U}$-modules $V_l = \mathrm{Span}\{|lm\rangle\}_{m=-l,\ldots,l}$.

The brackets denote $q$-integers: $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, provided $q \neq 1$.

## Action of $\mathcal{U}_q(su(2))$ on $\mathcal{A}(SU_q(2))$

Dual pairing $\langle .,. \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ induces left and right action of $h \in \mathcal{U}_q(su(2))$ on $x \in \mathcal{A}(SU_q(2))$:

$$\boxed{h \triangleright x := x_{(1)} \langle h, x_{(2)} \rangle \qquad x \triangleleft h := \langle h, x_{(1)} \rangle \, x_{(2)},}$$

where we use Sweedler's notation for the coproduct in $\mathcal{A}(SU_q(2))$:

$$\Delta x = x_{(1)} \otimes x_{(2)}, \qquad \big(x \in \mathcal{A}\big)$$

Using the antipode, the right action can be transformed into a left action, which we will denote by $h \cdot x$.

## Left regular representation of $\mathcal{A}(SU_q(2))$

We establish the left regular representation of $\mathcal{A}$ as an equivariant representation with respect to two copies of $\mathcal{U}$ acting via $\cdot$ and $\triangleright$ on the left.

**Definition.** *Let $\lambda$ and $\rho$ be mutually commuting representations of the Hopf algebra $\mathcal{U}$ on a vector space $V$. A representation $\pi$ of the algebra $\mathcal{A}$ on $V$ is $(\lambda, \rho)$-equivariant if the following compatibility relations hold:*

$$\lambda(h)\,\pi(x)\xi = \pi(h_{(1)} \cdot x)\,\lambda(h_{(2)})\xi,$$
$$\rho(h)\,\pi(x)\xi = \pi(h_{(1)} \triangleright x)\,\rho(h_{(2)})\xi,$$

*for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in V$.*

## Equivariant representation of $\mathcal{A}(SU_q(2))$

Representation space:

$$V := \bigoplus_{2l=0}^{\infty} V_l \otimes V_l \qquad (3)$$

The two copies of $\mathcal{U}_q(su(2))$ act via the irreducible representations $\sigma$ on the first and the second leg of the tensor product, respectively:

$$\lambda(h) = \sigma_l(h) \otimes \mathrm{id}, \qquad \rho(h) = \mathrm{id} \otimes \sigma_l(h) \qquad \text{on } V_l \otimes V_l. \qquad (4)$$

We abbreviate $|lmn\rangle := |lm\rangle \otimes |ln\rangle$, for $m, n = -l, \ldots, l$.

**Proposition.** *A* $(\lambda, \rho)$*-equivariant* $*$*-representation* $\pi$ *of* $\mathcal{A}(SU_q(2))$ *on* $V$ *is necessarily given by the left regular representation. Explicitly:*

$$\pi(a)\,|lmn\rangle = A^{+}_{lmn}|l+\tfrac{1}{2}, m+\tfrac{1}{2}, n+\tfrac{1}{2}\rangle + A^{-}_{lmn}|l-\tfrac{1}{2}, m+\tfrac{1}{2}, n+\tfrac{1}{2}\rangle,$$
$$\pi(b)\,|lmn\rangle = B^{+}_{lmn}|l+\tfrac{1}{2}, m+\tfrac{1}{2}, n-\tfrac{1}{2}\rangle + B^{-}_{lmn}|l-\tfrac{1}{2}, m+\tfrac{1}{2}, n-\tfrac{1}{2}\rangle,$$

*where for example the constants* $A^{\pm}_{lmn}$ *are given by*

$$A^{+}_{lmn} = q^{(-2l+m+n-1)/2}\left(\frac{[l+m+1][l+n+1]}{[2l+1][2l+2]}\right)^{\frac{1}{2}},$$
$$A^{-}_{lmn} = q^{(2l+m+n+1)/2}\left(\frac{[l-m][l-n]}{[2l][2l+1]}\right)^{\frac{1}{2}}.$$

# Spinor representation

We amplify representation $\pi$ of $\mathcal{A}$ to the spinor representation defined by $\pi' = \pi \otimes \mathrm{id}$ on $V \otimes \mathbb{C}^2$, and set $\rho' = \rho \otimes \mathrm{id}$, but $\lambda'$ as the tensor product of the representations $\lambda$ on $V$ and $\sigma_{\frac{1}{2}}$ on $V_{\frac{1}{2}} = \mathbb{C}^2$:

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)}). \tag{5}$$

**Proposition.** *The representation $\pi'$ of $\mathcal{A}$ is $(\lambda', \rho')$-equivariant.*

Clebsch-Gordan decomposition:

$$V \otimes \mathbb{C}^2 = \left( \bigoplus_{2l=0}^{\infty} V_l \otimes V_l \right) \otimes V_{\frac{1}{2}} \simeq V_{\frac{1}{2}} \oplus \bigoplus_{2j=1}^{\infty} (V_{j+\frac{1}{2}} \otimes V_j) \oplus (V_{j-\frac{1}{2}} \otimes V_j). \tag{6}$$

Basis vectors ($j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$):

$|j\mu n{\uparrow}\rangle := C_{j+1,\mu}\,|j+\frac{1}{2}, \mu-\frac{1}{2}, n\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle - S_{j+1,\mu}\,|j+\frac{1}{2}, \mu+\frac{1}{2}, n\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle,$

where $\mu = -j, \dots, j$ and $n = -(j+\frac{1}{2}), \dots, j+\frac{1}{2}$

$|j\mu n{\downarrow}\rangle := S_{j\mu}\,|j-\frac{1}{2}, \mu-\frac{1}{2}, n\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle + C_{j\mu}\,|j-\frac{1}{2}\mu+\frac{1}{2}n\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle,$

where $\mu = -j, \dots, j$ and $n = -(j-\frac{1}{2}), \dots, j-\frac{1}{2}$, and the $q$-Clebsch-Gordan coefficients come from the well-known representation theory of $\mathcal{U}_q(su(2))$:

$$C_{j\mu} := q^{-(j+\mu)/2}\,\frac{[j-\mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}, \qquad S_{j\mu} := q^{(j-\mu)/2}\,\frac{[j+\mu]^{\frac{1}{2}}}{[2j]^{\frac{1}{2}}}.$$

$\Longrightarrow$ expressions for $\pi'$ in basis $\{|j\mu n{\uparrow}\rangle, |j\mu n{\downarrow}\rangle\}$ contain off-diagonal terms.

# Invariant Dirac operator

**Proposition.** *Any self-adjoint operator on* $\mathcal{H} = (V \otimes \mathbb{C}^2)^{\mathrm{cl}}$*, that commutes with both actions* $\rho', \lambda'$ *of* $\mathcal{U}_q(su(2))$ *is of the form*

$$\boxed{D|j\mu n{\uparrow}\rangle = d_j^{\uparrow}\,|j\mu n{\uparrow}\rangle, \qquad D|j\mu n{\downarrow}\rangle = d_j^{\downarrow}\,|j\mu n{\downarrow}\rangle.}$$

Restrict form of eigenvalues by imposing bounded commutator condition:

$$[D, \pi'(x)] \in \mathcal{B}(\mathcal{H}), \qquad (x \in \mathcal{A}). \tag{7}$$

- $D$ with as eigenvalues $q$-analogues of the classical Dirac operator (like $[j]$) gives unbounded commutators (cf. [Bib-Kul]).

- 'Classical' Dirac operator with $D$ eigenvalues linear in $j$ with opposite signs on the $\uparrow$ and $\downarrow$-eigenspaces, respectively.

**Proposition.** *If $D$ has eigenvalues linear in $j$, the commutators $[D, \pi'(x)]$ ($x \in \mathcal{A}$) are bounded operators.*

The spectrum of $D$ coincides with that of the classical Dirac operator on the round sphere $S^3 \simeq SU(2)$. We make the following choice:

$$D|j\mu n\rangle\rangle = \begin{pmatrix} 2j + \frac{3}{2} & 0 \\ 0 & -2j - \frac{1}{2} \end{pmatrix} |j\mu n\rangle\rangle, \tag{8}$$

and conclude that $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ is a ($3^+$-summable) spectral triple.

Relation with [Gos], with unbounded commutators?
Difference in definition of spinor space: $\mathbb{C}^2 \otimes V$ (instead of $V \otimes \mathbb{C}^2$).
Define on $\mathbb{C}^2 \otimes V$:

$$\pi'(x) = \mathrm{id} \otimes \pi(x);$$
$$\rho'(h) = \mathrm{id} \otimes \rho(h);$$
$$\lambda'(h) = \sigma_{\frac{1}{2}}(h_{(1)}) \otimes \lambda(h_{(2)}).$$

Let us (naïvely) define the Dirac operator to be diagonal in the $\uparrow - \downarrow$ basis obtained from the Clebsch-Gordan decomposition, with $j$-linear eigenvalues. This is exactly [Gos]. A computation shows that $[D, \pi'(x)]$ is unbounded.

However, this $\pi'$ is not $(\lambda', \rho')$-equivariant, so that the choice of $\mathbb{C}^2 \otimes V$ is not allowed, because $\mathcal{U}_q(su(2))$ is not cocommutative.

# Real structure

A real structure $J$ on the spectral triple $(\mathcal{A}(SU_q(2)), \mathcal{H}, D)$ defines a representation of the opposite algebra $\mathcal{A}(SU_q(2))^\circ$:

$$\pi'^\circ(x) = J\pi'(x^*)J^{-1} \text{ satisiying } \pi'^\circ(xy) = \pi'^\circ(y)\pi'^\circ(x)$$

**Definition.** *The real structure $J$ is the antilinear operator on $\mathcal{H}$ which is defined on the orthonormal spinor basis by*

$$J\,|j\mu n{\uparrow}\rangle := i^{2(2j+\mu+n)}\,|j,-\mu,-n,\uparrow\rangle;$$
$$J\,|j\mu n{\downarrow}\rangle := i^{2(2j-\mu-n)}\,|j,-\mu,-n,\downarrow\rangle.$$

$\Longrightarrow$ The Dirac operator $D$ commutes with $J$.

Conditions such as the commutant property and first-order condition entail that $J$ maps both $\mathcal{A}$ and $[D, \mathcal{A}]$ to the commutant of $\mathcal{A}$.

In the case of $\mathcal{A}(SU_q(2))$, they are almost satisfied.

**Definition.** *The ideal $\mathcal{K}_q$ is defined as the two-sided ideal in $\mathcal{B}(\mathcal{H})$ generated by the positive traceclass operator: $L_q|j\mu n\rangle := q^j|j\mu n\rangle$.*

$\mathcal{K}_q$ is contained in the ideal of infinitesimals of order $\alpha$, that is, compact operators whose $n$-th singular value $\mu_n$ satisfies $\mu_n = O(n^{-\alpha})$, for all $\alpha > 0$.

**Proposition.** *With $D$ and $J$ as above, the commutant property and the first-order condition are satisfied up to infinitesimals of arbitrary order:*

$$[\pi'(x), J\pi'(y)J^{-1}] \in \mathcal{K}_q;$$
$$[\pi'(x), [D, J\pi'(y)J^{-1}]] \in \mathcal{K}_q; \qquad \big(\forall x, y \in \mathcal{A}(SU_q(2))\big)$$

## Local index formula for $SU_q(2)$ (sketchy!)

The local index formula of Connes-Moscovici has been applied to this spectral triple. It provides a method to compute the index of a twisted Dirac operator in terms of easier "local expressions".

Important here is that we can work modulo the infinitesimals of arbitrary high order. The quotient map is understood geometrically as a symbol map:

$$\rho : \mathcal{B} \to \mathcal{A}(D^2_{q+}) \otimes \mathcal{A}(D^2_{q-}) \tag{9}$$

which maps onto the "cosphere bundle" $\mathbb{S}^*_q$. Here $\mathcal{B}$ is the algebra $\delta^n(\mathcal{A}) \cup \delta^n([D,\mathcal{A}])$ for any $n$'th iteration of the derivation $\delta(x) = |D|x - x|D|$, and $D^2_{q\pm}$ are two noncommutative disks.

If $F$ denotes the sign of $D$ ($+$ on $\uparrow$ and - on $\downarrow$), we can express the Chern character $\mathrm{ch}(a_0, a_1) := \mathrm{tr}(a_0[F, a_1])$ of $\mathcal{A}(SU_q(2))$ in terms of three (relatively simple) linear functionals $\tau_0^{\uparrow}, \tau_0^{\downarrow}, \tau_1$ on the disks as

$$2(\tau_0^{\uparrow} \otimes \tau_0^{\downarrow})\big(\rho(a_0 \delta a_1)\big) - \big(\tau_1 \otimes \tau_0^{\downarrow} + \tau_0^{\uparrow} \otimes \tau_1\big)\big(\rho(a_0 \delta^2 a_1)\big) + \frac{2}{3}(\tau_1 \otimes \tau_1)\big(\rho(a_0 \delta^3 a_1)\big)$$

# References


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