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On One Method of Estimation of the L_1-Norm of an Exponential Sum

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In this work a method is proposed for finding the lower estimate of the L_1-norm of an exponential sum. The method is based on new multiplicative inequalities which are derived from the characterization, established by the author, of the space of functions of a bounded mean oscillation (BMO) via the La Vallée–Poussin expansion.

Let $\{V_n(x)\}_{n=1}^{\infty}$ be the Vallée–Poussin kernels, i.e. (see [1, p. 190]),

$$V_n(x) = 2K_{2n-1}(x) - K_{n-1}(x), \tag{1}$$

where $K_n(x)$ are Fejér's kernels,

$$K_n(x) = \frac{2}{n+1}\left(\frac{\sin\frac{1}{2}(n+1)x}{2\sin\frac{1}{2}x}\right)^2, \qquad n = 0, 1, 2, \dots .$$

We denote $Q_0(x) = V_1(x)$, and if $n \geq 1$, then

$$Q_n(x) = V_{2^n}(x) - V_{2^{n-1}}(x). \tag{2}$$

Any periodic function $f \in L(0, 2\pi)$ can be expanded in a convergent almost everywhere Vallée–Poussin series

$$f(x) = \frac{1}{\pi}\sum_{n=0}^{\infty}\int_0^{2\pi} f(t)Q_n(x-t)\,dt \equiv \sum_{n=0}^{\infty}\Delta_n(f, x). \tag{3}$$

The norm of the function $f \in L(0, 2\pi)$ in BMO is defined as follows (see [2, pp. 224–234]):

$$\|f\|_{\mathrm{BMO}} = \|f\|_1 + \sup_I\left(\frac{1}{|I|}\int_I\left|f(x) - \frac{1}{|I|}\int_I f(t)\,dt\right|^p dx\right)^{1/p}, \tag{4}$$

where $1 \leq p < \infty$, and I is an interval $[0, 2\pi]$ on the circle, and, according to the John–Nirenberg theorem, for all $1 \leq p < \infty$ expression (4) defines equivalent norms.

Using expansion (3), we can introduce, in the space of BMO, a new equivalent norm which does not vary when the signs of the terms of series (3) are arranged arbitrarily.

We set (see (1)–(3))

$$\|f\|_* = \sup_I\left(\frac{1}{|I|}\int_I\sum_{2^n|I|\geq 1}\Delta_n^2(f, x)\,dx\right)^{1/2}. \tag{5}$$

The following theorem is valid (see [3]).

Theorem 1. *Let* $f \in L(0,2\pi)$.

1. *If* $f \in \mathrm{BMO}$, *then*

$$\|f\|_* \ll \|f\|_{\mathrm{BMO}}. \tag{6}$$

2. *If* $\|f\|_* < \infty$, *then* $f \in \mathrm{BMO}$ *and*

$$\|f\|_{\mathrm{BMO}} \ll \|f\|_*. \tag{7}$$

By virtue of Theorem 1, series (3) unconditionally converges in BMO (see (5)–(7)), and since BMO is a conjugate of Hardy's real space $H^1_{\mathrm{Re}}(T)$, T is a unit circle, and Hardy's real space is isomorphic to Hardy's complex space $H^1(D)$, where D is a unit circle, with the norm

$$\|F\|_{H^1(D)} = \sup_{0<r<1} \int_0^{2\pi} |F(re^{it})|\,dt,$$

it follows that the Vallée–Poussin series unconditionally converges in the space $H^1(D)$ as well.

Therefore the Littlewood–Paley function of the Vallée–Poussin series in the space $H^1(D)$ satisfies the following relation (proved in [3]).

Theorem 2. *For any function* $F(z) \in H^1(D)$ *we have*

$$\left\| \left(\sum_{n=0}^{\infty} |\Delta_n(F)|^2 \right)^{1/2} \right\|_{L_1(T)} \sim \|F\|_{H^1(D)}. \tag{8}$$

Note that Theorems 1 and 2 are also valid for a polydisc.

The characterization of the space $H^1(D)$ (see (8)) served for finding the multiplicative lower estimate of the L_1-norm of the general exponential sum.

We denote $\delta_0(x) = c_1 e^{ix}$, and for $n \ge 1$ we have

$$\delta_n(x) = \sum_{k=2^{n-1}+1}^{2^n} c_k e^{ikx},$$

where c_k are arbitrary complex coefficients.

Theorem 3. *Let the positive multiplicator* $\{\mu_n\}$ *satisfy, for a certain* $A \ge 1$, *the relation*

$$\mu_{n+1} \le \mu_n \le A\mu_{n+1}, \qquad n = 0,1,2,\dots .$$

Then, for any $N = 1,2,\dots$, *the inequality*

$$\max_{n\le N} \left(\mu_n \sum_{m=0}^{n} \mu_m \|\delta_m\|_\infty^2 \right)^{1/2} \int_0^{2\pi} \left| \sum_{k=1}^{2^N} c_k e^{ikx} \right| dx \gg \int_0^{2\pi} \sum_{n=0}^{N} \mu_n |\delta_n(x)|^2\,dx$$

holds true.

Theorem 3 makes it possible to find the lower estimate of the L_1-norm of an exponential sum with an arbitrary spectrum expressed via quantities which characterize the arrangement of the points of the spectrum in binary intervals.

Let the sequence $\{d_k\}$ assume three values, namely, $d_k = 0, \pm 1$. We denote

$$\lambda_n = \bigl|\{k : 2^{n-1} < k \le 2^n,\ |d_k| = 1\}\bigr|, \qquad \Lambda_n = \bigl|\{k : 1 < k \le 2^n,\ |d_k| = 1\}\bigr|. \tag{9}$$

Theorem 4. *If the ratios Λ_n/λ_n do not decrease for $1 \le n \le N$ and the relation*

$$\lambda_n \le \lambda_{n+1} \le A\lambda_n, \qquad n = 1, \dots, N-1, \tag{10}$$

is satisfied for a certain $A \ge 1$, then we have the inequality

$$\int_0^{2\pi} \left| \sum_{k=1}^{2^N} d_k e^{ikx} \right| dx \gg N \left(\frac{\lambda_n}{\Lambda_n} \right)^{1/2}. \tag{11}$$

This estimate is exact both for a gapless spectrum and for a lacunary spectrum. For spectra that are intermediate between a gapless spectrum and a lacunary one estimate (11) gives the best result if the points of the spectrum are arranged uniformly within every binary interval $(2^n, 2^{n+1})$, $n = 0, 1, 2, \dots$.

As an example of the application of inequality (11) we give the estimate (see (9)–(11)) of an exponential sum, where $\varphi(n) = [2^{(\log n)^\beta}]$, $\beta \ge 1$, $[\,\cdot\,]$ is the integral part.

Theorem 5. *The inequality*

$$\int_0^{2\pi} \left| \sum_{n=1}^{N} \alpha_n \exp\bigl(i[2^{(\log n)^\beta}]x\bigr) \right| dx \gg (\log N)^{(\beta+1)/2} \tag{12}$$

holds for any coefficients $\alpha_n = \pm 1$ and any $\beta \ge 1$.

For $\beta = 1$ estimate (12) can be obtained from Hardy's inequality (see [1, p. 454]). If $\beta > 1$, then this estimate does not follow from the generalized Hardy inequality obtained by Konyagin [4] and Mc Gehee, Pigno, and Smith [5]. Note that estimate (12) is preserved for an arbitrary redistribution of the points of the spectrum within every binary interval. In this case the methods which use the arithmetic properties of the sequence $\varphi(n)$ cannot be applied either.

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