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On the Algebraic Independence of the Values of Functions Satisfying Systems of Functional Equations

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Let us consider different positive integers $l_1, \dots, l_m$ and set $l_{i+km} = l_i$ for any natural i, k. The symbol Tz denotes a change of variable z by z^t, where t is a positive integer not smaller than 2. Similarly, for any natural k, the symbol $T^k z$ denotes the raising of z to the power t^k. (We can also consider more general transformations T, see, e.g., [1].) Let $q(z)$ be a polynomial with complex coefficients. In this article use is made of the set of functions

$$
\begin{gathered}
f_1(z) = \sum_{n=1}^{\infty} q^{-l_1}(z) q^{-l_2}(Tz) \cdots q^{-l_n}(T^{n-1}z), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
f_m(z) = \sum_{n=1}^{\infty} q^{-l_m}(z) q^{-l_{m+1}}(Tz) \cdots q^{-l_{n+m-1}}(T^{n-1}z)
\end{gathered}
\tag{1}
$$

and

$$
\begin{gathered}
g_1(z) = q^{-l_1}(z) \cdots q^{-l_n}(T^{n-1}z) \cdots , \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
g_m(z) = q^{-l_m}(z) \cdots q^{-l_{n+m-1}}(T^{n-1}z) \cdots ,
\end{gathered}
\tag{2}
$$

and it is assumed that sets (1) and (2) consist of functions which are analytic in the domain $|z| < 1$. Certain conditions sufficient for this will be formulated below (see the formulations of Theorems 4 and 5).

The aim of the work is to prove the algebraic independence of the values of functions (1) and (2) at algebraic points α satisfying certain natural conditions. To this end, we use the modified Mahler method (see [1]). The author of [1] gives a number of examples of applications of the general theorems proved there. These examples are series of the form

$$
\sum_{h=0}^{\infty} q_i\bigl(T^h(z)\bigr), \qquad \sum_{h=0}^{\infty} \frac{1}{q_i(z) q_i(Tz) \cdots q_i(T^h(z))}, \qquad \sum_{h=0}^{\infty} \frac{(-1)^h}{q(T^h(z))}
$$

and the infinite product

$$
\prod_{h=0}^{\infty} q\bigl(T^h(z)\bigr).
$$

In the same work the reader can find the history of the question and references to articles which are close in the subject matter.

In the present article I did not pursue the objective of describing in detail the preceding works but restricted the discussion, as a complement of the review and references in [1], to an important remark to the point that Mahler's method was discussed in detail in Nishioka's book *Mahler Functions and Transcendence*, which will soon be published. My work is devoted to the construction of new types of functions to which the modified Mahler method can be applied. The functions considered are solutions of the systems of functional equations

$$f_1(z) = q^{-l_1}(z)\big(1 + f_2(Tz)\big),\ f_2(z) = q^{-l_2}(z)\big(1 + f_3(Tz)\big),\ \dots,\ f_m(z) = q^{-l_m}(z)\big(1 + f_1(Tz)\big) \quad (3)$$

and

$$g_1(z) = q^{-l_1}(z)g_2(Tz),\ g_2(z) = q^{-l_2}(z)g_3(Tz),\ \dots,\ g_m(z) = q^{-l_m}(z)g_1(Tz). \quad (4)$$

We set

$$F(x) = l_1x^m + \dots + l_{m-1}x + l_m. \quad (5)$$

Theorem 1. *Suppose that for any positive integers r and s all zeros of the polynomials $q(T^rz)$ and $q(T^sz)$ are different for $r \neq s$. Also suppose that none of the roots of unity of degree m is a root of polynomial* (5). *Then functions* (1) *are algebraically independent over $\mathbb{C}(z)$.*

Theorem 2. *Under the conditions of Theorem* 1, *functions* (2) *are also algebraically independent over $\mathbb{C}(z)$.*

The proofs of the two theorems have much in common, and therefore we shall begin with arguments that refer to these two theorems simulateously. We shall use the proof by contradiction and assume that functions (1) ((2), resp.) are connected by algebraic equations with coefficients from $\mathbb{C}(z)$. From these equations we shall choose those which have the smallest degree with respect to the set of variables and, from the chosen equations, we shall consider those which have the smallest number of homogeneous terms of the highest degree. Let us consider one of these equations, namely,

$$\sum C_{\mathbf{k}}(z)\mathbf{f}^{\mathbf{k}}(z) = 0, \quad (6)$$

where $\mathbf{k} = (k_1, \dots, k_m)$ and

$$\mathbf{f}^{\mathbf{k}}(z) = f_1^{k_1}(z) \cdots f_m^{k_m}(z).$$

The summation on the left-hand side of (6) is carried out over the finite set of collections $\mathbf{k}$. For functions (2) we shall consider the similar equation

$$\sum C_{\mathbf{k}}(z)\mathbf{g}^{\mathbf{k}}(z) = 0. \quad (7)$$

Since (6) and (7) are indentities with respect to z, for any natural n (in particular, for $n = m$) we have

$$\sum C_{\mathbf{k}}(T^nz)\mathbf{f}^{\mathbf{k}}(T^nz) = 0, \quad (8)$$

$$\sum C_{\mathbf{k}}(T^nz)\mathbf{g}^{\mathbf{k}}(T^nz) = 0. \quad (9)$$

It follows from the functional equations (3) that

$$\begin{aligned} f_i(z) = q^{-l_i}(z) + q^{-l_i}(z)q^{-l_{i+1}}(Tz) + \dots + q^{-l_i}(z) \cdots q^{-l_{i+m-1}}(T^{m-1}z) \\ + q^{-l_i}(z) \cdots q^{-l_{i+m-1}}(T^{m-1}z)f_i(T^mz), \end{aligned} \quad (10)$$

or, in the obvious notation,

$$f_i(z) = \sigma(i,z) + \pi(i,z) f_i(T^m z). \tag{11}$$

In the same notation we easily find from (4) that

$$g_i(z) = \pi(i,z) g_i(T^m z). \tag{12}$$

Since the coefficients in the leading terms of equations (6) ((7), resp.) are nonzero, we can assume that there is a term with the coefficient equal to 1 among them, and we shall denote the collection of exponents corresponding to it by $\mathbf{r} = (r_1, \dots, r_m)$. Then the term with this collection of exponents appears in (8) ((9), resp.) also with a coefficient equal to 1. On the other hand, if we replace in (6) ((7), resp.) $f_i(z)$ ($g_i(z)$, resp.) by the right-hand sides of the corresponding relation (11) or (12), then we obtain an equation that relates the functions $f_i(T^m z)$ ($g_i(T^m z)$, resp.) over $\mathbb{C}(z)$. In this case, the sets of the leading terms of the equations obtained and equations (8) and (9) coincide. This is only possible when the obtained equation is proportional to the corresponding equation (8) ((9), resp.), since, otherwise, we could obtain a new equation from the two equations which would either be of a lower degree with respect to the set of variables or would have a smaller number of terms of the highest power with respect to the set of variables, and this contradicts the definition of equations (6) and (7). The proportionality factor of the equations obtained upon the above-mentioned substitution (11) ((12), resp.) is equal to the coefficient which the term obtained from (6) ((7), resp.) will have with the collection of exponents $\mathbf{r}$, i.e.,

$$\pi(1,z)^{r_1} \cdots \pi(m,z)^{r_m}. \tag{13}$$

Since the functions $f_i(z)$ and $g_i(z)$ are not indentifically zero by construction, equations (6) and (7) must include at least one more term, say, $c_{\mathbf{s}}(z)\mathbf{f}^{\mathbf{s}}(z)$ or $c_{\mathbf{s}}(z)\mathbf{g}^{\mathbf{s}}(z)$ respectively. After the transformation of equations (6) and (7) with the use of relations (11) and (12) the equations obtained will contain, respectively, the terms

$$c_{\mathbf{s}}(z)\pi(1,z)^{s_1} \cdots \pi(m,z)^{s_m} f_1^{s_1}(T^m z) \cdots f_m^{s_m}(T^m z), \tag{14}$$

$$c_{\mathbf{s}}(z)\pi(1,z)^{s_1} \cdots \pi(m,z)^{s_m} g_1^{s_1}(T^m z) \cdots g_m^{s_m}(T^m z). \tag{15}$$

Comparing the obtained equations with equations (8) and (9) and taking into account their proportionality with cofficient (13), we get

$$c_{\mathbf{s}}(z)\pi(1,z)^{s_1} \cdots \pi(m,z)^{s_m} = c_{\mathbf{s}}(T^m z)\pi(1,z)^{r_1} \cdots \pi(m,z)^{r_m}. \tag{16}$$

Since $\mathbf{r} \neq \mathbf{s}$, it follows from relation (16) that

$$c_{\mathbf{s}}(z) = c_{\mathbf{s}}(T^m z)\pi(1,z)^{t_1} \cdots \pi(m,z)^{t_m},$$

where $t_1, \dots, t_m$ are integers, not all of which are equal to zero. By definition,

$$\pi(i,z) = q^{-l_i}(z) \cdots q^{-l_{i+m-1}}(T^{m-1}z).$$

Therefore

$$\pi(1,z)^{t_1} \cdots \pi(m,z)^{t_m} = q(z)^{-l_1 t_1 - \dots - l_m t_m} q(Tz)^{-l_2 t_1 - \dots - l_m t_{m-1} - l_1 t_m} \cdots \times q(T^{m-1}z)^{-l_m t_1 - l_1 t_2 - \dots - l_{m-1} t_m} = Q(z). \tag{17}$$

Then (16) gives

$$C(z) = C\prod_{i=0}^{\infty} Q(T^{im}z),$$

where C is a constant. Let us consider the exponents of terms appearing in product (17). If we assumed that all of them are zero, then we would find that the numbers $t_1, \dots, t_m$ satisfy the system of homogeneous equations

$$l_1t_1 + \cdots + l_mt_m = 0, \quad \dots, \quad l_mt_1 + l_1t_2 + \cdots + l_{m-1}t_m = 0. \tag{18}$$

The determinant of this system, the so-called circulant, is nonzero under the conditions of Theorem 1 (see [2, p. 36]). Therefore it would immediately follow from system (18) that $t_i = 0$, i.e., $r_i - s_i = 0$, $\mathbf{r} = \mathbf{s}$. Since this is not the fact by assumption, not all of the exponents in (17) are zero and, consequently, the function $C(z)$ has infinitely many zeros or poles as, by the hypothesis, the zeros of the functions $q(T^kz)$ and $q(T^lz)$ are different for $k \neq l$. However, $C(z)$ is a rational function. The contradiction obtained means that (6) has a single term of the highest degree with respect to the set of variables, and (7) consists of only one term. In oder to complete the proof of Theorem 2, we note that as follows from what has been proved (7) is not satisfied since functions (2) being considered are not identically zero.

Let us pass to (6). As was prove above, this equation has a single term of the highest degree with respect to the set of variables, $\mathbf{f}^{\mathbf{r}}(z)$. Since $r_1 + \cdots + r_m \geq 1$, there exists a number i such that $r_i \geq 1$. We can suppose that $r_1 \geq 1$. Let us assume that (6) includes the term

$$c(z)f_1^{r_1-1}(z)\cdots f_m^{r_m}(z). \tag{19}$$

Then (8) includes the term

$$c(T^mz)f_1^{r_1-1}(T^mz)\cdots f_m^{r_m}(T^mz)$$

and in the equation obtained from (6) upon the substitution (11) the term with this collection of exponents will be obtained only from the leading term and from term (19). The coefficient in this term is equal to

$$\big(r_1\sigma(1,z) + c(z)\big)\big(\pi(1,z)\big)^{r_1-1}\cdots\big(\pi(m,z)\big)^{r_m}.$$

Since both equations mentioned above are proportional with the proportionality factor (13), the relation

$$r_1\sigma(1,z) + c(z) = c(T^mz)\pi(1,z) \tag{20}$$

holds true. Recalling the definitions of the quantities $\sigma(1,z)$ and $\pi(1,z)$ (see (10) and (11)), we rewrite (20) in the form

$$\begin{aligned} c(z) = {} & -r_1\big(q^{-l_i}(z) + q^{-l_i}(z)q^{-l_{i+1}}(Tz) + \cdots + q^{-l_i}(z)\cdots q^{-l_{i+m-1}}(T^{m-1}z)\big) \\ & + q^{-l_i}(z)\cdots q^{-l_{i+m-1}}(T^{m-1}z)c(T^mz). \end{aligned} \tag{21}$$

It follows from equations (21) and (10) that

$$c(z) = \gamma f_1(z), \tag{22}$$

where γ is a constant, and this constant is nonzero since, otherwise, we would find from (20) that $\sigma(1,z)=0$, i.e., we would get an invalid relation. Relation (22) means that $f_1(z)$ is a rational function. We set $f_1(z)=a(z)/b(z)$, where $a(z)$, $b(z)$ are coprime polynomials. This means that there exist polynomials $u(z)$, $v(z)$ such that

$$u(z)a(z)+v(z)b(z)=1.$$

However, then

$$u(T^m z)a(T^m z)+v(T^m z)b(T^m z)=1,$$

as well, i.e., the polynomials $a(T^m z)$ and $b(T^m z)$ are also coprime. Then we can rewrite (11) for $i=1$ in the form

$$\frac{a(T^m z)}{b(T^m z)}=\left(\frac{a(z)}{b(z)}-\sigma(1,z)\right)\big(\pi(1,z)\big)^{-1},$$

or

$$a(T^m z)b(z)=b(T^m z)a(z)\big(\pi(1,z)\big)^{-1}-b(z)b(T^m z)\sigma(1,z)\big(\pi(1,z)\big)^{-1}.$$

Since $\big(\pi(1,z)\big)^{-1}$ and $\sigma(1,z)\big(\pi(1,z)\big)^{-1}$ are polynomials of z, the ploynomial $b(T^m z)$ divides the left-hand side of this relation. However, according to what was proved above, this polynomial is prime to the polynomial $a(T^m z)$. Consequently, it must divide the polynomial $b(z)$, and this is impossible since the degree of the polynomial $b(T^m z)$ is higher than that of the polynomial $b(z)$. The contradiction obtained shows that (6) is not satisfied and thus proves Theorem 1.

Now we shall formulate the numerical results which were obtained upon the application of the following general theorem to functions (1) and (2) that we have considered.

Theorem 3 [1, Corollary 2 of Theorem 3]. *Let $f_1(z),\dots,f_m(z)$ be functions which are holomorphic in a certain neighborhood of the point $z=0$, and algebraically independent over $\mathbb{C}(z)$, and $f_1(0),\dots,f_m(0)$ are algebraic numbers. Let $T(z)$ be a polynomial with algebraic coefficients,* $\deg T=d$, $T(0)=0$, $\operatorname{ord}_{z=0}T=\delta\ge 2$. *Suppose that the set of functions $\mathbf{f}=(f_1,\dots,f_m)$ constitutes a solution of the system of functional equations*

$$\mathbf{f}(z)=A(z)\mathbf{f}\big(T(z)\big)+\mathbf{B}(z),$$

where $A(z)$ is a regular matrix whose elements are polynomials with algebraic coefficients and $\mathbf{B}(z)$ is a vector whose coordinates are polynomials with algebraic coefficients. Let α be an algebraic number from the neighborhood in which the functions under consideration are holomorphic, and suppose that $T^k(\alpha)\ne 0$ and $\det A\big(T^k(\alpha)\big)\ne 0$ *for any nonnegative integer k. Let $d<\delta^{1+1/m}$.*

Then the numbers $f_1(\alpha),\dots,f_m(\alpha)$ are algebraically independent.

Theorem 4. *Suppose that the polynomial $q(z)$ has algebraic coefficients (we assume that $|q(0)|>1$) and suppose that the conditions of Theorem 1 are fulfilled. Let α be an algebraic number, $|\alpha|<1$. In addition, let α be different from zeros of the polynomial $q(T^n z)$ for any nonnegative integer n.*

Then the numbers $f_1(\alpha),\dots,f_m(\alpha)$ are algebraically independent.

Proof. The assumptions made about the polynomial $q(z)$ and the number α ensure the definiteness of all terms of each of the series $f_1(\alpha),\dots,f_m(\alpha)$ and the convergence of these series.

In addition, according to Theorem 1, functions (1) are algebraically independent over $\mathbb{C}(z)$. The other assumptions of Theorem 3 are also satisfied. Using this theorem, we get the statement of Theorem 4.

Theorem 5. *Let $q(z)$ be a polynomial with algebraic coefficients, with $q(0) = 1$, and let the conditions of Theorem 1 be fulfilled. Suppose that α is an algebraic number, $|\alpha| < 1$. In addition, let α be different from zeros of the polynomial $q(T^n z)$ for any nonnegative integer n.*

Then the numbers $g_1(\alpha), \dots, g_m(\alpha)$ are algebraically independent.

Proof. From the assumptions made about the polynomial q and from the conditions of the theorem it follows that all infinite products, which define the functions $g_1, \dots, g_m$, converge. According to Theorem 2, these functions are algebraically independent. Using Theorem 3, we get the required.

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