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Third-Degree Diophantine Equations

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1. INTRODUCTION

This work is devoted to the investigation of the solvability of the Diophantine equation

$$x^3 + y^3 + z^3 = nxyz, \tag{1}$$

where $n \notin \{0, 3\}$, n is an integer. The first results were obtained as long ago as 1856 by Sylvester (see [1]). He proved that for $n = -6$ (1) had no solutions in nonzero integers.

The next results were obtained almost eighty years later. In 1933–34 Hurwitz and Mordell showed that for $n = -1$ and for $n = 5$ (1) had the only solutions $(1, 1, -1)$ and $(1, 1, 2)$, respectively, with an accuracy to within the permutation and to the common factor (see [2, 3]).

Developing Mordell's method, Cassels showed in 1960 that for $n = 1$ (1) had no solutions in nonzero integers (see [4, 5]).

Note that for the integer n (1) is solvable in nonzero positive integers x, y, z if and only if the equation

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = n \tag{2}$$

is solvable in positive integers.

Considering (2) for $n \geq 4$, Serpinsky notes in [6] that he does not know whether this equation has a solution in positive integers for $n = 4$. It follows from Theorem 1, Section 2 of this work that (2) has no solutions in the positive integers x, y, z not only for $n = 4$ but also for any values of n which are multiples of 4 as well as for many odd values of $n > 4$.

In Section 3 we consider various Diophantine equations which are connected, to a certain extent, with (1).

2. ON THE EQUATION $x^3 + y^3 + z^3 = nxyz$

Theorem 1. *Let $n \in \{4k, 8k-1, 2^{2m+1}(2k-1)+3\}$, where k and m run through all positive integers. Then* (1) *has no solutions in the positive integers x, y, z.*

Corollary. *For the values of n indicated in Theorem* 1 (2) *has no solutions in the positive integers x, y, z.*

In order to prove the theorem, we shall need some auxiliary statements which we shall formulate as lemmas. Lemma 1 belongs to Sylvester (see [1]).

Lemma 1. *Let A, B, C, D, α, β, γ be arbitrary real numbers for which*

$$A\alpha^3 + B\beta^3 + C\gamma^3 = D\alpha\beta\gamma.$$

Then

$$ABCf^3 + g^3 + h^3 = Dfgh,$$

where

$$h = A^2B\alpha^6\beta^3 + B^2C\beta^6\gamma^3 + C^2A\gamma^6\alpha^3 - 3ABC\alpha^3\beta^3\gamma^3,$$
$$g = AB^2\alpha^3\beta^6 + BC^2\beta^3\gamma^6 + CA^2\gamma^3\alpha^6 - 3ABC\alpha^3\beta^3\gamma^3,$$
$$f = \alpha\beta\gamma(A^2\alpha^6 + B^2\beta^6 + C^2\gamma^6 - AB\alpha^3\beta^3 - AC\alpha^3\gamma^3 - BC\beta^3\gamma^3).$$

Consider the functions

$$f(x,y,z) = xyz(x^6 + y^6 + z^6 - x^3y^3 - y^3z^3 - z^3x^3),$$
$$g(x,y,z) = x^3y^6 + y^3z^6 + z^3x^6 - 3x^3y^3z^3,$$
$$h(x,y,z) = x^6y^3 + y^6z^3 + z^6x^3 - 3x^3y^3z^3.$$

In this notation, Lemma 1 means that if (1) is valid, then

$$f^3(x,y,z) + g^3(x,y,z) + h^3(x,y,z) = nf(x,y,z)g(x,y,z)h(x,y,z).$$

Lemma 2. *Suppose that $n \neq 3$, n is an integer, and (x_1, y_1, z_1) is the solution of* (1). *Then*

(a) *if x_1, y_1, z_1 are positive integers, then $f(x_1,y_1,z_1)$, $g(x_1,y_1,z_1)$, $h(x_1,y_1,z_1)$ are also positive integers,*

(b) *if x_1, y_1, z_1 are nonzero integers, then $f(x_1,y_1,z_1)$, $g(x_1,y_1,z_1)$, $h(x_1,y_1,z_1)$ are also nonzero integers.*

Proof. Item (a) immediatly follows from the relations

$$u^3 + v^3 + w^3 - 3uvw = \frac{1}{2}(u+v+w)\big((u-v)^2 + (v-w)^2 + (w-u)^2\big),$$
$$u^2 + v^2 + w^2 - uv - vw - wu = \frac{1}{2}\big((u-v)^2 + (v-w)^2 + (w-u)^2\big).$$

Item (b) follows from the fact that if (1) is valid, where x, y, z, n are integers, then $(x,y) = (y,z) = (z,x) = (x,y,z)$. Therefore, if $xy^2 + yz^2 + zx^2 = 0$, then $|x| = |y| = |z|$, $x + y + z = 0$, but this is impossible for $xyz \neq 0$.

Let $n \neq 3$, (x_1, y_1, z_1) be a triple of nonzero integers satisfying (1), and $\mathrm{GCD}(x_1,y_1,z_1) = 1$ (hence, $(x_1,y_1) = (y_1,z_1) = (z_1,x_1) = 1$).

We define the sequence of the triples (x_i, y_i, z_i) as follows:

$$x_{k+1} = \frac{f_k}{(f_k, g_k, h_k)}, \qquad y_{k+1} = \frac{g_k}{(f_k, g_k, h_k)}, \qquad z_{k+1} = \frac{h_k}{(f_k, g_k, h_k)},$$

where

$$f_k = f(x_k, y_k, z_k), \qquad g_k = g(x_k, y_k, z_k), \qquad h_k = f(x_k, y_k, z_k),$$

and $(f_k, g_k, h_k) := \mathrm{GCD}(f_k, g_k, h_k)$.

It follows from what was said above that the following lemma is valid.

Lemma 3. *For any positive integer i the triple (x_i, y_i, z_i) is the solution of* (1) *and* $(x_i, y_i) = (y_i, z_i) = (z_i, x_i) = 1$.

Lemma 4. *For any positive integer i the congruence*

$$x_{i+1} \equiv 0 \ \big(\mathrm{mod}(x_i y_i z_i)\big)$$

holds true.

Proof. By definition,

$$(f_i, g_i, h_i) x_{i+1} = f_i = x_i y_i z_i (x_i^6 + y_i^6 + z_i^6 - x_i^3 y_i^3 - y_i^3 z_i^3 - z_i^3 x_i^3),$$

and therefore

$$(f_i, g_i, h_i) x_{i+1} \equiv 0 \ \big(\mathrm{mod}(x_i y_i z_i)\big)$$

and, all the more so,

$$g_i x_{i+1} \equiv 0 \ \big(\mathrm{mod}(x_i y_i z_i)\big).$$

If we assume that $x_{i+1} \not\equiv 0 \ \big(\mathrm{mod}(x_i y_i z_i)\big)$, then there exists a prime number p for which $x_i y_i z_i \equiv 0 \ (\mathrm{mod}\, p)$, $g_i \equiv 0 \ (\mathrm{mod}\, p)$. Suppose, for instance, that $x_i \equiv 0 \ (\mathrm{mod}\, p)$. Then it follows from

$$x_i^3 y_i^6 + y_i^3 z_i^6 + z_i^3 x_i^6 - 3 x_i^3 y_i^3 z_i^3 = g_i \equiv 0 \ (\mathrm{mod}\, p)$$

that $y_i^3 z_i^6 \equiv 0 \ (\mathrm{mod}\, p)$, i.e., either y_i or z_i is divisible by p, and this, together with $x_i \equiv 0 \ (\mathrm{mod}\, p)$, contradicts Lemma 3.

A simillar contradiction can be found in the case where $y_i \equiv 0 \ (\mathrm{mod}\, p)$ or $z_i \equiv 0 \ (\mathrm{mod}\, p)$.

Lemma 5. *If $i \neq j$, then*

$$(y_i z_i,\ y_j z_j) = 1. \tag{3}$$

Proof. Indeed, suppose, for definiteness, that $i < j$. Then Lemma 4 yields $x_j \equiv 0 \ \big(\mathrm{mod}(y_i z_i)\big)$. According to Lemma 3, $(x_j, y_j z_j) = 1$. Therefore relation (3) holds true.

Lemma 6. *For any positive integer N there exists a number i such that $(y_i z_i, N) = 1$.*

Proof. Indeed, let $N = p_1^{k_1} \cdots p_s^{k_s}$ be a prime factorization of the number N. The statement of the lemma follows from the fact that the terms of the sequence of $s+1$ numbers $y_1 z_1, \dots, y_{s+1} z_{s+1}$ are, according to Lemma 5, pairwise prime numbers.

Thus the following fundamental lemma is valid.

Fundamental lemma. *Let n be an integer and let* (1) *be solvable in nonzero positive integers. Then, for any positive integer N, there exists a triple (x_N, y_N, z_N) of nonzero positive integers which is the solution of* (1) *and which satisfies the conditions*

$$(x_N, y_N) = (y_N, z_N) = (z_N, x_N) = (y_N z_N, N) = 1.$$

Lemma 7. *Let n be an integer, $n \neq 3$, and let x, y, z be nonzero pairwise prime integers satisfying relation* (1). *Suppose, furthermore, that $d = (x+y, z)$, i.e.,*

$$x + y = da, \quad z = db, \quad (a, b) = 1. \tag{4}$$

Then there exists an integer α such that

$$n^3 - 27 \equiv 0 \ (\operatorname{mod} \alpha), \tag{5}$$

$$(\alpha, a) = (\alpha, b) = 1, \tag{6}$$

$$4 \equiv 0 \ \big(\operatorname{mod} (\alpha, nb - a)\big), \tag{7}$$

$$\alpha\big(4z^2 + n(x-y)^2\big) = a(n^2ab - na^2 + 12b^2), \tag{8}$$

$$\alpha\big(z^2 + n(x-y)^2\big) = (nb - a)(na^2 - 3b^2). \tag{9}$$

When x, y, z are positive integers, the numbers α, a, b, $nb - a$ are also positive integers.

Proof. According to the hypothesis,

$$(x+y)^3 + z^3 = xy(3(x+y) + nz), \quad x + y = da, \quad z = db, \quad (a, b) = 1, \quad d = (x+y, z),$$

whence it follows that

$$d^2(a^3 + b^3) = (3a + nb)xy.$$

Since $(x, y) = (x + y, xy) = 1$, we note that $(d^2, xy) = 1$. Therefore there exists an integer α such that

$$\begin{cases} 3a + nb = \alpha d^2 \\ a^3 + b^3 = \alpha xy. \end{cases} \tag{10}$$

We write the second relation of system (10) in the form

$$4a^3 + 4b^3 = 4\alpha xy = \alpha(x+y)^2 - \alpha(x-y)^2 = \alpha d^2 a^2 - \alpha(x-y)^2 = (3a + nb)a^2 - \alpha(x-y)^2.$$

Consequently,

$$\begin{cases} \alpha d^2 = 2a + nb \\ \alpha(x-y)^2 = na^2 b - a^3 - 4b^3, \end{cases}$$

whence, after elementary transformations, we get (8), (9).

We can see from system (10) that (6) is satisfied for α and

$$(n^3 - 27)b^3 \equiv 0 \ (\operatorname{mod} \alpha),$$

i.e., (5) is satisfied as well.

Next it follows, also from (10), that $4a$ is divisible by $(\alpha, nb - a)$. Together with $(\alpha, a) = 1$ this means that relation (7) holds true.

If x, y, z are positive integers, then d, a, b are positive integers, and therefore α is also a positive integer ($n \geq 4$ from (10)) In addition, in this case we have

$$d(nb - a) = nz - (x+y) = \frac{x^3 + y^3 + z^3}{xy} - (x+y) = \frac{(x+y)(x-y)^2 + z^3}{xy} > 0,$$

i.e., $nb - a > 0$. We proved the lemma.

Proof of Theorem 1. We shall carry out the proof by contradiction. Let (1) be solvable in positive integers x, y, z for a certain n from the hypothesis of the theorem. Taking $N = 2n$ in the fundamental lemma, we find that (1) is solvable in positive integers which we shall also denote by x, y, z and which satisfy the conditions

$$(x, y) = (y, z) = (z, x) = 1, \qquad (yz, 2n) = 1.$$

Using Lemma 7, we have (4), and there exists a positive integer α for which (5)–(7) are satisfied and

$$\alpha\bigl(4z^2 + n(x - y)^2\bigr) \equiv 0 \pmod a, \tag{11}$$

$$\alpha\bigl(z^2 + n(x - y)^2\bigr) \equiv 0 \bigl(\mathrm{mod}(nb - a)\bigr), \tag{12}$$

$$(yz, 2n) = 1. \tag{13}$$

It is easy to note that $(n, a) = (n, nb - a) = 1$. Indeed, if p is a prime number or unity and is such that $n \equiv 0 \pmod p$, $a \equiv 0 \pmod p$, then

$$x + y = da \equiv 0 \pmod p, \tag{14}$$

and $z \equiv 0 \pmod p$ according to (1) . From (13) we find that $p = 1$. This means that $(n, nb - a) = (n, a) = 1$.

We divide the possible values of n into three variants, namely,

(1) $n = 4k$,

(2) $n = 8k - 1$,

(3) $n = 2^{2m+1}(2k - 1) + 3$.

(1) Since $(\alpha, a) = 1$, it follows from (11) that

$$4z^2 + n(x - y)^2 \equiv 0 \pmod a. \tag{15}$$

It follows from (5) that α is an odd number, and therefore conditions (7) and (12) mean that

$$z^2 + n(x - y)^2 \equiv 0 \bigl(\mathrm{mod}(nb - a)\bigr), \tag{16}$$

It follows from (13) that y, z are odd numbers and (1) implies the evenness of the number x ($n = 4k$). All this means that the numbers a and b are odd.

Let us show that

$$(x - y, a) = (x - y, nb - a) = 1. \tag{17}$$

Indeed, if p is a prime number or unity and $x - y \equiv 0 \pmod p$, $a \equiv 0 \pmod p$, then relation (14) is satisfied, whence it follows that $2x \equiv 0 \pmod p$, $2y \equiv 0 \pmod p$. Since $(x, y) = 1$, we get either $p = 2$ or $p = 1$. The case $p = 2$ cannot hold since a is an odd number. Only the case $p = 1$ remains. This means that $(x - y, a) = 1$.

Now if $x - y \equiv 0 \pmod p$, $nb - a \equiv 0 \pmod p$, then we find from (16) that $z \equiv 0 \pmod p$, $x^3 + y^3 = nxyz - z^3 \equiv 0 \pmod p$. Therefore $2x^3 \equiv 0 \pmod p$, $2y^3 \equiv 0 \pmod p$, i.e., either $p = 2$ or $p = 1$. Since z is odd, only $p = 1$ remains.

We have thus proved (17). Taking this into account in congruences (15) and (16), we find that there exist positive integers z_1 and z_2 for which

$$z_1^2 \equiv -n \pmod{a}, \qquad z_2^2 \equiv -n \bigl(\operatorname{mod}(nb-a)\bigr).$$

Next, $n = 4k$, a is odd, $nb - a$ is odd, $(n, a) = (n, nb - a) = 1$. Therefore

$$\left(\frac{-n}{a}\right) = \left(\frac{-n}{nb-a}\right) = 1,$$

where $\left(\frac{p}{q}\right)$ is Jacobi's symbol.

Let us write n as $n = 2^m n_0$, $m \geq 2$, where m is a positive integer and n_0 is an odd positive integer,

$$1 = \left(\frac{-n}{a}\right) = (-1)^{(a-1)/2}\left(\frac{2^m n_0}{a}\right) = (-1)^{(a-1)/2}(-1)^{(a^2-1)m/8}\left(\frac{n_0}{a}\right).$$

Therefore

$$\left(\frac{a}{n_0}\right) = (-1)^{(a-1)/2}(-1)^{(a^2-1)m/8}\left(\frac{n_0}{a}\right)\left(\frac{a}{n_0}\right) = (-1)^{(a-1)/2+(a^2-1)m/8+(a-1)(n_0-1)/2}.$$

Similarly, we find from $\left(\frac{-n}{nb-a}\right) = 1$ that

$$\begin{aligned}\left(\frac{nb-a}{n_0}\right) &= (-1)^{(nb-a-1)/2+((nb-a)^2-1)m/8+(nb-a-1)(n_0-1)/2}\\ &= (-1)^{(-a-1)/2+(-2^{m+1}n_0ab+a^2-1)m/8-(a+1)(n_0-1)/2}.\end{aligned}$$

Taking into account that $2^{m+1}m/8$ is an odd number for $m \geq 2$, $n \equiv 0 \pmod{n_0}$, we have

$$\begin{aligned}(-1)^{(n_0-1)/2} &= \left(\frac{a}{n_0}\right)\left(\frac{nb-a}{n_0}\right)\\ &= (-1)^{(a-1)/2+(a^2-1)m/8+(a-1)(n_0-1)/2-(a+1)/2+(a^2-1)m/8-(a+1)(n_0-1)/2}\\ &= (-1)^{1+(n_0-1)/2} = -(-1)^{(n_0-1)/2},\end{aligned}$$

i.e., $(-1)^{(n_0-1)/2} = -(-1)^{(n_0-1)/2}$.

We have obtained a contradiction which proves the theorem for case (1).

(2) Let $a = 2^{s_1}a_1$, $nb - a = 2^{s_2}b_1$, where $s_1 \geq 0$, $s_2 \geq 0$ are integers and a_1, b_1 are odd positive integers. From (6), (7), (11), (12) we obtain

$$4z^2 + n(x-y)^2 \equiv 0 \pmod{a_1}, \qquad z^2 + n(x-y)^2 \equiv 0 \pmod{b_1}.$$

Just as in case (1), it is easy to show that $(x - y, a_1) = (x - y, b_1) = 1$. Taking into account that $(n, a_1) = (n, b_1) = 1$, we again have

$$\left(\frac{-n}{a_1}\right) = \left(\frac{-n}{b_1}\right) = 1,$$

$$\left(\frac{a_1}{n}\right) = \left(\frac{-n}{a_1}\right)\left(\frac{a_1}{n}\right) = (-1)^{(a_1-1)/2}(-1)^{((a_1-1)/2)((n-1)/2)} = (-1)^{(a_1-1)/2}(-1)^{(a_1-1)/2} = 1,$$

i.e., $\left(\frac{a_1}{n}\right) = 1$.

Similarly, $\left(\frac{b_1}{n}\right) = 1$. Since $n = 8k - 1$, it follows that $\left(\frac{2}{n}\right) = (-1)^{(n^2-1)/8} = 1$. Therefore

$$\left(\frac{a}{n}\right) = \left(\frac{2^{s_1}a_1}{n}\right) = 1, \qquad \left(\frac{nb-a}{n}\right) = \left(\frac{2^{s_2}b_1}{n}\right) = 1,$$

whence it follows that

$$1 = \left(\frac{a}{n}\right)\left(\frac{nb-a}{n}\right) = (-1)^{(n-1)/2}\left(\frac{a}{n}\right)\left(\frac{a}{n}\right) = -1.$$

The contradiction obtained proves the theorem for case (2).

(3) We shall first prove that the number x is even.

Let us assume that this is not the case. Then, since $(yz, 2n) = 1$, all numbers x, y, z are odd. We rewrite (1) in the form

$$(x+y+z)\frac{(2x-y-z)^2 + 3(y-z)^2}{4} = 2^{2m+1}(2k-1)xyz.$$

Since the numbers $x+y+z$ and xyz are odd, we have

$$u^2 + 3v^2 \equiv 0 \ (\operatorname{mod} 2^{2m+3}), \qquad u^2 + 3v^2 \equiv 0 \ (\operatorname{mod} 2^{2m+4}),$$

where $u = 2x - y - z$, $v = y - z$. It is easy to see that this is impossible for any integers u, v and any positive integers m.

Thus x is an even number and yz is an odd number. Next, it follows from the relation

$$(x+y+z)(x^2+y^2+z^2-xy-yz-zx) = 2^{2m+1}(2k-1)xyz$$

that $x + y + z \equiv 0 \ (\operatorname{mod} 16)$.

Therefore, since z and d are odd, we find from the relation

$$d(nb-a) = nz - (x+y) = \left(2^{2m+1}(2k-1) + 4\right)z - (x+y+z)$$

that the number $nb - a$, which is divisible by 4, is not divisible by 8.

Thus a is an odd number and $nb - a = 4b_1$, where b_1 is also an odd number.

Just as in cases (1) and (2), we have

$$(\alpha, a) = (\alpha, b_1) = 1, \qquad (x-y, a) = (x-y, b_1) = 1, \qquad (n, a) = (n, b_1) = 1$$

and

$$\left(\frac{-n}{a}\right) = \left(\frac{-n}{b_1}\right) = 1,$$

$$\left(\frac{a}{n}\right) = \left(\frac{-n}{a}\right)\left(\frac{a}{n}\right) = (-1)^{(a-1)/2}(-1)^{((a-1)/2)((n-1)/2)} = (-1)^{(a-1)/2}(-1)^{(a-1)/2} = 1,$$

i.e., $\left(\frac{a}{n}\right) = 1$. Similarly, $\left(\frac{b_1}{n}\right) = 1$.

Therefore

$$-1 = (-1)^{(n-1)/2} = \left(\frac{-1}{n}\right) = \left(\frac{-a}{n}\right) = \left(\frac{nb-a}{n}\right) = \left(\frac{4b_1}{n}\right) = \left(\frac{b_1}{n}\right) = 1.$$

The contradiction obtained proves the theorem in case (3). This completes the proof of Theorem 1.

The corollary can be proved either with the use of Lemma 1 or in any other way.

3. ON THE DIOPHANTINE EQUATION $x^3 + y^3 + 1 = nxy$

Lemma 8. *If the relation*

$$x_0^3 + y_0^3 + 1 = n_0 x_0 y_0$$

is satisfied for the positive integers x_0, y_0, n_0, then

$$x_1^3 + y_1^3 + 1 = n_1 x_1 y_1, \tag{18}$$

where

$$x_1 = n_0 x_0 - y_0^2, \qquad y_1 = x_0, \qquad n_1 = n_0^2 x_0 - n_0 y_0^2 - y_0 x_0^2,$$

where x_1, y_1, z_1 are positive integers. In this case, if $x_0 \geq y_0$, then $x_1 > y_1$, $n_1 > n_0$.

This lemma can be proved by means of simple calculations.

According to Theorem 1, if $n \in \{4k, 8k-1, 2^{2m+1}(2k-1)+3\}$, where k and m run through all positive integers, then the equation

$$x^3 + y^3 + 1 = nxy \tag{19}$$

has no solutions in positive integers.

Theorem 2. *For any $i \in \{1, 2, 3, 5, 6\}$ there exists an infinite number of positive integers n of the form $8k + i$ for which (19) has a solution in the positive integers x, y.*

Proof. Let $(x, y, n) = (x_0, y_0, n_0)$ be triples of positive integers satisfying (19), $x_0 \geq y_0$. We shall define the sequence (x_j, y_j, n_j) of triples of positive integers in accordance with Lemma 8:

$$x_j = n_{j-1} x_{j-1} - y_{j-1}^2, \quad y_j = x_{j-1}, \quad n_j = n_{j-1}^2 x_{j-1} - n_{j-1} y_{j-1}^2 - y_{j-1} x_{j-1}^2, \qquad j = 1, 2, \dots .$$

According to Lemma 8,

$$x_j^3 + y_j^3 + 1 = n_j x_j y_j, \qquad j = 1, 2, \dots, \quad n_0 < n_1 < n_2 < \cdots .$$

We take $x_0 = 3$, $y_0 = 2$, $n_0 = 6$ and write the remainders of the division of the sequence of triples (x_j, y_j, n_j) by 8 in a row. Obviously, the remainders can be calculated from the same formulas which define the sequence (x_j, y_j, n_j)

$$(3, 2, 6) \to (6, 3, 2) \to (3, 6, 2) \to (2, 3, 6) \to (3, 2, 6) \to \cdots .$$

We see that the remainders are repeated, and we have

$$n_{4j} \equiv n_{4j-1} \equiv 6 \pmod 8, \qquad n_{4j+1} \equiv n_{4j+2} \equiv 2 \pmod 8$$

for any positive integer j.

And now we take $x_0 = 1$, $y_0 = 1$, $n_0 = 3$ and carry out the same procedure. We get

$$(1, 1, 3) \to (2, 1, 5) \to (1, 2, 1) \to (5, 1, 3) \to (6, 5, 1) \to (5, 6, 1)$$
$$\to (1, 5, 3) \to (2, 1, 1) \to (1, 2, 5) \to (1, 1, 3) \to \cdots .$$

In this case we have

$$n_{9j} \equiv n_{9j+3} \equiv n_{9j+6} \equiv 3 \pmod 8, \qquad n_{9j+1} \equiv n_{9j+8} \equiv 5 \pmod 8,$$
$$n_{9j+2} \equiv n_{9j+4} \equiv n_{9j+5} \equiv n_{9j+7} \equiv 1 \pmod 8.$$

We have proved the theorem.

Theorem 3. *The equation*

$$x^3 + 64y^3 + 1 = 4xyz \tag{20}$$

has no solutions in positive integers x, y, z.

Proof. We shall carry out the proof by contradiction. Let the positive integers x, y, z be such that (20) is satisfied. According to Lemma 8, relation (18) holds true, where

$$x_1 = zx - 16y^2, \qquad y_1 = x, \qquad n_1 = z^2x - 16zy^2 - 4yx^2,$$

x_1, y_1, n_1 are positive integers.

If z is an even number, then n_1 is divisible by 4, and in that case relation (18) contradicts Theorem 1. Therefore z is an odd number. Then $z^2 \equiv 1 \pmod 8$. This means, by virtue of the oddness of x, that $n_1 \equiv x - 4y \pmod 8$.

It follows from (20) that $x^3 + 1 \equiv 4y \pmod 8$. Hence,

$$n_1 \equiv x - 4y \equiv x - x^3 - 1 \equiv -1 \pmod 8,$$

i.e., $n_1 \equiv -1 \pmod 8$.

The last congruence, together with relation (18), contradicts Theorem 1.

Theorem 4. *The equation*

$$t^2 = 4xyzuv - xu^2 - yv^2 \tag{21}$$

has no solutions in positive integers t, x, y, z, u, v.

Proof. We carry out the proof by contradiction. Let the positive integers t, x, y, z, u, v be such that (21) is satisfied.

Obviously, we can assume, without loss of generality, that $(u, v) = 1$. Then one of the numbers u, v is odd.

Let u be an odd number. We denote by d the greatest common divisor of the numbers u and y, i.e., let

$$u = du_1, \qquad y = dy_1, \qquad (u_1, y_1) = 1, \qquad t^2 = d(4xy_1zdu_1v - xdu_1^2 - y_1v^2).$$

We set $a = (d, y_1)$, i.e.,

$$d = ad_1, \qquad y_1 = ay_2, \qquad (d_1, y_2) = 1.$$

We have

$$t^2 = a^2d_1(4xy_2zad_1u_1v - xd_1u_1^2 - y_2v^2),$$

$$(u, v) = 1 \implies (d_1, v) = 1,\ (d_1, y_2) = 1,\ (d_1, 4xy_2zad_1u_1v - xd_1u_1^2 - y_2v^2) = 1$$

$$\implies d_1 = d_2^2, 4xy_2zad_2^2u_1v - xd_2^2u_1^2 - y_2v^2 = t_1^2,$$

where d_2, t_1 are certain positive integers.

We denote $d_2u_1 = u_2$, $zad_2 = z_1$ and obtain

$$t_2^2 = 4xy_2z_1u_2v - xu_2^2 - y_2v^2.$$

In addition to the fact that u_2 is an odd number, $(u_2, v) = 1$, we have obtained the equality $(u_2, y_2) = 1$.

Next,

$$t_2^2 \equiv -y_2 v^2 \ \bigl(\operatorname{mod}\bigl(u_2(4y_2z_1v - u_2)\bigr)\bigr),$$

whence it follows that

$$\left(\frac{-y_2}{u_2}\right) = \left(\frac{-y_2}{4y_2z_1v - u_2}\right) = 1.$$

Reasoning as in case (1) of Section 2, we arrive at a contradiction.

Corollary. *Let the integers a, b, c, d be such that*

$$ab \equiv c \equiv d \equiv 1 \ (\operatorname{mod} 4).$$

Then the equation

$$(a^2x)^3 + (b^2y)^3 + 1 = ab(cx + dy)xy \tag{22}$$

has no solutions in positive integers.

Proof. Indeed, if this is not the fact, i.e., if the positive integers x, y satisfy (22), then

$$(a^2x)^3 + (b^2y)^3 + 1 \equiv 0 \ \bigl(\operatorname{mod}(4abxy)\bigr),$$

Hence, denoting $t = (a^2x + b^2y)^2$, we obtain

$$t^2 \equiv -xa^2 - yb^2 \ \bigl(\operatorname{mod}(4abxy)\bigr),$$

and this contradicts Theorem 4.

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REFERENCES

1. Dickson, L. E., *History of the Theory of Numbers*, vol. 2, New York, 1934.
2. Hurwitz, A., Zahlentheorie, Algebra und Geometrie, in: *Mathematische Werke*, Basel, 1933, vol. 2.
3. Mordell, L. J., The Diophantine Equation $x^3 + y^3 + z^3 + kxyz = 0$, in: *Colloque sur la theorie des nombers*, Bruxelles, 1955, pp. 67–76.
4. Cassels, J. W. S., On a Diophantine Equation, *Acta Arith.*, 1960, vol. 6, pp. 47–51.
5. Serpinskii, V., *250 zadach po elementarnoi teorii chisel* (*250 Problems in Elementary Number Theory*), Moscow: Prosveshchenie, 1968.

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