Proceedings of the Steklov Institute of Mathematics, Vol. 218, 1997, pp. 104–116.
Translated from Trudy Matematicheskogo Instituta imeni V. A. Steklova, Vol. 218, 1997, pp. 109–121.

Limit Theorem with Weight for the Lerch Zeta Function in the Space of Analytic Functions

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Received February, 1997

Let $s = \sigma + i\tau$ be a complex variable and $L(\lambda, \alpha, s)$ denote *the Lerch zeta function* that, for $\sigma > 1$, is defined by the Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}$$

and outside of the plane $\sigma > 1$ by means of an analytic continuation. Here $\lambda \in \mathbb{R}$, $0 < \alpha \leq 1$ are certain fixed parameters ($\mathbb{R}$ is the set of real numbers). As usual, $\mathbb{Z}$ is the set of integers. The function $L(\lambda, \alpha, s)$ was introduced in the work [8], where it was found that for $\lambda \notin \mathbb{Z}$ it was an entire function. To be more precise, it was shown that for all s

$$L(\lambda, \alpha, s) = e^{-s\pi i}\Gamma(1-s)\frac{1}{2\pi i}K(\lambda, \alpha, s),$$

where, as usual, $\Gamma(s)$ denotes Euler's gamma-function, and

$$K(\lambda, \alpha, s) = \int_L \frac{e^{-\lambda z} z^{s-1}\, dz}{1 - e^{2\pi i \alpha - z}}.$$

The circuit L begins and terminates at infinity bypassing the point $z = 0$ along a circle. The points $2\pi i(\alpha + k)$, $k \in \mathbb{Z}$, do not lie on the circle L or inside it. The article [8] also contains a functional equation for the function $L(\lambda, \alpha, s)$.

When $\lambda \in \mathbb{Z}$, the Lerch zeta-function obviously becomes Hurwitz zeta-function $\zeta(s, \alpha)$, and when $\lambda \in \mathbb{Z}$ and $\alpha = 1$, it become Riemannian zeta-function $\zeta(s)$. Thus the Lerch zeta-function is a generalization of the well-known classical functions.

The asymptotics of the moments of Lerch zeta-function was studied by Klusch [5, 6] and the asymptotics of the quantity

$$\int_0^1 |L(\lambda, \alpha, s) - \alpha^{-s}|^2\, d\alpha$$

was studied by Zhang [10] and Katsurada [4]. We became interested in the last work and began to investigate in [2] the values of the function $L(\lambda, \alpha, s)$ by statistical methods, where, in the case of $\lambda \notin \mathbb{Z}$, proved the limit theorem for $L(\lambda, \alpha, s)$ in the complex plane $\mathbb{C}$. The aim of this article is to obtain the limit theorem for $L(\lambda, \alpha, s)$ in the space of analytic functions.

Let $D = \{s \in \mathbb{C} : \sigma > 1/2\}$ and let $H(D)$ denote the space of functions analytic in D with the topology of uniform convergence on compacts. We denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Let T_0 be a fixed positive number and let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$ such that its variation $V_a^b w$ satisfies the inequality $V_a^b w \le c w(a)$ with a certain constant $c > 0$ for all $b > a \ge T_0$. We set

$$U = U(T, w) = \int_{T_0}^{T} w(t)\, dt$$

and assume that

$$\lim_{T \to \infty} U(T, w) = \infty.$$

Let us denote the probability measure

$$P_{T,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : L(\lambda, \alpha, s + i\tau) \in A\}}\, d\tau, \qquad A \in \mathcal{B}\big(H(D)\big),$$

where I_A is the indicator of the set A. We assume, in what follows, that $\lambda \notin \mathbb{Z}$ and α is a transcendental number.

Theorem. *There exists on $\big(H(D), \mathcal{B}(H(D))\big)$ a probability measure P_w such that the measure $P_{T,w}$ weakly converges to P_w as $T \to \infty$.*

A limit theorem of this kind for the Riemannian zeta-function was obtained in [7].

We preface the proof of the theorem with several lemmas. Note that the system of numbers $\{\log(m + \alpha),\ m = 0, 1, 2, \dots\}$ is linearly independent over the field of rational numbers. Only in this statement do we use the transcendence of the number α.

We begin with proving the limit theorem for the Dirichlet polynomial

$$p_n(s, \alpha) = \sum_{m=0}^{n} \frac{a_m}{(m + \alpha)^s},$$

where $a_m \in \mathbb{C}$. When proving this theorem, we shall use the following properties of the weak convergence of measures.

Let S and S_1 be two metric spaces and let $h \colon S \to S_1$ be a measurable function. Then every probability measure P on $\big(S, \mathcal{B}(S)\big)$ induces on $\big(S_1, \mathcal{B}(S_1)\big)$ the unique probability measure Ph^{-1} defined by the relation $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_1)$.

Lemma 1. *Let $h \colon S \to S_1$ be a continuous function and P_n and P be probability measures on $\big(S, \mathcal{B}(S)\big)$. If P_n weakly converges to P as $n \to \infty$, then $P_n h^{-1}$ also weakly converges to Ph^{-1} as $n \to \infty$.*

Proof. The lemma is a corollary of Theorem 5.1 from [1].

Suppose that $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ and P is a probability measure on $\big(\gamma^r, \mathcal{B}(\gamma^r)\big)$, where r is a positive integer. The Fourier transform $g(k_1, \dots, k_r)$ of the measure P is defined by the relation

$$g(k_1, \dots, k_r) = \int_{\gamma^r} x_1^{k_1} \cdots x_r^{k_r}\, dP,$$

where $k_j \in \mathbb{Z}$, $x_j \in \gamma$, $j = 1, \dots, r$.

Lemma 2. *Let $\{P_n\}$ be a sequence of probability measures on $\big(\gamma^r, \mathcal{B}(\gamma^r)\big)$ and $\{g_1(k_1, \dots, k_r)\}$ be the sequence of the corresponding Fourier transforms. We assume that for every collection of integers $(k_1, \dots, k_r)$ there exists a limit*

$$g(k_1, \dots, k_r) = \lim_{n\to\infty} g_n(k_1, \dots, k_r).$$

Then there exists on $\big(\gamma^r, \mathcal{B}(\gamma^r)\big)$ a probability measure P such that P_n weakly converges to P as $n \to \infty$. In this case, $g(k_1, \dots, k_r)$ is a Fourier transform of the measure P.

Proof. The lemma is a special case of the continuity theorems for compact Abelian groups, see, e.g., [11].

Let us define on $\big(H(D), \mathcal{B}(H(D))\big)$ the probability measure

$$P_{T,n,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : p_n(s+i\tau,\alpha)\in A\}}\, d\tau.$$

Lemma 3. *There exists on $\big(H(D), \mathcal{B}(H(D))\big)$ a probability measure $P_{n,w}$ such that the measure $P_{T,n,w}$ weakly converges to $P_{n,w}$ as $T \to \infty$.*

Proof. Let

$$\Omega_n = \prod_{m=0}^{n} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m = 0, 1, \dots, n$. We define the function $h\colon \Omega_n \to H(D)$ by the relation

$$h(x_0, \dots, x_n) = \sum_{m=0}^{n} \frac{a_m}{(m+\alpha)^s x_m}, \qquad (x_0, \dots, x_n) \in \Omega_n.$$

Then the function h is continuous, and we have the relation

$$p_n(s+i\tau, \alpha) = h\big(\alpha^{i\tau}, (1+\alpha)^{i\tau}, \dots, (n+\alpha)^{i\tau}\big). \tag{1}$$

Let us define the probability measure

$$Q_{T,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau : (\alpha^{i\tau}, (1+\alpha)^{i\tau}, \dots, (n+\alpha)^{i\tau}) \in A\}}\, d\tau, \qquad A \in \mathcal{B}(\Omega_n),$$

and denote by $g_T(k_0, \dots, k_n)$, $k_j \in \mathbb{Z}$, $j = 0, \dots, n$, its Fourier transform. Then we have

$$g_T(k_0, \dots, k_n) = \int_{\Omega_n} x_0^{k_0} \cdots x_n^{k_n}\, dQ_{T,w} = \frac{1}{U} \int_{T_0}^{T} w(\tau) \prod_{m=0}^{n} (m+\alpha)^{i\tau k_m}\, d\tau.$$

Integrating by parts and using the properties of the function $w(\tau)$ and the linear independence of the system $\{\log(m+\alpha),\ m=0,1,\dots\}$, we find that

$$\int_{T_0}^{T} w(\tau)\prod_{m=0}^{n}(m+\alpha)^{i\tau k_m}\,d\tau=\Big(i\sum_{m=0}^{n}k_m\log(m+\alpha)\Big)^{-1}\int_{T_0}^{T} w(\tau)\,d\exp\Big\{i\tau\sum_{m=0}^{n}k_m\log(m+\alpha)\Big\}$$

$$=B\Big|\sum_{m=0}^{n}k_m\log(m+\alpha)\Big|^{-1}-\Big(i\sum_{m=0}^{n}k_m\log(m+\alpha)\Big)^{-1}\int_{T_0}^{T}\exp\Big\{i\tau\sum_{m=0}^{n}k_m\log(m+\alpha)\Big\}\,dw(\tau)$$

$$=B\Big|\sum_{m=0}^{n}k_m\log(m+\alpha)\Big|^{-1}$$

for $(k_0,\dots,k_n)\neq(0,\dots,0)$. Here and in the sequel B_η denotes a quantity (not always the same) bounded modulo by a constant dependent on the parameter η. Thus

$$g_T(k_0,\dots,k_n)=\begin{cases}1, & (k_0,\dots,k_n)=(0,\dots,0)\\ B\Big(U\Big|\sum_{m=0}^{n}k_m\log(m+\alpha)\Big|\Big)^{-1}, & (k_0,\dots,k_n)\neq(0,\dots,0).\end{cases}$$

Consequently,

$$\lim_{T\to\infty}g_T(k_0,\dots,k_n)=\begin{cases}1, & (k_0,\dots,k_n)=(0,\dots,0)\\ 0, & (k_0,\dots,k_n)\neq(0,\dots,0).\end{cases}$$

By virtue of Lemma 2, we find that the measure $Q_{T,w}$ weakly converges to a certain measure m_n on $\big(\Omega_n,\mathcal{B}(\Omega_n)\big)$ as $T\to\infty$ (the measure m_n is the Haar measure). Therefore we find from Lemma 1, the continuity of the function h, and relation (1) that the measure $P_{T,n,w}$ weakly converges to m_nh^{-1} as $T\to\infty$.

We can assume, without loss of generality, that $0<\lambda<1$. In order to approximate the function $L(\lambda,\alpha,s)$ in the mean by the absolutely convergent Dirichlet series, we shall need the following two lemmas. The first of them is an analog of the approximation of the Riemannian zeta-function by a finite sum, see, for instance, [3].

Lemma 4. *Let $0<\sigma_0\le\sigma\le2$ and $|t|\le\pi\lambda x$. Then*

$$L(\lambda,\alpha,s)=\sum_{0\le m\le x}\frac{e^{2\pi i\lambda m}}{(m+\alpha)^s}+B_{\sigma_0}x^{-\sigma}.$$

The **proof** of the lemma for the case $0<\sigma_0\le\sigma\le1$, $0\le t\le\pi\lambda x$ is given in [2]. However, it is easy to see that it remains valid in the present formulation.

Lemma 5. *Let $\sigma>1/2$ be a fixed number. Then, for every $\tau\in\mathbb{R}$, we have*

$$\int_{T_0+\tau}^{T+\tau} w(t-\tau)|L(\lambda,\alpha,\sigma+it)|^2\,dt=B_{\lambda,\alpha}U(1+|\tau|)^2.$$

Proof. It is obvious that for $\sigma > 2$ the integral of the lemma is estimated as BU. Therefore we shall only consider the case $\sigma \le 2$. Setting $x = (t + |\tau|)\lambda^{-1}$ in Lemma 4, we obtain

$$\int\limits_{T_0+\tau}^{T+\tau} w(t-\tau)|L(\lambda,\alpha,\sigma+it)|^2\,dt = B\int\limits_{T_0}^{T} w(t)\Bigg|\sum_{m\le(t+|\tau|)\lambda^{-1}} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^{\sigma+it+i\tau}}\Bigg|^2\,dt + B_\lambda U. \tag{2}$$

Let $\max(m,k) = (T_1 + |\tau|)\lambda^{-1}$, where $T_1 = T_1(m,k)$. Then

$$\begin{aligned}
&\int\limits_{T_0}^{T} w(t)\Bigg|\sum_{m\le(t+|\tau|)\lambda^{-1}} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^{\sigma+it+i\tau}}\Bigg|^2\,dt \\
&= \sum\sum_{\substack{(T_0+|\tau|)\lambda^{-1}<m\le(T+|\tau|)\lambda^{-1}\\(T_0+|\tau|)\lambda^{-1}<k\le(T+|\tau|)\lambda^{-1}}} \frac{e^{2\pi i\lambda(m-k)}}{(m+\alpha)^{\sigma+i\tau}(k+\alpha)^{\sigma-i\tau}}\int\limits_{T_1}^{T} w(t)\left(\frac{k+\alpha}{m+\alpha}\right)^{it}dt + B_\lambda U(1+|\tau|) \\
&= \sum_{(T_0+|\tau|)\lambda^{-1}<m\le(T+|\tau|)\lambda^{-1}} \frac{1}{(m+\alpha)^{2\sigma}}\int\limits_{T_1}^{T} w(t)\,dt \\
&\quad + B\sum\sum_{(T_0+|\tau|)\lambda^{-1}<m<k\le(T+|\tau|)\lambda^{-1}} \frac{w(k-|\tau|)}{(m+\alpha)^{\sigma}(k+\alpha)^{\sigma}\log\frac{k+\alpha}{m+\alpha}} + B_\lambda U(1+|\tau|) \\
&= B_\lambda U(1+|\tau|) + B\sum\sum_{T_0\lambda^{-1}<m<k\le T\lambda^{-1}} \frac{w(k)}{(m+|\tau|)^{\sigma}(k+|\tau|)^{\sigma}\log\frac{k+|\tau|+\alpha}{m+|\tau|+\alpha}}.
\end{aligned} \tag{3}$$

First suppose that $m + |\tau| + \alpha < (k + |\tau| + \alpha)/2$. Then

$$\log\frac{k+|\tau|+\alpha}{m+|\tau|+\alpha} > \log 2,$$

and, consequently,

$$\begin{aligned}
\sum\sum_{\substack{T_0\lambda^{-1}<m<k\le T\lambda^{-1}\\ m+|\tau|+\alpha<(k+|\tau|+\alpha)/2}} \frac{w(k)}{(m+|\tau|)^{\sigma}(k+|\tau|)^{\sigma}\log\frac{k+|\tau|+\alpha}{m+|\tau|+\alpha}} &= B\sum_{T_0\lambda^{-1}<m<k\le T\lambda^{-1}} \frac{w(k)}{m^\sigma k^\sigma} \\
&= B\sum_{T_0\lambda^{-1}<k\le T\lambda^{-1}} w(k) = B_\lambda U,
\end{aligned} \tag{4}$$

since

$$\begin{aligned}
\sum_{T_0\lambda^{-1}<k\le T\lambda^{-1}} w(k) &= \int\limits_{T_0\lambda^{-1}}^{T\lambda^{-1}} w(\omega)\,d[u] = B_\lambda + w(T\lambda^{-1})[T\lambda^{-1}] - \int\limits_{T_0\lambda^{-1}}^{T\lambda^{-1}} (u-\{u\})\,dw(u) \\
&= B_\lambda + w(T\lambda^{-1})([T\lambda^{-1}] - T\lambda^{-1}) + \int\limits_{T_0\lambda^{-1}}^{T\lambda^{-1}} w(u)\,du = B_\lambda U.
\end{aligned} \tag{5}$$

If $m+|\tau|+\alpha \geq (k+|\tau|+\alpha)/2$, then we set $m = k - r$, where $1 \leq r \leq (k+|\tau|+\alpha)/2$. Then

$$\log \frac{k+|\tau|+\alpha}{m+|\tau|+\alpha} = -\log \frac{k-r+|\tau|+\alpha}{k+|\tau|+\alpha} = -\log\left(1 - \frac{r}{k+|\tau|+\alpha}\right) > \frac{r}{k+|\tau|+\alpha}.$$

Concequently, by virtue of (5), we have

$$\begin{aligned}
&\sum_{\substack{T_0\lambda^{-1}<m<k\leq T\lambda^{-1} \\ m+|\tau|+\alpha\geq(k+|\tau|+\alpha)/2}}\sum \frac{w(k)}{(m+|\tau|)^\sigma (k+|\tau|)^\sigma \log \frac{k+|\tau|+\alpha}{m+|\tau|+\alpha}} \\
&\quad = B \sum_{T_0\lambda^{-1}<k\leq T\lambda^{-1}} \ \sum_{r\leq(k+|\tau|+\alpha)/2} \frac{w(k)(k+|\tau|)}{(k+|\tau|-r)^\sigma k^\sigma r} \\
&\quad = B \sum_{T_0\lambda^{-1}<k\leq T\lambda^{-1}} \frac{w(k)}{k^{2\sigma-1}} \sum_{r\leq(k+|\tau|+\alpha)/2} \frac{1}{r} + B(1+|\tau|) \sum_{T_0\lambda^{-1}<k\leq T\lambda^{-1}} \frac{w(k)}{k^{2\sigma}} \sum_{r\leq(k+|\tau|+\alpha)/2} \frac{1}{r} \\
&\quad = B(1+|\tau|) \sum_{T_0\lambda^{-1}<k\leq T\lambda^{-1}} w(k)\frac{\log k}{k^{2\sigma-1}} + B(1+|\tau|)^2 \sum_{T_0\lambda^{-1}<k\leq T\lambda^{-1}} w(k)\frac{\log k}{k^{2\sigma}} \\
&\quad = B_\lambda(1+|\tau|)^2 U.
\end{aligned}$$

From this and from (2)–(4) we get the statement of the lemma.

Suppose now that $\sigma_1 > 1/2$. In the strip $\sigma \in [-\sigma_1, \sigma_1]$ we shall define the function

$$l_n(\alpha, s) = \frac{s}{\sigma_1}\Gamma\left(\frac{s}{\sigma_1}\right)(n+\alpha)^s, \qquad n = 1, 2, \dots .$$

In the half-plane $\sigma > 1/2$ we set

$$L_n(\lambda, \alpha, s) = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} L(\lambda, \alpha, s+z) l_n(\alpha, z)\, \frac{dz}{z}. \tag{6}$$

It follows from Lemma 4 that

$$L(\lambda, \alpha, s) = B_{\lambda,\alpha}(1+|t|).$$

for $\sigma \geq 1/2$. Therefore, by virtue of the estimate

$$\Gamma(\sigma + it) = Be^{-c_1|t|}, \qquad |t| \geq c_2,$$

which is uniform with respect to σ, $|\sigma| \leq c_3$, the integral in (6) exists.

Lemma 6. *The relation*

$$L_n(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s} \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1}\right\}$$

holds true, and the series converges absolutely for $\sigma > 1/2$.

Proof. It is obvious that $\sigma + \sigma_1 > 1$ for $\sigma > 1/2$. Therefore the function $L(\lambda, \alpha, s+z)$ on the straight line $\operatorname{Re} z = \sigma_1$ is represented by the absolutely convergent Dirichlet series

$$L(\lambda, \alpha, s+z) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^{s+z}}.$$

We set

$$a_n(\alpha, m) = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{l_m(\alpha, s)\, ds}{s(m+\alpha)^s}$$

and consider the series

$$\sum_{m=0}^{\infty} \frac{a_n(\alpha, m) e^{2\pi i \lambda m}}{(m+\alpha)^s}. \tag{7}$$

It is clear that

$$a_n(\alpha, m) = B(m+\alpha)^{-\sigma_1} \int_{-\infty}^{\infty} |l_n(\alpha, \sigma_1 + it)|\, dt = B(m+\alpha)^{-\sigma_1}.$$

This estimate gives the absolute convergence of series (7) for $\sigma > 1/2$. Consequently, changing the order of integration and summation in (6), we obtain

$$L_n(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{a_n(\alpha, m) e^{2\pi i \lambda m}}{(m+\alpha)^s}. \tag{8}$$

By virtue of the well-known property of the gamma-function

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \Gamma(s) a^{-s}\, ds = e^{-a},$$

which is valid for all positive a and b, we find that

$$a_n(\alpha, m) = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \Gamma\left(\frac{s}{\sigma_1}\right) \left(\frac{m+\alpha}{n+\alpha}\right)^{-s} d\left(\frac{s}{\sigma_1}\right) = \exp\left\{ -\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1} \right\}.$$

The last relation, together with (8), proves the lemma.

We shall prove now that the function $L(\lambda, \alpha, s)$ in D can be approximated by the series $L_n(\lambda, \alpha, s)$.

Lemma 7. *Let K be a compact set of the half-plane D. Then*

$$\lim_{n\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{U} \int_{T_0}^{T} w(\tau) \sup_{s\in K} |L(\lambda, \alpha, s+i\tau) - L_n(\lambda, \alpha, s+i\tau)|\, d\tau = 0.$$

Proof. Let us shift the integration line in relation (6). The integrand function has a simple pole at the point $z=0$. Let $\sigma \in [1/2+\eta, A]$, $\eta>0$, when $s \in K$. We set $\sigma_2 = 1/2+\eta/2$. Then, in accordance with the residue theorem, we obtain

$$L_n(\lambda,\alpha,s) = L(\lambda,\alpha,s) + \frac{1}{2\pi i}\int\limits_{\sigma_2-\sigma-i\infty}^{\sigma_2-\sigma+i\infty} L(\lambda,\alpha,s+z) l_n(\alpha,z)\,\frac{dz}{z}. \tag{9}$$

Let L be a simple closed circuit which lies in D and encircles the set K and δ be the distance from L to K. Then, by virtue of Cauchy's integral formula, we obtain

$$\sup_{s\in K} |L(\lambda,\alpha,s+i\tau) - L_n(\lambda,\alpha,s+i\tau)| \le \frac{1}{2\pi\delta}\int\limits_L |L(\lambda,\alpha,z+i\tau) - L_n(\lambda,\alpha,z+i\tau)|\,|dz|.$$

Let $|L|$ be the length of the circuit L. Then it follows that

$$\begin{aligned}
&\frac{1}{U}\int\limits_{T_0}^{T} w(\tau)\sup_{s\in K}|L(\lambda,\alpha,s+i\tau) - L_n(\lambda,\alpha,s+i\tau)|\,d\tau \\
&\quad = \frac{B}{U\delta}\int\limits_L |dz| \int\limits_{T_0+\operatorname{Im} z}^{T+\operatorname{Im} z} w(\tau-\operatorname{Im} z)|L(\lambda,\alpha,\operatorname{Re} z+i\tau) - L_n(\lambda,\alpha,\operatorname{Re} z+i\tau)|\,d\tau \\
&\quad = \frac{B|L|}{U\delta}\sup_{\sigma+iu\in L}\int\limits_{T_0+u}^{T+u} w(t-u)|L(\lambda,\alpha,\sigma+it) - L_n(\lambda,\alpha,\sigma+it)|\,dt. \qquad (10)
\end{aligned}$$

We can choose the circuit L such that the inequalities $\inf\{\sigma : s\in L\} \ge 1/2+3\eta/4$, $\delta \ge \eta/4$ are satisfied. From relation (9) we have

$$L(\lambda,\alpha,\sigma+it) - L_n(\lambda,\alpha,\sigma+it) = B\int\limits_{-\infty}^{\infty} |L(\lambda,\alpha,\sigma_2+it+i\tau)| \times |l_n(\alpha,\sigma_2-\sigma+i\tau)|\,d\tau.$$

Thus

$$\begin{aligned}
&\frac{1}{U}\int\limits_{T_0+u}^{T+u} w(t-u)|L(\lambda,\alpha,\sigma+it) - L_n(\lambda,\alpha,\sigma+it)|\,dt \\
&\quad = B\int\limits_{-\infty}^{\infty} |l_n(\alpha,\sigma_2-\sigma+i\tau)| \frac{1}{U}\int\limits_{T_0+u+\tau}^{T+u+\tau} w(t-u-\tau)|L(\lambda,\alpha,\sigma_2+it)|\,dt\,d\tau. \qquad (11)
\end{aligned}$$

Using the Cauchy inequality, we obtain

$$\int\limits_{T_0+u+\tau}^{T+u+\tau} w(t-u-\tau)|L(\lambda,\alpha,\sigma_2+it)|\,dt \le \Bigg(\int\limits_{T_0}^{T} w(t)\,dt \int\limits_{T_0+u+\tau}^{T+u+\tau} w(t-u-\tau)|L(\lambda,\alpha,\sigma_2+it)|^2\,dt\Bigg)^{1/2}. \tag{12}$$

Since the quantity u is bounded, we have, by virtue of Lemma 5,

$$\int\limits_{T_0+u+\tau}^{T+u+\tau} w(t-u-\tau)|L(\lambda,\alpha,\sigma_2+it)|^2\,dt = BU(1+|\tau|)^2.$$

From this and from (11) and (12) we find that the left-hand side of function (10) does not exceed

$$B\sup_{\sigma+iu\in L}\int\limits_{-\infty}^{\infty}|l_n(\alpha,\sigma_2-\sigma+i\tau)|(1+|\tau|)\,d\tau = B\sup_{\sigma\in[-A,-\eta/4]}\int\limits_{-\infty}^{\infty}|l_n(\alpha,\sigma_2-\sigma+i\tau)|(1+|\tau|)\,d\tau.$$

However, it follows from the definition of $l_n(\alpha,s)$ that

$$\lim_{n\to\infty}\sup_{\sigma\in[-A,-\eta/4]}\int\limits_{-\infty}^{\infty}|l_n(\alpha,\sigma_2-\sigma+i\tau)|(1+|\tau|)\,d\tau = 0.$$

We have proved the lemma.

Now we shall prove the limit theorem for the function $L_n(\lambda,\alpha,s)$. We shall use the following well-known results.

Lemma 8. *If the family of functions regular in the domain $\mathcal{G}$ is uniformly bounded inside $\mathcal{G}$, then it is compact in $\mathcal{G}$.*

Proof. The lemma is known as compactness principle, its proof can be found, for instance, in [9].

Let Y_n, $X_{1n}, X_{2n},\ldots$ be (S,ϱ)-valued random elements defined on the same probability space $(\Omega,\mathcal{F},\mathsf{P})$. We assume that S is a separable space.

Lemma 9. *Let $X_{kn}\xrightarrow{\mathcal{D}}X_k$, as $n\to\infty$, for every k and let $X_k\xrightarrow{\mathcal{D}}X$ as $k\to\infty$. In addition, suppose that*

$$\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\{\varrho(X_{kn},Y_n)\geq\varepsilon\}=0$$

for every $\varepsilon>0$. Then $Y_n\xrightarrow{\mathcal{D}}X$ as $n\to\infty$.

The **proof** is given in [1].

Let us define on $\big(H(D),\mathcal{B}(H(D))\big)$ the probability measure by means of the function $L_n(\lambda,\alpha,s)$,

$$Q_{T,n,w}(A)=\frac{1}{U}\int\limits_{T_0}^{T} w(\tau)I_{\{\tau:L_n(\lambda,\alpha,s+i\tau)\in A\}}\,d\tau,$$

and consider its weak convergence as $T\to\infty$.

Lemma 10. *There exists on $\big(H(D),\mathcal{B}(H(D))\big)$ a probability measure $Q_{n,w}$ such that the measure $Q_{T,n,w}$ weakly converges to $Q_{n,w}$ as $T\to\infty$.*

Proof. Also suppose that

$$L_{n,N}(\lambda,\alpha,s)=\sum_{m\leq N}\frac{e^{2\pi i\lambda m}}{(m+\alpha)^s}\exp\bigg\{-\bigg(\frac{m+\alpha}{n+\alpha}\bigg)^{\sigma_1}\bigg\}$$

and

$$Q_{T,n,N,w}(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau: L_{n,N}(\lambda,\alpha,s+i\tau)\in A\}}\, d\tau, \qquad A \in \mathcal{B}\big(H(D)\big).$$

Then it follows from Lemma 3 that as $T\to\infty$ the measure $Q_{T,n,N,w}$ weakly converges to a certain measure $Q_{n,N,w}$ on $\big(H(D), \mathcal{B}(H(D))\big)$. We shall show that the family $\{Q_{n,N,w}\}$ is dense for every fixed n.

Let η be a random variable on $(\widetilde{\Omega}, \mathcal{F}, \mathsf{P})$ such that

$$\mathsf{P}\{\eta\in A\} = \frac{1}{U}\int_{T_0}^{T} w(t) I_A\, dt, \qquad A\in\mathcal{B}(\mathbb{R}).$$

We define the $H(D)$-valued random element $X_{T,n,N}$ by the relation

$$X_{T,n,N}(s) = L_{n,N}(\lambda,\alpha,s+i\eta).$$

Then we find that

$$\mathsf{P}\{X_{T,n,N}(s)\in A\} = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau: L_{n,N}(\lambda,\alpha,s+i\tau)\in A\}}\, d\tau, \qquad A \in \mathcal{B}\big(H(D)\big).$$

Consequently,

$$X_{T,n,N} \xrightarrow[T\to\infty]{\mathcal{D}} X_{n,N}, \tag{13}$$

where $X_{n,N}$ is a $H(D)$-valued random element with distribution $Q_{n,N,w}$.

Let us introduce on $H(D)$ the metric ϱ as follows:

$$\varrho(f_1,f_2) = \sum_{n=1}^{\infty} 2^{-n}\frac{\varrho_n(f_1,f_2)}{1+\varrho_n(f_1,f_2)}, \qquad f_1,f_2\in H(D),$$

where

$$\varrho_n(f_1,f_2) = \sup_{s\in K_n}|f_1(s)-f_2(s)|$$

and $\{K_n\}$ is a sequence of compact subsets from D such that

$$D = \bigcup_{n=1}^{\infty} K_n, \quad K_n\subset K_{n+1}, \qquad n=1,2,\dots,$$

and if K is a compact subset from D, then $K\subseteq K_n$ for some n. It is easy to see that the metric that we introduced induces the topology of the space $H(D)$.

The Chebyshev inequality yields

$$\mathsf{P}\Big\{\sup_{s\in K_l}|X_{T,n,N}(s)| > M_l\Big\} \le \frac{1}{M_l U}\int_{T_0}^{T} w(\tau)\sup_{s\in K_l}|L_{n,N}(\lambda,\alpha,s+i\tau)|\, d\tau. \tag{14}$$

It follows from the absolute convergence of the series for $L_n(\lambda,\alpha,s)$ in the half-plane D that

$$\sup_{N\geq 1}\varlimsup_{T\to\infty}\frac{1}{U}\int\limits_{T_0}^{T} w(\tau)\sup_{s\in K_l}|L_{n,N}(\lambda,\alpha,s+i\tau)|\,d\tau\leq A_l<\infty.$$

Consequently, setting $M_l=A_l 2^l/\varepsilon$, we find from (14) for every $\varepsilon>0$ that

$$\varlimsup_{T\to\infty}\mathsf{P}\Big\{\sup_{s\in K_l}|X_{T,n,N}(s)|>M_l\Big\}\leq\frac{\varepsilon}{2^l} \tag{15}$$

for all positive integers l. The function $h\colon H(D)\to\mathbb{R}$, defined by the relation

$$h(f)=\sup_{s\in K_l}|f(s)|,\qquad f\in H(D),$$

is continuous. Thus Lemma 1 and relation (13) imply

$$\sup_{s\in K_l}|X_{T,n,N}(s)|\xrightarrow[T\to\infty]{\mathcal{D}}\sup_{s\in K_l}|X_{n,N}(s)|.$$

From this and from (15) we find that

$$\mathsf{P}\Big\{\sup_{s\in K_l}|X_{n,N}(s)|>M_l\Big\}\leq\frac{\varepsilon}{2^l} \tag{16}$$

for all positive integers l. We set

$$H_\varepsilon=\Big\{f\in H(D):\sup_{s\in K_l}|f(s)|\leq M_l,\ l\geq 1\Big\}.$$

Then, from the properties of the sets K_l and from Lemma 8 we find that H_ε is a compact set of the space $H(D)$, and, by virtue of (16), it follows that

$$\mathsf{P}\{X_{n,N}(s)\in H_\varepsilon\}\geq 1-\varepsilon$$

for all $N\geq 1$, or

$$Q_{n,N,w}(H_\varepsilon)\geq 1-\varepsilon$$

for all $N\geq 1$. The latter means that the family of probability measures $\{Q_{n,N,w}\}$ is dense. From this we find, in accordance with the Prokhorov theorem, that $\{Q_{n,N,w}\}$ is relatively compact.

It is clear that $L_{n,N}(\lambda,\alpha,s)$ converges to $L_n(\lambda,\alpha,s)$ uniformly as $N\to\infty$ when $\sigma\geq\frac{1}{2}+\delta$ for any $\delta>0$. Consequently,

$$\lim_{N\to\infty}\varlimsup_{T\to\infty}\frac{1}{U}\int\limits_{T_0}^{T} w(\tau)I_{\{\tau:\varrho(L_{n,N}(\lambda,\alpha,s+i\tau),L_n(\lambda,\alpha,s+i\tau))\geq\varepsilon\}}\,d\tau$$

$$\leq\lim_{N\to\infty}\varlimsup_{T\to\infty}\frac{1}{\varepsilon U}\int\limits_{T_0}^{T} w(\tau)\varrho\big(L_{n,N}(\lambda,\alpha,s+i\tau),L_n(\lambda,\alpha,s+i\tau)\big)\,d\tau=0.$$

Suppose, in addition, that

$$X_{T,n}(s) = L_n(\lambda, \alpha, s + i\eta).$$

Then, again using absolute convergence of the series for $L_n(\lambda, \alpha, s)$ and Chebyshev's inequality, we find that

$$\lim_{N\to\infty} \overline{\lim_{T\to\infty}} \, \mathsf{P}\bigl\{\varrho\bigl(X_{T,n,N}(s), X_{T,n}(s)\bigr) \geq \varepsilon\bigr\} = 0. \tag{17}$$

It follows from the relative compactness of the family $\{Q_{n,N,w}\}$ that there exists a subsequence $\{Q_{n,N_1,w}\}$ that weakly converges as $N_1 \to \infty$, say, to $Q_{n,w}$. Then we have

$$X_{n,N_1} \xrightarrow[N_1\to\infty]{\mathcal{D}} Q_{n,w}. \tag{18}$$

The space $H(D)$ is separable, and therefore it follows from (13), (17), (18), and from Lemma 9 that

$$X_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} Q_{n,w}. \tag{19}$$

The latter means that as $T \to \infty$, the measure $Q_{T,n,w}$ weakly converges to the measure $Q_{n,w}$.

Proof of the Theorem. We shall show that the family of probability measures $\{Q_{n,w},\ n \geq 1\}$ is dense. From (19) we have

$$X_{T,n} \xrightarrow[T\to\infty]{\mathcal{D}} X_n, \tag{20}$$

where X_n is a $H(D)$-valued random element with distribution $Q_{n,w}$. Reasoning as in the proof of Lemma 9 and preserving its notation, we find that

$$\mathsf{P}\{X_n(s) \in H_\varepsilon\} \geq 1 - \varepsilon$$

for all $n \geq 1$, i.e., we find that the family $\{Q_{n,w}\}$ is dense.

We define a subsequence $\{Q_{n_1,w}\}$ from $\{Q_{n,w}\}$ such that as $n_1 \to \infty$, $Q_{n_1,w}$ weakly converges, say, to Q_w. Then

$$X_{n_1} \xrightarrow[n_1\to\infty]{\mathcal{D}} Q_w. \tag{21}$$

Using Lemma 7 and Chebyshev's inequality, we derive

$$\lim_{n\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau:\varrho\left(L(\lambda,\alpha,s+i\tau),L_n(\lambda,\alpha,s+i\tau)\right)\geq\varepsilon\}}\, d\tau$$

$$\leq \lim_{n\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{\varepsilon U} \int_{T_0}^{T} w(\tau) \varrho\bigl(L(\lambda, \alpha, s + i\tau), L_n(\lambda, \alpha, s + i\tau)\bigr)\, d\tau = 0.$$

Setting

$$X_T(s) = L(\lambda, \alpha, s + i\eta),$$

we find that

$$\lim_{n\to\infty} \overline{\lim_{T\to\infty}} \, \mathsf{P}\bigl\{\varrho\bigl(X_{T,n}(s), X_T(s)\bigr) \geq \varepsilon\bigr\} = 0.$$

The last relation, (20), (21), and Lemma 9 yield

$$X_T \xrightarrow[T\to\infty]{\mathcal{D}} Q_w,$$

and this proves the theorem.

ACKNOWLEDGMENTS

This work was supported by the Lithuanian Foundation for Research and Studios.

REFERENCES

1. Billingsley, P., *Skhodimost' veroyatnostnykh mer* (Convergence of Probability Measures), Moscow: Nauka, 1977.
2. Garuncstis, P. and Laurinchicas, A., On Lerch Zeta-Function, *Lit. Mat. Sb.* (to appear).
3. Karatsuba, A. A., *Osnovy analiticheskoi teorii chisel* (Foundation of the Analytic Number Theory), Moscow: Nauka, 1975.
4. Katsurada, M., An Application of Mellin–Barnes' Type Integrals to the Mean Square of Lerch Zeta-Functions, *Preprint*, *Kagoshima*, 1995.
5. Klusch, D., Asymptotic Equalities for the Lipschitz–Lerch Zeta-Function, *Arch. Math.* (*Basel*), 1987, vol. 49, pp. 38–43.
6. Klusch, D., A Hybrid Version of a Theorem of Atkinson, *Rev. Roumaine Math. Pures Appl.*, 1989, vol. 34, pp. 721–728.
7. Laurinchicas, A. and Misyavichus, G., Limit Theorem with Weight for the Riemannian Zeta-Function in the Space of Analytic Functions, *Lit. Mat. Sb.*, 1984, vol. 34, no. 2, pp. 211–225.
8. Lerch, M., Note sur la fonction $K(w,x,s)=\sum_{n\geq 0}\exp\{2\pi inx\}(n+w)^{-s}$, *Acta Math.*, 1887, vol. 11, pp. 19–24.
9. Shabat, B. V., *Vvedenie v kompleksnyi analiz* (Introduction to Complex Analysis), Moscow: Nauka, 1969.
10. Zhang, W., On the Mean Square Value Formula of Lerch Zeta-Function, *Adv. Math.* (*China*), 1993, vol. 22, pp. 367–369.
11. Heyer, H., *Probability Measures on Locally Compact Groups*, Heidelberg: Springer, 1977.

Translated by I. Aleksandrova