Proceedings of the Steklov Institute of Mathematics, Vol. 218, 1997, pp. 129–145.
Translated from Trudy Matematicheskogo Instituta imeni V. A. Steklova, Vol. 218, 1997, pp. 134–150.

On the Zeros of Linear Combinations of the Analogs of Riemann's Function

S. A. Gritsenko

Received February, 1997

Selberg defined in [1] the class S of Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}a(n)n^{-s},\qquad \operatorname{Re}s>1,$$

satisfying the following conditions:

(1) there exists a nonnegative integer m such that the function $(s-1)^m F(s)$ is entire,

(2) the Dirichlet coefficients $a(n)$ satisfy the inequalities $a(n)\ll_\varepsilon n^\varepsilon$ for any positive ε, and $a(1)=1$,

(3) there exists a function $\gamma_F(s)$ of the form

$$\gamma_F(s)=\varepsilon_1 Q^s\prod_{\nu=1}^{k}\Gamma(\lambda_\nu s+\mu_\nu),$$

where $|\varepsilon_1|=1$, $k=k_F$ is a positive integer, $Q>0$, $\lambda_\nu>0$, $\operatorname{Re}\mu_\nu\geq 0$, the function being such that the identity $\Phi(s)=\overline{\Phi}(1-\overline{s})$ is valid for the function $\Phi(s)=\gamma_F(s)F(s)$,

(4) for $\operatorname{Re}s>1$ the function $F(s)$ is decomposed in the Euler product

$$F(s)=\prod_p(1+a(p)p^{-s}+a(p^2)p^{-2s}+\cdots),$$

and p runs through the prime numbers

$$\log F(s)=\sum_{n=1}^{\infty}b(n)n^{-s},$$

where $b(n)=0$ if n is not equal to the positive power of the prime number, with $b(n)\ll n^\theta$ for a certain $\theta\leq 1/2$.

The class S includes the Dirichlet series which are defined by a functional equation of the Riemann type and a decomposition in the Euler product, in particular, in Riemann's zeta function, in Dirichlet's L-functions etc.

In the article [2], for any function $F(s)$ from S the degree of $F(s)$ is defined as

$$d_F=2\sum_{\nu=1}^{k}\lambda_\nu.$$

For primitive functions from this class, i.e., functions which are not representable as

$$F_1(s)F_2(s), \qquad F_1(s), F_2(s) \in \mathrm{S},$$

Selberg stated in [1] a number of conjectures, in particular, the following conjectures.

1. As $x \to \infty$, the asymptotic formula

$$\sum_{p \le x} |a(p)|^2 p^{-1} = \log\log x + O(1)$$

holds true.

2. As $x \to \infty$, the estimate

$$\sum_{p \le x} a(p)\overline{a}'(p)p^{-1} = O(1),$$

where p runs through the prime numbers and $a(n)$ and $a'(n)$, which are Dirichlet coefficients of the primitive functions $F(s)$ and $F'(s)$, $F(s)$ and $F'(s)$, are distinct, holds true.

Let $F(s)$ be a function from the class S. We say that the zeros of $F(s)$ which coincide with the poles of the function $\gamma_F(s)$ are trivial and the other zeros are nontrivial.

It is known (see [1, p. 368]) that the nontrivial zeros of $F(s)$ lie in the strip $1 - A \le \operatorname{Re} s \le A$, $A = A_F > 0$.

In addition, (see [1, p. 368]), since $F(s)$ satisfies a functional equation of the Riemann type, it follows that

$$N(T) \sim cT\log T, \qquad c = c_F > 0,$$

where $N(T)$ is the number of nontrivial zeros of $F(s)$ in the rectangle $1 - A \le \operatorname{Re} s \le A$, $0 < \operatorname{Im} s \le T$.

We define the following three functions:

$$\rho(s) = \overline{\gamma}_F(1-\overline{s})\gamma_F^{-1}(s) = \overline{\varepsilon}_1^2 Q^{1-2s} \prod_{\nu=1}^{k} \Gamma(\lambda_\nu(1-s) + \overline{\mu}_\nu)\Gamma^{-1}(\lambda_\nu s + \mu_\nu),$$

$$\exp\bigl(i\theta(t)\bigr) = \Bigl(\rho\Bigl(\frac{1}{2} + it\Bigr)\Bigr)^{-1/2}, \qquad Z(t) = \exp\bigl(i\theta(t)\bigr)F\Bigl(\frac{1}{2} + it\Bigr).$$

It follows from the functional equation that $Z(t)$ assumes real values for real t. The function $Z(t)$ is an analog of Riemann's function

$$\Xi(t) = -\frac{1}{2}(T^2+1)\pi^{-1/4-it/2}\Gamma\Bigl(\frac{1}{4} + \frac{1}{2}it\Bigr)\zeta\Bigl(\frac{1}{2} + it\Bigr)$$

(see [3, pp. 249–250, 253] and [4, pp. 218–219]).

Let us define $f(t)$, which is a linear combination of the analogs of Riemann's function:

$$f(t) = \Phi_1\Bigl(\frac{1}{2} + it\Bigr) = \sum_{j=1}^{N} a_j\Bigl(\rho_j\Bigl(\frac{1}{2} + it\Bigr)\Bigr)^{-1/2} F_j\Bigl(\frac{1}{2} + it\Bigr) = \sum_{j=1}^{N} a_j Z_j(t). \tag{1}$$

In what follows we assume that $T \ge T_0 > 0$, where T_0 is a sufficiently large constant which depends only on $F_1(s), \dots, F_N(s)$.

Let $N_0(T)$ be the number of zeros of the function $\Phi_1(s)$ on the interval $(1/2, 1/2+iT]$ or, what is the same, the number of zeros of the function $f(t)$ on the interval $(0, T]$.

The following inequality is "expected" for the quantity $N_0(T)$: $N_0(T) \geq c_4 T \log T$, where $c_4 > 0$ is a constant.

Note that if $F_1(s), \dots, F_N(s)$ are functions of degree 1 from the class S, then the application of the Hardy–Littlewood–Landau method to the function $f(t)$ yields the inequality $N_0(T) \geq c_5 T$ (c_5 is an absolute positive constant).

For the first time a principally more exact estimate of the quantity $N_0(T)$, which corresponds to the Davenport–Heilbronn function, was obtained by Voronin, who proved in 1980, in the article [5], that

$$N_0(T) \geq c_6 T \exp\Big(\frac{1}{20}\sqrt{\log\log\log\log T}\Big),$$

(c_6 is an absolute positive constant). The last inequality means that the straight line $\operatorname{Re} s = 1/2$ is a "special set" for the zeros of the Davenport–Heilbronn function. The *Davenport–Heilbronn function* is defined in the half-plane $\operatorname{Re} s > 0$ by the Dirichlet series

$$\sum_{n=1}^{\infty} r(n) n^{-s}, \qquad \text{where} \quad r(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 5 \\ \varkappa & \text{if } n \equiv 2 \pmod 5 \\ -\varkappa & \text{if } n \equiv 3 \pmod 5 \\ -1 & \text{if } n \equiv 4 \pmod 5, \end{cases} \qquad \varkappa = \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}.$$

It is intensively studied as the simplest Dirichlet series for which the analog of Riemann's conjecture is not satisfied.

In 1989 Karatsuba worked out a new method for estimating from below the number of zeros of certain Dirichlet series on segments of a critical line and used this method to prove the following statement for the Davenport–Heilbronn functions [6]:

$$N_0(T) \geq T(\log T)^{0.5}(\log T)^{-\varepsilon}, \tag{2}$$

where $\varepsilon > 0$ is an arbitrarily small fixed number, $T \geq T_0(\varepsilon) > 0$.

Then, in his article [7], Karatsuba used his method to estimate the number of zeros on the critical line of the function $\Phi_1(s)$ for the case where $F_1(s), \dots, F_N(s)$ are Dirichlet's L-functions which correspond to the irreducible characters (in our terminology they refer to functions from the class S of degree 1) and got the following estimates for the quantity $N_0(T)$:

$$N_0(T) \geq T(\log T)^{2\beta}(\log T)^{-\varepsilon}, \tag{3}$$

if Dirichlet characters $\chi_1, \dots, \chi_N$ are of the same parity,

$$N_0(T) \geq T(\log T)^{\beta}\,(\log T)^{-\varepsilon} \tag{4}$$

otherwise, here $\beta = \big(\varphi(K)\big)^{-1}$, K is the least common multiple of the positive integers $k_1, \dots, k_N$ which are moduli of the characters $\chi_1, \dots, \chi_N$, $\varphi(K)$ is Euler's function, ε is an arbitrarily small fixed number, $T \geq T_0(\varepsilon) > 0$.

Later, in the work [8] the factor $(\log T)^{-\varepsilon}$ on the right-hand sides of inequalities (2)–(4) was replaced by the slower decreasing factor $\exp(-c_4\sqrt{\log\log T}\,)$, where c_4 is an absolute positive constant.

At present, estimates (2)–(4) are the strongest among the known estimates.

Suppose now that the functions $F_1(s),\dots,F_N(s)$ in relation (1) belong to the Selberg class and are of degree 2. The problem of estimating the quantity $N_0(T)$ from below was considered in the works [9–11], where the following theorems were proved.

Theorem 1. *Let N be an arbitrary positive integer and $a_1,\dots,a_N$ be arbitrary real numbers, $T \ge T_0 > 0$. Suppose that $f(t)$ is the function defined by relation (1) and $F_1(s),\dots,F_N(s)$ are functions of degree 2 from S which satisfy the following conditions.*

1. *As $x \to \infty$, the asymptotic formulas*

$$\sum_{p\le x} |a_j(p)|^2 \log p = x + O\big(x(\log x)^{-1}\big),$$

hold true, where $a_j(n)$ are Dirichlet coefficients of the functions $F_j(s)$ and p runs through the prime numbers.

2. *As $x \to \infty$, the estimates*

$$\sum_{p\le x} a_j(p)\overline{a}_i(p)p^{-1} = O(1)$$

are valid, where $a_j(n)$ and $a_i(n)$ are Dirichlet coefficients of any functions $F_j(s)$, $F_i(s)$, $1 \le j \ne i \le N$.

3. *For any j, $1 \le j \le N$, any prime numbers p, and any real t there exist positive constants c_{F_j} such that*

$$|1 + a_j(p)p^{-1/2-it} + a_j(p^2)p^{-1-2it} + \cdots| > c_{F_j}.$$

4. *Let $F(s)$ be any one of the functions $F_1(s),\dots,F_N(s)$. Then, as $x \to \infty$, the asymptotic formula*

$$\sum_{n\le x} |a(n)|^2 n^{-1} = A_F \log x + B_F + O\big((\log x)^{-4}\big)$$

holds true, where $a(n)$ are Dirichlet coefficients of the function $F(s)$, $A_F > 0$, B_F are constants dependent only on $F(s)$.

Then the inequality

$$N_0(T) > T\exp(\sqrt{\log\log\log T}\,)$$

holds true.

Theorem 2. *Suppose that $F_1(s),\dots,F_N(s)$ are Hecke's L-functions corresponding to Hecke's complex characters and $f(t)$ is the function defined by relation (1). Then the inequality*

$$N_0(T) > T\exp(\sqrt{\log\log\log T}\,)$$

holds true.

In this article we continue to study the distributions of zeros of the function $f(t)$. We formulate our main results.

Theorem 3. *Suppose that N is an arbitrary positive integer, $a_1,\dots,a_N$ are arbitrary real numbers, $T \ge T_0 > 0$. Let $f(t)$ be the function defined by relation* (1) *and $F_1(s),\dots,F_N(s)$ be functions of degree* 2 *from* S *which satisfy the following conditions.*

1. *As $x\to\infty$, the asymptotic formulas*
$$\sum_{p\le x}|a_j(p)|^2\log p = x+O\big(x(\log x)^{-1}\big)$$
hold true, where $a_j(n)$ are Dirichlet coefficients of the functions $F_j(s)$ and p runs through prime numbers.
2. *As $x\to\infty$, the estimates*
$$\sum_{p\le x}a_j(p)\overline{a}_i(p)p^{-1}=O(1)$$
hold true, where $a_j(n)$ and $a_i(n)$ are Dirichlet coefficients of any functions $F_j(s)$, $F_i(s)$, $1\le j\ne i\le N$.
3. *For any j, $1\le j\le N$, any prime numbers p, and any real t there exist positive constants c_{F_j} such that*
$$|1+a_j(p)p^{-1/2-it}+a_j(p^2)p^{-1-2it}+\cdots|>c_{F_j}.$$
4. *Let $F(s)$ be any one of the functions $F_1(s),\dots,F_N(s)$. Then, as $x\to\infty$, the asymptotic formula*
$$\sum_{n\le x}|a(n)|^2n^{-1}=A_F\log x+B_F+O\big((\log x)^{-4}\big)$$
holds true, where $a(n)$ are Dirichlet coefficients of the function $F(s)$, $A_F>0$, B_F are constant dependent only on $F(s)$.

Then there exists a positive constant η, dependent only on $F_1(s),\dots,F_N(s)$, such that the inequality
$$N_0(T)>T(\log\log T)^\eta$$
holds true.

Theorem 4. *Let $F_1(s),\dots,F_N(s)$ be Hecke's L-functions corresponding to Hecke's complex characters and let $f(t)$ be the function defined by relation* (1). *Then the inequality*
$$N_0(T)>T(\log\log T)^\eta$$
holds true, where η is the same constant as in Theorem 3.

It follows from our estimates that the straight line $\operatorname{Re}s=1/2$ is a special set for the zeros of the function $f(t)$. Obviously, in addition, they are less exact than Karatsuba's estimates (2)–(4).

Theorem 3 can be proved by Voronin's method (see [5]) which he worked out in 1980 in order to estimate the number of zeros of the Davenport–Heilbronn function which lie on a segment of the critical line.

We begin with a brief description of the proof of Theorem 3. It is based on the idea, which belongs to Hardy, Littlewood, and Landau, of comparing integrals
$$I(t)=\left|\int_0^h\int_0^h f(t+u_1+u_2)\,du_1\,du_2\right|\qquad\text{and}\qquad J(t)=\int_0^h\int_0^h|f(t+u_1+u_2)|\,du_1\,du_2,$$

where h is a small parameter whose exact value will be chosen later and $f(t+u_1+u_2)$ is a linear combination of the analogs of Riemann's function defined by (1).

First the interval $[T,2T]$ is divided into two subsets

$$\mathcal{S}=\{t\in[T,2T]\mid I(t)=J(t)\},\qquad E=[t,2T]\setminus\mathcal{S}.$$

The integral

$$J=\int\limits_{\mathcal{S}} J(t)\,dt$$

is estimated from above and from below. By virtue of the definition of the set S, we have

$$J=\int\limits_{\mathcal{S}} J(t)\,dt\le\int\limits_{T}^{2T} I(t)\,dt=I$$

and the integral I is estimated, with the standard use of the Hardy–Littlewood–Landau method, as $O(Th^{3/2})$.

Let us estimate the integral J from below. With the use of elementary arguments, we derive the inequality

$$J\ge h^2J_1-\sum_{j=1}^{N}\sum_{k=1}^{N}|a_ja_k|I_{jk},$$

where

$$J_1=\min_{0\le u\le 2h}\int\limits_{\mathcal{S}}\frac{\left|\sum_{j=1}^{N}a_j\exp(i\theta_j(t+u))P_j(1/2+it+iu)\right|^2}{1+\sum_{j=1}^{N}|a_j|\times|P_j(1/2+it+iu)|}\,dt,$$

$$I_{jk}=\int\limits_{T}^{2T}\left|\int\limits_{0}^{h}\int\limits_{0}^{h}\frac{P_j(1/2+it+iu)\overline{P}_k(1/2+it+iu)}{1+\sum_{j=1}^{N}|a_j|\times|P_j(1/2+it+iu)|}\right.$$

$$\left.\times\exp\bigl(i(\theta_j(t+u_1+u_2)-\theta_k(t+u_1+u_2))\bigr)\Bigl(Q_j\Bigl(\frac12+it+iu\Bigr)-1\Bigr)\,du_1\,du_2\right|\,dt,$$

$$P_j(s)=\prod_{p\le y}(1+a_j(p)p^{-s}+a_j(p^2)p^{-2s}+\cdots),\qquad Q_j(s)=F_j(s)P_j^{-1}(s),\qquad y=(\log T)^{\varepsilon}$$

($\varepsilon>0$ is an arbitrarily small number which is smaller than 0.001).

The interals I_{jk} are estimated from above. The more difficult part of the solution of this problem is the estimation of the sum V of the form

$$V=\sum_{\substack{n\le U\\(n,\overline{\mathbf{P}})=1}}|a_j(n)|^2,\qquad\text{where}\quad\mathbf{P}=\prod_{p\le y}p.$$

For any U, $y<U$, we must prove that $V=O\bigl(U(\log y)^{-1}\bigr)$. To this end, we must use the small sieve method since for "insufficiently large" values of U, as compared to y, the method of generating functions does not work.

For the sum $\sum_{j=1}^{N}\sum_{k=1}^{N}|a_j a_k| I_{jk}$ we get the estimate

$$\sum_{j=1}^{N}\sum_{k=1}^{N}|a_j a_k| I_{jk} = O(Th^{3/2}),$$

from which, by virtue of $J = O(Th^{3/2})$ and

$$J \geq h^2 J_1 - \sum_{j=1}^{N}\sum_{k=1}^{N}|a_j a_k| I_{jk},$$

it follows that $h^2 J_1 = O(Th^{3/2})$.

In order to estimate J_1 from below, we show that there exists a set E_1 of a large measure $\operatorname{mes}(E_1) > c_0 T(\log\log T)^{-2\eta}$ such that for any t from E_1 the value of the function

$$\frac{\left|\sum_{j=1}^{N} a_j \exp(i\theta_j(t)) P_j(1/2+it)\right|^2}{1+\sum_{j=1}^{N}|a_j| \times |P_j(1/2+it)|}$$

is "large", to be more precise,

$$\frac{\left|\sum_{j=1}^{N} a_j \exp(i\theta_j(t)) P_j(1/2+it)\right|^2}{1+\sum_{j=1}^{N}|a_j| \times |P_j(1/2+it)|} > (\log\log T)^{4\eta},$$

whence it follows that

$$J_1 \gg (\log\log T)^{4\eta} \operatorname{mes}\bigl(\mathcal{S} \cup (E_1 - u)\bigr).$$

If we had the inequality

$$\operatorname{mes}\bigl(\mathcal{S} \cup (E_1 - u)\bigr) > \frac{1}{2}\operatorname{mes}(E_1) > T\ (\log\log T)^{-2\eta},$$

then it would follow that

$$J_1 > \frac{1}{2} T(\log\log T)^{2\eta},$$

and this contradicts the estimate $h^2 J_1 = O(Th^{3/2})$ for $h = (\log\log t)^{-3.5\eta}$. Hence

$$\operatorname{mes}\bigl(\mathcal{S} \cup (E_1 - u)\bigr) \leq \frac{1}{2}\operatorname{mes}(E_1),$$

and, consequently, the inequality

$$\operatorname{mes}(E) > \frac{1}{4} T(\log\log t)^{-2\eta}$$

holds true, from which it is easy to obtain the statement of the theorem.

Note that in problems on the estimation of the number of zeros of Dirichlet series on a critical line one usually tries to get the inequality

$$\operatorname{mes}(E) > c_0 T, \qquad \text{where} \quad c_0 > 0 \quad \text{is a constant.} \tag{5}$$

With the use of our approach, we can prove the validity of estimate (5) only for

$$h \geq \exp(-c_1\sqrt{\log\log y}\,),$$

where $c_1 > 0$ is a constant.

It turns out that the result is somewhat better if we replace estimate (5) by an estimate, which is less exact but which acts for smaller values of the parameter h.

The proof of Theorem 1 is given in the article [9], and in order to prove Theorem 3, it suffices to use the same scheme only with the application of the following lemmas at appropriate places.

Lemma 1. *Suppose that N is an arbitrary positive integer, $F_1(s),\dots,F_N(s)$ are pairwise distinct primitive functions of degree 2 from Selberg's class* S *which satisfy the conditions of Theorem* 1 *and $a_j(n)$ are their Dirichlets coefficients, $j=1,\dots,N$.*

Let a and c be nonnegative real numbers with a satisfying the inequalities $0 < a \leq \xi\sqrt{\log\log y}$, where $\xi = \min(0.01, \xi_1)$ and ξ_1 is such a small positive number that

$$\xi_1 \sum_{p\leq y}\sum_{\alpha=3}^{\infty} |a_j(p^\alpha)p^{-0.5\alpha} \leq 0.01$$

for any $1 \leq j \leq N$. Let b assume one of the three values, $0, a, 2a$.

Let

$$g_1(t) = (\log\log y)^{-1/2}\,\mathrm{Re}\biggl(\sum_{p\leq y} a_1(p)p^{-1/2-it}\biggr) - c,$$

$$g_j(t) = -(\log\log y)^{-1/2}\,\mathrm{Re}\biggl(\sum_{p\leq y} a_j(p)p^{-1/2-it}\biggr) - c, \qquad j=2,\dots,N,$$

$y = (\log T)^\varepsilon$ (ε is an arbitrarily small number, smaller than 0.001*).*

Let n be any positive integer such that $1 \leq n \leq N$.

Then the inequalities

$$0.9T\exp\biggl(\frac{a^2}{4}(N-1)+\frac{b^2}{4}-ca(N-1)-2b\biggr) \leq I \leq 1.1T\exp\biggl(\frac{a^2}{4}(N-1)+\frac{b^2}{4}-ca(N-1)-2b\biggr)$$

are valid for I,

$$I = \int_T^{2T} \exp\Biggl(a\sum_{\substack{j=1\\ j\neq n}}^{N} g_j(t) + bg_n(t)\Biggr)\,dt.$$

Proof. We introduce the notation

$$\beta_j(p) = \frac{\delta(j)}{\sqrt{\log\log y}}\,\mathrm{Re}\biggl(\frac{a_j(p)}{\sqrt{p}}p^{-it}\biggr), \qquad \text{where} \quad \delta(j) = \begin{cases} \;\;1 & \text{if } j=1\\ -1 & \text{if } j\neq 1.\end{cases}$$

Then

$$\exp\Biggl(a\sum_{\substack{j=1\\ j\neq n}}^{N} g_j(t) + bg_n(t)\Biggr) = \exp\bigl(-2a(N-1)-2b\bigr)\prod_{p\leq y}\prod_{\substack{j=1\\ j\neq n}}^{N}\exp\bigl(a\beta_j(p)\bigr)\exp\bigl(b\beta_n(p)\bigr).$$

According to the hypothesis,

$$|a\beta_j(p)| \leq 0.5, \quad |b\beta_n(p)| \leq 0.5.$$

For any real α, $|\alpha| \leq 0.5$, the inequality

$$\left(1+\alpha+\frac{\alpha^2}{2}\right)(1-|\alpha|^3) \leq \exp(\alpha) \leq \left(1+\alpha+\frac{\alpha^2}{2}\right)(1+|\alpha|^3)$$

holds true, and therefore

$$\begin{aligned}
\mathrm{Pr}_1 \times \prod_{p\leq y}\prod_{\substack{j=1\\ j\neq n}}^{N}&\left(1+a\beta_j(p)+\frac{a^2}{2}\beta_j^2(p)\right)\left(1+b\beta_n(p)+\frac{b^2}{2}\beta_n^2(p)\right)\\
&\leq \prod_{p\leq y}\prod_{\substack{j=1\\ j\neq n}}^{N}\exp\bigl(a\beta_j(p)\bigr)\exp\bigl(b\beta_n(p)\bigr)\\
&\leq \mathrm{Pr}_2 \times \prod_{p\leq y}\prod_{\substack{j=1\\ j\neq n}}^{N}\left(1+a\beta_j(p)+\frac{a^2}{2}\beta_j^2(p)\right)\left(1+b\beta_n(p)+\frac{b^2}{2}\beta_n^2(p)\right),
\end{aligned}$$

where

$$\mathrm{Pr}_1 = \prod_{p\leq y}\prod_{\substack{j=1\\ j\neq n}}^{N}\left(1-\frac{a^3|a_j(p)|^3}{(p\log\log y)^{3/2}}\right)\left(1-\frac{b^3|a_n(p)|^3}{(p\log\log y)^{3/2}}\right),$$

$$\mathrm{Pr}_2 = \prod_{p\leq y}\prod_{\substack{j=1\\ j\neq n}}^{N}\left(1+\frac{a^3|a_j(p)|^3}{(p\log\log y)^{3/2}}\right)\left(1+\frac{b^3|a_n(p)|^3}{(p\log\log y)^{3/2}}\right).$$

Since $\mathrm{Pr}_1 \geq 0.99$, $\mathrm{Pr}_2 \leq 1.01$ by the hypothesis of Lemma 1, it follows that in order to prove Lemma 1, it suffices to derive for

$$I' = \int_T^{2T}\prod_{p\leq y}\prod_{\substack{j=1\\ j\neq n}}^{N}\left(1+a\beta_j(p)+\frac{a^2}{2}\beta_j^2(p)\right)\left(1+b\beta_n(p)+\frac{b^2}{2}\beta_n^2(p)\right)dt$$

an asymptotic formula of the form

$$I' = T\exp\left(\frac{a^2}{4}(N-1)+\frac{b^2}{4}\right)\bigl(1+O\bigl((\log\log y)^{-1}\bigr)\bigr).$$

By definition we have

$$\begin{aligned}
1+a\beta_j(p)+\frac{a^2}{2}\beta_j^2(p) &= 1+\frac{a}{2}\delta(j)(\log\log y)^{-1/2}\frac{a_j(p)}{\sqrt{p}}p^{-it}+\frac{a}{2}\delta(j)(\log\log y)^{-1/2}\frac{\overline{a}_j(p)}{\sqrt{p}}p^{it}\\
&\quad+\frac{a^2}{8}(\log\log y)^{-1}\frac{a_j^2(p)}{p}p^{-2it}+\frac{a^2}{8}(\log\log y)^{-1}\frac{\overline{a}_j^2(p)}{p}p^{2it}\\
&\quad+\frac{a^2}{4}(\log\log y)^{-1}\frac{|a_j(p)|^2}{p},\qquad j=1,\dots,N,\quad j\neq n,\\
1+b\beta_n(p)+\frac{b^2}{2}\beta_n^2(p) &= 1+\frac{b}{2}\delta(n)(\log\log y)^{-1/2}\frac{a_n(p)}{\sqrt{p}}p^{-it}+\frac{b}{2}\delta(n)(\log\log y)^{-1/2}\frac{\overline{a}_n(p)}{\sqrt{p}}p^{it}\\
&\quad+\frac{b^2}{8}(\log\log y)^{-1}\frac{a_n^2(p)}{p}p^{-2it}+\frac{b^2}{8}(\log\log y)^{-1}\frac{\overline{a}_n^2(p)}{p}p^{2it}\\
&\quad+\frac{b^2}{4}(\log\log y)^{-1}\frac{|a_n(p)|^2}{p}.
\end{aligned}\tag{6}$$

We take the product Pr_3,

$$\mathrm{Pr}_3=\prod_{p\le y}\prod_{\substack{j=1\\ j\neq n}}^{N}\Bigl(1+a\beta_j(p)+\frac{a^2}{2}\beta_j^2(p)\Bigr)\Bigl(1+b\beta_n(p)+\frac{b^2}{2}\beta_n^2(p)\Bigr)\,dt.$$

By virtue of relations (6), it can be represented as the sum of $6^{N\pi(y)}$ terms. We combine the terms independent of t (nonoscillating); their sum is

$$\begin{aligned}
\mathrm{Pr}_4=\prod_{p\le y}\Biggl(&1+\frac{a^2}{4}(\log\log y)^{-1}\sum_{\substack{j=1\\ j\neq n}}^{N}\frac{|a_j(p)|^2}{p}+\frac{b^2}{4}(\log\log y)^{-1}\frac{|a_n(p)|^2}{p}\\
&+\frac{a^2}{4}(\log\log y)^{-1}\sum_{\substack{j=1\\ j\neq n}}^{N}\sum_{\substack{m=1\\ m\neq n}}^{N}\delta(j)\delta(m)p^{-1}\bigl(a_j(p)\overline{a}_m(p)+\overline{a}_j(p)a_m(p)\bigr)\\
&+\frac{ab}{4}(\log\log y)^{-1}\sum_{\substack{j=1\\ j\neq n}}^{N}\delta(j)\delta(n)p^{-1}\bigl(a_j(p)\overline{a}_n(p)+\overline{a}_j(p)a_n(p)\bigr)+O\bigl((\log\log y)^{-3/2}p^{-4/3}\bigr)\Biggr).
\end{aligned}$$

The terms dependent on t (oscillating) (they are less in number than $6^{N\pi(y)}$) have the form

$$d\Bigl(\frac{l}{g}\Bigr)^{\pm it},\tag{7}$$

where $|d|\ll 1$, l and g are positive integers not exceeding

$$\prod_{p\le y}p^2<\exp(3y),$$

and therefore the integral over t of the terms of (7) can be estimated as $O\big(\exp(3y)\big)$, and the integral of the sum of the oscillating terms can be estimated as

$$O(6^{N\pi(y)}\exp(3y)) = O(T^{\varepsilon})$$

for any positive ε. Consequently, $I' = \mathrm{Pr}_4 T + O(T^{\varepsilon})$.

Finally, we get for Pr_4 the asymptotic formula

$$\mathrm{Pr}_4 = \exp\left(\frac{a^2}{4}(N-1) + \frac{b^2}{4}\right) \times \big(1 + O\big((\log\log y)^{-1}\big)\big).$$

We have proved Lemma 1.

Corollary 1. *Suppose that the conditions of Lemma* 1 *are satisfied and, in addition,*

$$c = \frac{\xi}{16N}\sqrt{\log\log y}, \qquad a = 4c(1+0.01\xi), \qquad D = \frac{3}{4}a^2 - ca + 0.01\xi a.$$

Then the inequality

$$\mathrm{mes}(E_2) > \frac{1}{2}T\exp\left(\frac{a^2}{4}N - DN - caN\right)$$

holds true.

The proof follows immediately from Lemma 1, as well as from Lemma 7 and the corollary of Lemma 9 of the work [9] (it should only be pointed out that the function $H_1(x)$ in Lemma 7 [9] must be defined as $H_1(x) = \exp(ax - D)$).

Setting now $\eta = 0.99\xi/(64N)$, we have $\mathrm{mes}(E_1) > T(\log\log T)^{-2\eta}$, where

$$E_1 = \left\{t \in [T,2T] \;\Big|\; \frac{\left|\sum_{j=1}^{N} a_j\exp(i\theta_j(t+u))P_j(1/2+it+iu)\right|^2}{1+\sum_{j=1}^{N}|a_j|\times|P_j(1/2+it+iu)|} > (\log\log T)^{4\eta}\right\}.$$

We have shown how the integral J can be estimated from below. In order to prove Theorem 3, it is necessary to estimate the integrals I_{jk} from above. The estimation is carried out just as in the article [9]. However, since the definitions that we gave here do not completely coincide with those from [9], we must additionally use the following lemma.

Lemma 2. *Suppose that N is an arbitrary positive integer, $F_1(s),\dots,F_N(s)$ are Dirichlet series from the Selberg class, and $a_j(n)$ are their Dirichlet coefficients, $j=1,\dots,N$.*

Suppose that there exist positive constants $c_{F_1}, c_{F_2},\dots,c_{F_N}$ such that for any real t and any prime number p the inequalities

$$\left|1 + \sum_{k=1}^{\infty}\frac{a_j(p^k)}{p^{k/2}}p^{-ikt}\right| > c_{F_j}, \qquad j = 1,2,\dots,N,$$

hold true.

Let

$$P_j\Big(\frac{1}{2}+it\Big) = \prod_{p\le y}\left(1+\sum_{k=1}^{\infty}\frac{a_j(p^k)}{p^{k/2}}p^{-ikt}\right) \qquad j=1,2,\dots,N,$$

$y = (\log T)^{\varepsilon}$, ε *is an arbitrary positive number smaller than* 0.001.

Let

$$g_1\Big(\frac{1}{2}+it\Big) = \bigg(1+\sum_{j=1}^{N}|a_j| \times \Big|P_j\Big(\frac{1}{2}+it\Big)\Big|\bigg)^{-1},$$

where $a_1,\dots,a_N$ *are arbitrary constants.*

Then there exists a number sequence $x_{m,n}$ *such that for any* $A>0$ *the relation*

$$g_1\Big(\frac{1}{2}+it\Big) = \sum_{\substack{m,n\in M\\(m,n)=1\\m,n\le T^{1/200}}} x_{m,n}\Big(\frac{m}{n}\Big)^{it} + O\big((\log T)^{-A}\big)$$

is valid, where M *is a set consisting of* 1 *and those positive integers all of whose prime divisors do not exceed* y.

Proof. 1. Let $2,3,\dots,p,\dots,q$ be all prime numbers not exceeding y and let $\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q$ be real variables indexed by prime numbers.

Let us consider the function $\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q)$ of the form

$$\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q) = \big(1+\Sigma(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q)\big)^{-1},$$

where

$$\begin{aligned}\Sigma(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q) &= \sum_{j=1}^{N}|a_j|\prod_{p\le y}\bigg|1+\sum_{k=1}^{\infty}\frac{a_j(p^k)}{p^{k/2}}\exp(2\pi ik\theta_p)\bigg|\\ &= \sum_{j=1}^{N}|a_j|\times\bigg|\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\cdots\sum_{k_p=0}^{\infty}\cdots\sum_{k_q=0}^{\infty}\frac{a_j(2^{k_2}3^{k_3}\cdots p^{k_p}\cdots q^{k_q})}{\sqrt{2^{k_2}3^{k_3}\cdots p^{k_p}\cdots q^{k_q}}}\\ &\qquad\times\exp\big(2\pi i(k_2\theta_2+k_3\theta_3+\dots+k_p\theta_p+\dots+k_q\theta_q)\big)\bigg|.\end{aligned}$$

It follows from the conditions of the lemma that

$$\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q) = g_1\Big(\frac{1}{2}+it\Big)$$

for $\theta_p = -(t\log p)/(2\pi)$, $2\le p\le y$. Moreover, by the hypothesis we have

$$\bigg|1+\sum_{k=1}^{\infty}\frac{a_j(p^k)}{p^{k/2}}\exp(2\pi ik\theta_p)\bigg| > c_{F_j},$$

and therefore the function

$$\bigg|1+\sum_{k=1}^{\infty}\frac{a_j(p^k)}{p^{k/2}}\exp(2\pi ik\theta_p)\bigg|$$

is infinitely differentiable with respect to θ_p.

Consequently, the function $\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q)$ is infinitely differentiable with respect to each variable and periodic with respect to each variable with period 1. By virtue of the well-known theorem from analysis, it can be expanded in the Fourier series which can be termwise differentiated with respect to each variable any number of times.

The sum

$$\sum_{\substack{m,n\in M\\(m,n)=1\\m,n\le T^{1/200}}} x_{m,n}\left(\frac{m}{n}\right)^{it}$$

that appears in the formulation of the lemma is a partial sum of this Fourier series at the point

$$\left(-\frac{t\log 2}{2\pi},-\frac{t\log 3}{2\pi},\dots,-\frac{t\log p}{2\pi},\dots,-\frac{t\log q}{2\pi}\right).$$

2. Since $F_1(s),\dots,F_N(s)$ are functions from the Selberg class, there exists a constant $d>1$ such that the inequality $|a_j(n)|\le dn^{0.01}$ holds true for any j, $1\le j\le N$, and any n; if, in addition, $d^{100}<n$, then $|a_j(n)|\le n^{0.02}$.

Let $\xi=\max(d^{100},e^{25\pi})$, and $2,3,\dots,p_1$ be prime numbers not exceeding ξ.

Using the boundedness of all partial derivatives of the function $\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q)$, whose orders are bounded, we obtain the relation

$$\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q)=\sum_{|k_2|\le(\log T)^{0.01}}\cdots\sum_{|k_{p_1}|\le(\log T)^{0.01}}\exp\bigl(2\pi i(k_2\theta_2+\dots+k_{p_1}\theta_{p_1})\bigr)$$

$$\times\int_0^1\cdots\int_0^1\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q)\exp\bigl(-2\pi i(k_2\theta_2+\dots+k_{p_1}\theta_{p_1})\bigr)\,d\theta_2\cdots d\theta_{p_1}+O\bigl((\log T)^{-A}\bigr).$$

3. Let $p_2,\dots,p,\dots,q$ be prime numbers lying between ξ and y. Let

$$\Phi(\theta_{p_2},\dots,\theta_p,\dots,\theta_q)=\int_0^1\cdots\int_0^1\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q)\exp\bigl(-2\pi i(k_2\theta_2+\dots+k_{p_1}\theta_{p_1})\bigr)\,d\theta_2\cdots d\theta_{p_1}.$$

We expand $\Phi(\theta_{p_2},\dots,\theta_p,\dots,\theta_q)$ in the Fourier series

$$\Phi(\theta_{p_2},\dots,\theta_p,\dots,\theta_q)=\sum_{\overline{k}}b_{\overline{k}}\exp\bigl((\overline{k},\overline{\theta})\bigr),$$

where

$$\overline{k}=(k_{p_2},\dots,k_p,\dots,k_q),\qquad \overline{\theta}=(\theta_{k_{p_2}},\dots,\theta_{k_p},\dots,\theta_{k_q}),$$

$$b_{\overline{k}}=\int_0^1\cdots\int_0^1\Phi(\theta_{p_2},\dots,\theta_p,\dots,\theta_q)\exp\bigl(-2\pi i(\overline{k},\overline{\theta})\bigr)\,d\theta_{p_2}\cdots d\theta_p\cdots d\theta_q.$$

Then, by virtue of the relation obtained at the end of the preceding section, we have

$$\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q)=\sum_{|k_2|\le(\log T)^{0.01}}\cdots\sum_{|k_{p_1}|\le(\log T)^{0.01}}\exp\bigl(2\pi i(k_2\theta_2+\dots+k_{p_1}\theta_{p_1})\bigr)$$

$$\times\sum_{k_{p_2}=-\infty}^{\infty}\cdots\sum_{k_p=-\infty}^{\infty}\cdots\sum_{k_q=-\infty}^{\infty}b_{\overline{k}}\exp(k_{p_2}\theta_{k_{p_2}}+\dots+k_p\theta_{k_p}+\dots+k_q\theta_{k_q})+O\bigl((\log T)^{-A}\bigr)\bigr).$$

Setting $\theta_p = -(t\log p)/(2\pi)$, $2 \le p \le y$, and introducing the notation

$$m = \prod_{\substack{p\le\xi\\ 0\le k_p\le(\log T)^{0.01}}} p^{k_p} \prod_{\substack{\xi<p\le y\\ 0\le k_p}} p^{k_p}, \qquad n = \prod_{\substack{p\le\xi\\ 0<-k_p\le(\log T)^{0.01}}} p^{k_p} \prod_{\substack{\xi<p\le y\\ 0<-k_p}} p^{k_p},$$

and $x_{m,n}$, which is the corresponding Fourier coefficient, we obtain

$$g_1\Big(\frac{1}{2}+it\Big) = \Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q) = \sum_{\substack{m,n\in M\\ (m,n)=1}} x_{m,n}\Big(\frac{m}{n}\Big)^{-it} + O\big((\log T)^{-A}\big).$$

Let us estimate the sum

$$\sum_{\substack{m,n\in M\\ (m,n)=1\\ \max(m,n)>T^{1/200}}} x_{m,n}\Big(\frac{m}{n}\Big)^{-it}$$

trivially:

$$\sum_{\substack{m,n\in M\\ (m,n)=1\\ \max(m,n)>T^{1/200}}} x_{m,n}\Big(\frac{m}{n}\Big)^{-it} \ll (\log T)^{0.01\xi} \sum_{\substack{\overline{k}\\ |k_{p_2}|+\dots+|k_q|>(1/400)\log T(\log y)^{-1}}} |b_{\overline{k}}|.$$

Indeed, let, for instance, $m > T^{1/200}$. Then the inequality

$$|k_{p_2}| + \dots + |k_q| \le \frac{1}{400}\log T(\log y)^{-1}$$

is not satisfied since otherwise it would yield

$$\log m \le \xi(\log T)^{0.01} + \frac{1}{400}\log T < \frac{1}{200}\log T.$$

4. Let us estimate the sum W,

$$W = \sum_{\substack{\overline{k}\\ |k_{p_2}|+\dots+|k_q|>(1/400)\log T(\log y)^{-1}}} |b_{\overline{k}}|.$$

Let l be a positive integer whose value will be chosen later. Integrating by parts, we arrive at the inequality

$$|b_{\overline{k}}| \le \frac{1}{(2\pi)^{2l}} \int_0^1\!\!\cdots\!\int_0^1 \Big| \sum_{\xi<p\le y} \frac{\partial^{2l}}{\partial\theta_p^{2l}} \Phi(\theta_{p_2},\dots,\theta_p,\dots,\theta_q) \Big|\, d\theta_{p_2}\cdots d\theta_p\cdots d\theta_q \times (k_{p_2}^{2l}+\dots+k_p^{2l}+\dots+k_q^{2l})^{-1},$$

from which it follows that

$$W \le \frac{1}{(2\pi)^{2l}} \int_0^1 \cdots \int_0^1 \sum_{\xi<p\le y} \Big| \frac{\partial^{2l}}{\partial\theta_p^{2l}} \Phi(\theta_{p_2},\dots,\theta_p,\dots,\theta_q) \Big|\, d\theta_{p_2}\cdots d\theta_p\cdots d\theta_q \times U,$$

where

$$U = \sum_{|k_{p_2}|+\cdots+|k_q|>(1/400)\log T(\log y)^{-1}} (k_{p_2}^{2l}+\cdots+k_p^{2l}+\cdots+k_q^{2l})^{-1}.$$

In order to estimate W, it suffices now to estimate the quantities

$$\sum_{\xi<p\le y}\left|\frac{\partial^{2l}}{\partial\theta_p^{2l}}\Phi(\theta_{p_2},\dots,\theta_p,\dots,\theta_q)\right|$$

and U, the statement of the lemma will follow immediately from the inequality $W \le (\log T)^{-2A}$, where A is an arbitrary positive number.

5. We begin with estimating

$$\left|\frac{\partial^{2l}}{\partial\theta_p^{2l}}\Phi(\theta_{p_2},\dots,\theta_p,\dots,\theta_q)\right|.$$

We use the fact that by virtue of the choice of ξ, for any $p>\xi$ and any j, $1\le j\le N$, the inequalities

$$\left|\sum_{k=1}^{\infty}\frac{a_j(p^k)}{p^{k/2}}\exp(2\pi ik\theta_p)\right| \le \sum_{k=1}^{\infty}p^{-0.48k} < \frac{1}{2}$$

hold true, from which it follows that

$$\left(1+\sum_{k=1}^{\infty}\frac{a_j(p^k)}{p^{k/2}}\exp(2\pi ik\theta_p)\right)^{1/2} = 1+\sum_{n=1}^{\infty}\frac{\alpha_j(p^n)}{p^{n/2}}\exp(2\pi in\theta_p),$$

where

$$\alpha_j(p^n) = \sum_{l=1}^{n}\binom{1/2}{l}\sum_{\substack{k_1=1\\ k_1+\cdots+k_l=n}}^{n}\cdots\sum_{k_l=1}^{n} a_j(p^{k_1})\cdots a_j(p^{k_l})$$

with

$$|\alpha_j(p^n)| \le \sum_{l=1}^{n}\binom{n-1}{l-1}p^{0.02n} = (2p^{0.02})^n.$$

From the relations obtained we have

$$\Psi(\theta_2,\theta_3,\dots,\theta_p,\dots,\theta_q) = \left(1+\sum_{j=1}^{N}|a_j|\prod_{\substack{p_3\le y\\ p_3\ne p}}\left|1+\sum_{k=1}^{\infty}\frac{a_j(p_3^k)}{p_3^{k/2}}\exp(2\pi ik\theta_{p_3})\right|G_{1j}(\theta_p)G_{2j}(\theta_p)\right)^{-1},$$

where

$$G_{1j}(\theta_p) = 1+\sum_{k=1}^{\infty}\frac{\alpha_j(p^k)}{p^{k/2}}\exp(2\pi ik\theta_p), \qquad G_{2j}(\theta_p) = 1+\sum_{k=1}^{\infty}\frac{\overline{\alpha}_j(p^k)}{p^{k/2}}\exp(-2\pi ik\theta_p).$$

Let z_p be a complex number such that $|z_p|\le 1$ and θ_p be an arbitrary real number.

Using the inequalities $\xi < p$ and $|\alpha_j(p^n)| \le (2p^{0.02})^n$, we get one more inequality:

$$\left|\sum_{k=1}^{\infty} \frac{\alpha_j(p^k)}{p^{k/2}} \exp\bigl(2\pi i k(\theta_p + z_p)\bigr)\right| < \sum_{k=1}^{\infty} \frac{|\alpha_j(p^k)|}{p^{k/2}} \exp(2\pi k) < \sum_{k=1}^{\infty} \left(\frac{2p^{0.02}}{\sqrt{p}}\right)^k < \sum_{k=1}^{\infty} p^{-0.3k} < \frac{1}{4}.$$

It follows from the last inequality that

$$\begin{aligned}
&\mathrm{Re}\bigl(G_{1p}(\theta_p + z_p)G_{2p}(\theta_p + z_p)\bigr) \\
&\quad > 1 - \left|\sum_{k=1}^{\infty} \frac{\alpha_j(p^k)}{p^{k/2}} \exp\bigl(2\pi i k(\theta_p + z_p)\bigr)\right| - \left|\sum_{k=1}^{\infty} \frac{\overline{\alpha}_j(p^k)}{p^{k/2}} \exp\bigl(-2\pi i k(\theta_p + z_p)\bigr)\right| \\
&\qquad - \left|\sum_{k=1}^{\infty} \frac{\alpha_j(p^k)}{p^{k/2}} \exp\bigl(2\pi i k(\theta_p + z_p)\bigr)\right| \times \left|\sum_{k=1}^{\infty} \frac{\overline{\alpha}_j(p^k)}{p^{k/2}} \exp\bigl(-2\pi i k(\theta_p + z_p)\bigr)\right| \\
&\quad > 1 - \frac{1}{2} - \frac{1}{16} = \frac{7}{16} > 0.
\end{aligned}$$

We have proved that in the circle $|z_p| \le 1$ the function

$$\Psi(\theta_2, \theta_3, \dots, \theta_p + z_p, \dots, \theta_q)$$

is analytic, with

$$|\Psi(\theta_2, \theta_3, \dots, \theta_p + z_p, \dots, \theta_q)| \le 1.$$

(the last inequality follows immediately from

$$\mathrm{Re}\left(1 + \sum_{j=1}^{N} |a_j| \prod_{\substack{p_3 \le y \\ p_3 \ne p}} \left|1 + \sum_{k=1}^{\infty} \frac{a_j(p_3^k)}{p_3^{k/2}} \exp(2\pi i k\theta_{p_3})\right| G_{1j}(\theta_p) G_{2j}(\theta_p)\right) \ge 1.)$$

In order to estimate

$$\left|\frac{\partial^{2l}}{\partial\theta_p^{2l}} \Phi(\theta_2, \dots, \theta_p, \dots, \theta_q)\right|,$$

we apply the integral Cauchy formula. We have

$$\frac{\partial^{2l}}{\partial\theta_p^{2l}} \Phi(\theta_2, \dots, \theta_p, \dots, \theta_q) = \frac{(2l)!}{2\pi i} \int\limits_{\gamma} \frac{\Phi(\theta_2, \dots, \zeta_p, \dots, \theta_q)}{(\zeta_p - \theta_p)^{2l+1}}\, d\zeta_p,$$

where γ is a circle with center at the point θ_p of radius 1 which is traversed counterclockwise. From this we find that

$$\left|\frac{\partial^{2l}}{\partial\theta_p^{2l}} \Phi(\theta_2, \dots, \theta_p, \dots, \theta_q)\right| \le (2l)!\,.$$

6. We have thus obtained the estimate

$$W \le \frac{(2l)!}{(2\pi)^{2l}} \pi(y) U,$$

where

$$U = \sum_{|k_{p_2}|+\cdots+|k_q|>(1/400)\log T(\log y)^{-1}} (k_{p_2}^{2l} + \cdots + k_p^{2l} + \cdots + k_q^{2l})^{-1}.$$

For the given natural N, under the condition $\max(|k_{p_2}|, \ldots, |k_q|) = N$, the number of integer points $(k_{p_2}, \ldots, k_q) \in \mathbb{R}^{\pi(y)-\pi(\xi)}$ does not exceed $2\pi(y)(2N+1)^{\pi(y)-1}$, and therefore, setting $l = \pi(y)$, we obtain

$$U \le 2\pi(y)3^{\pi(y)} \sum_{N>(1/400)\log T\,(\log y)^{-1}} N^{-1-\pi(y)} \le 4\pi(y)\left(\frac{\log T}{1200\log y}\right)^{-\pi(y)},$$

and, consequently,

$$W \le 4\pi(y)\left(\frac{\log T}{4800\pi^2(y)\log y}\right)^{-\pi(y)}.$$

Since $y \le (\log T)^{0.001}$ by the hypothesis, the inequalities

$$4800\pi^2(y)\log y < \sqrt{\log T}, \qquad W \le (\log T)^{-2A}$$

are now valid for any $A > 0$. We have proved the lemma.

ACKNOWLEDGMENTS

The work was supported by the Russian Foundation for Fundamental Research (project no. 97-01-00721).

REFERENCES

1. Selberg, A., Old and New Conjectures and Results about a Class of Dirichlet Series, in: *Proc. of the Amalfi Conf. on Analytic Number Theory*, Univ. di Salerno, 1992, pp. 365–387.
2. Conrey, J.B. and Ghosh, A., On the Selberg Class of Dirichlet Series: Small Degrees, *Duke Math. J.*, 1993, vol. 72, no. 3, pp. 673–695.
3. Titchmarsh, E., *The Zeta-Function of Riemann*, Oxford, 1953.
4. Riemann, B., *Collected Works* [*Russ. transl.*], Moscow, Leningrad, 1948.
5. Voronin, S. M., On the Zeros of Some Dirichlet Series Lying on a Critical Line, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1980, vol. 44, no. 1, pp. 63–91.
6. Karatsuba, A. A., On the Zeros of the Davenport–Heilbronn Function Lying on a Critical Line, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1990, vol. 54, no. 2, pp. 303–315.
7. Karatsuba, A. A., On the Zeros of a Special Kind of Functions Related to Dirichlet Series, *Izv. Akad. Nauk SSSR, Ser. Mat.*, 1991, vol. 55, no. 3, pp. 483–514.
8. Karatsuba, A. A., A New Approach to the Problem of Zeros of Some Dirichlet Series, *Tr. Mat. Inst. Akad. Nauk*, 1994, vol. 207, pp. 180–196.
9. Gritsenko, S. A., On the Zeros of Linear Combinations of a Special Kind of Functions Related to Dirichlet Series from Selberg's Class, *Izv. Ross. Akad. Nauk, Ser. Mat.*, 1996, vol. 60, no. 4, pp. 3–42.
10. Gritsenko, S. A., On the Zeros of a Special Kind of Functions Related to Dirichlet Series from Selberg's Class, *Mat. Zametki*, 1996, vol. 60, no. 4, pp. 601–605.
11. Gritsenko, S.A., On the Zeros of Linear Combinations of a Special Kind of Functions Related to Hecke L-Functions of Imaginary Quadratic Fields, *Izv. Ross. Akad. Nauk, Ser. Mat.*, 1997, vol. 61, no. 1, pp. 45–68.

Translated by I. Aleksandrova