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On the Exponent of Convergence of Trigonometric Integrals

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The method of trigonometric sums is one of the most powerful tools in analytic number theory. Trigonometric integrals play an especially important role. Moreover, many problems in both mathematical physics and the theory of probabilities require the investigation of trigonometric integrals. Namely, the asymptotic behavior, the estimation and summation exponent of convergence with respect to parameters of these integrals are of importance.

In this paper we consider multidimensional trigonometric integrals with polynomial phases and get a lower estimate for the exponent of convergence of functions defined by these integrals. Moreover, we obtain an explicit value of the exponent of convergence in special cases.

1. INTRODUCTION

Let $Q(x,a) \in \mathbb{R}[x]$ (where $x \in \mathbb{R}^{m+1}$) be a polynomial with real coefficients $a \in \mathbb{R}^N$. We consider the trigonometric integral given by

$$J(a) = \int\limits_{\Pi} \exp\bigl(2\pi i Q(x,a)\bigr)\,dx, \tag{1}$$

where $\Pi = [0,1]^{m+1}$ is an $(m+1)$-dimensional parallelepiped. A special integral of Tarry's problem is the integral

$$\theta = \int\limits_{\mathbb{R}^N} |J(a)|^g\,da \tag{2}$$

which appears as the coefficient of asymptotic representation in a number of integer solutions of Diophantine systems [4, 13]. Therefore it is important to find the minimal value of the parameter g such that the singular integral is convergent.

The real number γ is called an *exponent of convergence* of the singular integral if integral (2) is convergent for every $g > \gamma$ and divergent for every $g < \gamma$. It should be noted that the exponent of convergence essentially depends on the form of the polynomials $Q(x,a)$.

Let $Q(x,a)$ be a polynomial of the form

$$Q(x,a) = \sum_{k=1}^{n} Q_k(x,a), \tag{3}$$

where $Q_k(x,a)$ is a homogeneous polynomial of degree k, i.e.,

$$Q_k(x,a) = \sum_{|\nu|=k} a_\nu x^\nu,$$

where $\nu = (\nu_1, \dots, \nu_{m+1}) \in \mathbb{Z}_+^{m+1}$, $|\nu| = \nu_1 + \nu_2 + \dots + \nu_{m+1}$, $x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots x_{m+1}^{\nu_{m+1}}$. In this case, the following upper estimate for γ is well known [4]:

$$\gamma \le (m+1)\binom{n+m+1}{m+2} + 1, \qquad \text{where} \quad \binom{N}{M} = \frac{N!}{(N-M)!M!}.$$

We obtain a lower bound for γ in the following theorem.

Theorem 1. *Let the phase function $Q(x,a)$ be a polynomial of the form* (3) *and γ be the exponent of convergence. Then we have the following estimate for the exponent of convergence*:

$$\gamma \ge \binom{n+m+1}{m+2} + 1.$$

Let $Q(x,a)$ be a polynomial of the form

$$Q(x,a) = \sum_{\nu_1=0}^{n_1} \cdots \sum_{\nu_{m+1}=0}^{n_{m+1}} a_\nu x^\nu, \tag{4}$$

where $\nu = (\nu_1, \dots, \nu_{m+1})$ is a multi-index, $x^\nu = x_1^{\nu_1} \cdots x_{m+1}^{\nu_{m+1}}$, and $a_{0\dots0} = 0$.

Note that in this case the dimension of the space of the polynomial is

$$M = (\nu_1 + 1) \cdots (\nu_{m+1} + 1) - 1.$$

Arkhipov, Karatsuba, and Chubarikov [1–4, 14, 15] proved the following upper estimate for the exponent of convergence:

$$\gamma \le M \max\{\nu_1, \dots \nu_{m+1}\}.$$

We get a lower estimate for γ in the following theorem.

Theorem 2. *If the phase function $Q(x,a)$ of the trigonometric integral* (1) *has the form* (4), *then the estimate*

$$\gamma \ge \frac{1}{2}(M+1)\max\{\nu_1, \dots, \nu_{m+1}\} + 1$$

holds for the exponent of convergence γ.

If some coefficients of the polynomial $Q(x,a)$ are absent, then the system of Diophantine equations is incomplete. We consider one of these cases.

Let $(x,y) \in \mathbb{R} \times \mathbb{R}^m$ and let the phase function $Q(x,y,a)$ be a polynomial of the form

$$Q(x,y,a) = \sum_{k=[n/2]+1}^{n} Q_k(y,a) x^k, \tag{5}$$

where $[n/2]$ is the integer part of $n/2$ and $Q_k(y,a)$ is a polynomial of the form

$$Q_k(y,a) = \sum_{\nu_1=0}^{k} \cdots \sum_{\nu_m}^{k} a_\nu y^\nu.$$

Then we have the following result.

Theorem 3. *If the phase function $Q(x,y,a)$ in integral* (1) *has the form* (5), *then the exponent of convergence is defined by the formula*

$$\gamma = \sum_{k=[n/2]+1}^{n} k(k+1)^m.$$

2. ON THE ASYMPTOTIC BEHAVIOR OF ONE-DIMENSIONAL TRIGONOMETRIC INTEGRALS

Let the function $F(x)$ be n times continuously differentiable and let a be a function of bounded variation on $[\xi,\eta]$, where $-\infty \le \xi < \eta \le +\infty$. We consider the trigonometric integral

$$J = \int_{\xi}^{\eta} a(x)\exp\bigl(2\pi i F(x)\bigr)\,dx.$$

The following generalized van der Corput lemma is needed for the sequel.

Lemma 1 (see [4, 7, 10, 11]). *If the nth derivative of the function F has no more than k nulls, then the following estimate holds*:

$$|J| \le \frac{c\|a\|_v}{H(F)},$$

where

$$H(F) = \inf_{x\in[\xi,\eta]} \sum_{k=1}^{n} \left|\frac{F^{(k)}(x)}{k!}\right|^{1/k}, \qquad \|a\|_v = |a(\xi)| + V_{[\xi,\eta]}[a],$$

$V_{[\xi,\eta]}[a]$ *is the total variation of the function a, and c is a constant dependent only on n and k.*

Let $a<0, b>0$ be fixed real numbers. We consider the trigonometric integral

$$J = \int_{\xi}^{\eta} \exp\bigl(2\pi i(\alpha_n x^n + \alpha_{n-1}x^{n-1} + \dots + \alpha_1 x)\bigr)\,dx, \tag{6}$$

where $\xi \le a$, $\eta \ge b$ are fixed parameters.

Lemma 2. *Suppose that the coefficients of the polynomial $q(x) = \alpha_n x^n + \dots + \alpha_1 x$ are defined by the system of inequalities*

$$P^n \le \alpha_n \le (2P)^n, \quad |\alpha_{n-1}| \le \delta P^{n-1}, \quad \dots, \quad |\alpha_1| \le \delta P.$$

Then integral (6) *has the asymptotic representation*

$$J = \frac{c_1(\alpha)}{\alpha_n^{1/n}} + O(P^{1-n}) \qquad \text{as} \quad P\to\infty.$$

Moreover, the coefficient $c_1(\alpha)$ satisfies the relation

$$c_1(\alpha) = c + o(\delta) \qquad \text{as} \quad \delta \to 0,$$

where c is a nonvanishing number dependent only on n.

Proof. We apply the change of variable $x \to x\alpha_n^{-1/n}$ in the integration and have

$$J = \alpha_n^{-1/n} \int_{\xi\alpha_n^{1/n}}^{\eta\alpha_n^{1/n}} \exp\bigl(2\pi i(x^n + \beta_{n-1}x^{n-1} + \dots + \beta_1 x)\bigr)\,dx,$$

where $\beta_k = \alpha_k \alpha_n^{k/n}$, $k = 1, 2, \dots, n-1$.

It follows from the hypothesis of Lemma 1 that $|\beta_k| \le \delta$, where $k = 1, 2, \dots, n-1$. Next, we consider the estimation of the integral

$$J_1 = \int\limits_{\eta\alpha_n^{1/n}}^{\infty} \exp\bigl(2\pi i(x^n + \beta_{n-1}x^{n-1} + \dots + \beta_1 x)\bigr)\, dx.$$

If the number P is sufficiently large and δ is sufficiently small, then it follows from Lemma 1 that the integral J_1 has the estimate

$$|J_1| \le c\alpha_n^{(2-n)/n},$$

where c is a positive number. Similarly, the integral

$$J_2 = \int\limits_{-\infty}^{\xi\alpha_n^{1/n}} \exp\bigl(2\pi i(x^n + \beta_{n-1}x^{n-1} + \dots + \beta_1 x)\bigr)\, dx$$

has an estimate

$$|J_2| \le c\alpha_n^{(2-n)/n}.$$

Let

$$J_k(\beta) = \int\limits_{\mathbb{R}} \exp\bigl(2\pi i(x^n + \beta_{n-1}x^{n-1} + \dots + \beta_1 x)\bigr)\, dx.$$

It is well known that the integral $J_k(\beta)$ is conditionally convergent and has an analytical continuation on $\mathbb{C}^{n-1}$ with respect to the parameters $(\beta_{n-1}, \dots, \beta_1)$, moreover $J_k(0) \ne 0$ (see [5, 6, 9]). In fact, the integral $J_k(0)$ is calculated and expressed by Euler integrals. It follows that

$$J(\alpha) = \alpha_n^{-1/n} J_k(\beta) + O(\alpha_n^{(1-n)/n}),$$

or, equivalently,

$$J(\alpha) = \alpha_n^{-1/n} J_k(\beta) + O(P^{1-n}).$$

It is clear that the function $c_k(\alpha) = J_k\bigl(\beta(\alpha)\bigr)$ satisfies the conditions of Lemma 2. This completes the proof of Lemma 2.

3. PROOF OF THEOREM 1

We will prove Theorem 1 in the two-dimensional case. In the general case it can be proved in a similar way. Let the number P be a sufficiently large positive integer. We define the number x_r by P: $x_r = 1/4 + r/2P$, $r = 1, 2, \dots, P$. Let us denote by $\|q\|$ the norm of the polynomial $q(x)$ in the space $C[0,1]$, i.e.,

$$\|q\| = \max_{x \in [0,1]} |q(x)|.$$

The expansion of the polynomial $Q(x, y, a)$ about the number $x = x_r$ has the form

$$Q(x, y, a) = \beta_n(y)(x - x_r)^n + \beta_{n-1}(x - x_r)^{n-1} + \dots + \beta_1(y)(x - x_r) + \beta_0(y),$$

where $\beta_k(y)$ is a polynomial of degree $n-k$. It should be noted that $\beta_n(y)=\alpha_n^n$ and that it does not depend on y and x_r. We define the set

$$\Omega_r=\bigl\{\alpha\in\mathbb{R}^N : P^n\le\alpha_n^n\le(2P)^n,\ \|\beta_k(y)\|\le\delta P^k,\ k=1,2,\dots,n-1\bigr\},$$

where δ is a sufficiently small positive number.

It is easy to prove that $\Omega_{r_1}\cap\Omega_{r_2}=\varnothing$ whenever $r_1\ne r_2$. There exists a positive number $c(n)$ such that for every number r, $r=1,2,\dots,P$, the following inequality holds:

$$\mu(\Omega_r)\ge c(n)P^{n(n+1)(n+2)/6},$$

where $\mu(\Omega_r)$ is the Lebesgue measure of Ω_r.

We represent the trigonometric integral (2) as an iterated integral

$$J=\int_0^1 dy\int_0^1 \exp\bigl(2\pi i Q(x,y)\bigr)\,dx.$$

Let

$$J_0(\alpha,y)=\int_0^1 \exp\bigl(2\pi i(\beta_n(y)(x-x_r)^n+\beta_{n-1}(y)(x-x_r)^{n-1}+\dots+\beta_1(y)(x-x_r))\bigr)\,dx.$$

We use a change of variable in the integration and have

$$J_0(\alpha,y)=\int_{-x_r}^{1-x_r} \exp\bigl(2\pi i(\beta_n(y)x^n+\beta_{n-1}(y)x^{n-1}+\dots+\beta_1(y)x)\bigr)\,dx.$$

It follows from Lemma 2 that the integral $J_0(\alpha,y)$ has the asymptotic representation

$$J_0(\alpha,y)=c(\alpha,y)\alpha_n^{-1/n}+O(P^{1-n}),$$

where $\alpha\in\Omega_r$. Therefore, if the number P is sufficiently large and the number δ is sufficiently small, then, for $\alpha\in\Omega_r$ and $\|\beta_0(y)\|\le\delta$, the integral $J(\alpha)$ has the estimate

$$|J(\alpha)|\ge cP^{-1},$$

where c is a fixed positive number. Let the last inequality hold for $P\ge 2^{m_0}$, where m_0 is a fixed positive integer.

Then the lower estimate of the singular integral θ is

$$\theta\ge\sum_{m=m_0}^{\infty}\sum_{r=1}^{2^m}\int_{\Omega_r}|J(\alpha)|^{2k}\,d\alpha\ge\sum_{m=m_0}^{\infty}c(n)2^{(n(n+1)(n+2)/6+1-2k)m}.$$

It is obvious that if $2k\le\frac{n(n+1)(n+2)}{6}+1$, then the integral θ is divergent. This completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

We assume, as above, that $x \in \mathbb{R}^2$ and $Q(x,y,a)$ is a polynomial of the form

$$Q(x,y,a) = \sum_{\nu=0}^{m} \sum_{\mu=0}^{n} a_{\nu\mu} x^{\nu} y^{\mu}$$

and $a_{00} = 0$.

Furthermore, we can assume without loss of generality that $n \geq m$, i.e., $\max\{n,m\} = n$. As above, we suppose that P is a sufficiently large positive integer. There exists a number $\delta > 0$ such that the following inequalities hold:

$$1.5P^n \leq |a_{n0}| \leq (2P)^n, \qquad |a_{nl}| \leq \delta P^n, \quad \text{where } l = 1, 2, \dots, n,$$

and

$$\min_{y \in [0,1]} |\sum_{l=0}^{m} a_{nl} y^l| \geq P.$$

By analogy, the polynomial $Q(x,y,a)$ can be represented in the form

$$Q(x,y,a) = \beta_n(y)(x - x_r)^n + \beta_{n-1}(x - x_r)^{n-1} + \dots + \beta_1(y)(x - x_r) + \beta_0(y),$$

where $x_r = 1/4 + r/(2P)$.

It should be noted that

$$\beta_n(y) = \sum_{l=0}^{m} a_{nl} y^l,$$

i.e., $\beta_n(y)$ does not depend on x_r. We define the following set of fixed numbers r and P:

$$\Omega_r = \{\alpha \in \mathbb{R}^M : 1.5P^n \leq |a_{n0}| \leq (2P)^n, \ |a_{nl}| \leq \delta P^n, \ \|\beta_k(y)\| \leq \delta P^k, \ k = 1, \dots, n-1\}.$$

It is easy to prove that if δ is small and positive, then $\Omega_{r_1} \cap \Omega_{r_2} = \varnothing$ whenever $r_1 \neq r_2$, and the following inequality holds:

$$\mu(\Omega_r) \geq c(n) P^{(M+1)n/2},$$

where $\mu(\Omega_r)$ is the Lebesgue measure of Ω_r and $c(n)$ is a positive number.

As above, the trigonometric integral (2) is represented as an iterated integral

$$J = \int_0^1 dy \int_0^1 \exp(2\pi i Q(x,y))\, dx.$$

Let

$$J_0(\alpha, y) = \int_0^1 \exp\bigl(2\pi i(\beta_n(y)(x - x_r)^n + \beta_{n-1}(y)(x - x_r)^{n-1} + \dots + \beta_1(y)(x - x_r))\bigr)\, dx.$$

On the other hand, by a change of variable we have

$$J_0(\alpha, y) = \int_{-x_r}^{1-x_r} \exp(2\pi i(\beta_n(y)x^n + \beta_{n-1}(y)x^{n-1} + \dots + \beta_1(y)x))\, dx.$$

It follows from Lemma 2 that the integral $J_0(\alpha, y)$ has the asymptotic representation

$$J_0(\alpha, y) = c(\alpha, y)\alpha_n^{-1/n} + O(P^{1-n})$$

whenever $\alpha \in \Omega_r$.

Consequently, if P is sufficiently large and δ is sufficiently small, then the integral $J(\alpha)$ has a lower estimate

$$|J(\alpha)| \geq \frac{c}{P},$$

where c is a positive number.

Let the last estimation hold for every $P \geq 2^{\nu_0}$, where the number ν_0 is a positive integer. Then the singular integral θ has a lower estimate

$$\theta \geq \sum_{\nu=\nu_0}^{\infty} \sum_{r=1}^{2^\nu} \int_{\Omega_r} |J(\alpha)|^{2k}\, d\alpha \geq \sum_{\nu=\nu_0}^{\infty} c(n)\big(2^{(M+1)n/2+1-2k}\big)^{\nu}.$$

It is clear that if $2k \leq (M+1)n/2+1$, then the singular integral θ is divergent. This completes the proof.

5. AN INCOMPLETE CASE

Let $Q(x, y, a)$ be a polynomial of the form

$$Q(x, y, a) = \sum_{k=[n/2]+1}^{n} Q_k(y, a)x^k, \tag{7}$$

where $Q_k(y, a)$ is a polynomial of degree $\leq k$, i.e.,

$$Q_k(y, a) = \sum_{\nu=0}^{k} a_\nu y^\nu.$$

The following result is valid.

Proposition 1. *If the phase function Q in integral* (2) *has the form* (7), *then the exponent of convergence is defined by the expression*

$$\gamma = \frac{n(n+1)(n+2) - ([n/2]([n/2]+1)([n/2]+2))}{3}.$$

We shall begin with considering some properties of polynomials. Let $f(x, a)$ be a polynomial of the form

$$f(x, a) = \sum_{k=[n/2]+1}^{n} a_k x^k. \tag{8}$$

The following lemma is needed for the sequel.

Lemma 3. *If the polynomial $f(x,a)$ has the form* (8) *and $[\xi,\eta]=[0,1]$, then there exists a positive real number c such that the following inequality holds*:

$$H(f)\ge c\sum_{k=[n/2]+1}^{n}|a_k|^{1/k}. \tag{9}$$

Proof. We assume for definiteness that the number n is odd and set $n=2m+1$. We denote by Hi Hilbert's D-invariant of the translation group [12]. In our case Hilbert's D-invariant is defined by the relation

$$\mathrm{Hi}(f')=f'(x)f^{(n)}(x)-f''(x)f^{(n-1)}(x)+\cdots+(-1)^m\frac{(f^{(m+1)}(x))^2}{2}.$$

It is easy to prove that the inequality

$$|\,\mathrm{Hi}(f')|^{1/(n+1)}\le c\sum_{k=1}^{n}|f^{(k)}(x,a)|^{1/k}$$

holds for every $x\in[0,1]$, where c is a positive number. At the same time, the function $\mathrm{Hi}(f')$ is an invariant of the translation group, and therefore it does not depend on x. Thus we have

$$|\,\mathrm{Hi}(f')|^{1/(n+1)}\le cH(f).$$

On the other hand, we have $\mathrm{Hi}(f')=(-1)^m(a_{m+1})^2/2$. Consequently, we obtain

$$|a_{m+1}|^{1/m+1}\le cH(f).$$

In the same way we get the estimates

$$|\,\mathrm{Hi}(f^{(k)})|^{1/(n+2k-1)}\le c_kH(f),$$

or, equivalently,

$$|a_{m+k}|^{1/(m+k)}\le c_kH(f),$$

where $k=1,2,\dots,m+1$. Finally, summing up the last estimates for $k=1,2,\dots,m+1$, we obtain inequality (9). The proof is omitted in the case $n=2m$ since this is similar. The proof of Lemma 3 is completed.

Let $Q_k(y,a)$ be a polynomial of degree k, i.e.,

$$Q_k(y,a)=a_k^ky^k+a_{k-1}^ky^{k-1}+\cdots+a_0^k,\qquad k=1,2,\dots,n.$$

We define $N(a)$ and $h(Q)$ by the expressions

$$N(a)=\sum_{k,l}|a_k^l|^{1/l}\qquad\text{and}\qquad h(Q)=\sum_{k=1}^{n}|Q_k(y)|^{1/k}.$$

The following lemma is proved by analogy with [8, p. 182].

Lemma 4. *Let $\varepsilon < 1$ be a fixed positive number. Then there exists a number $c = c(\varepsilon)$ such that the inequality*

$$\int_0^1 h(Q)^{-\varepsilon}\, dy \le c\big(N(a)\big)^{-\varepsilon}$$

holds true.

Proof. The following estimation holds:

$$\int_0^1 h(Q)^{-\varepsilon}\, dy \le \min_k \Big\{ \int_0^1 |Q_k(y)|^{-\varepsilon/k} dy \Big\}.$$

From the lemma given in [8, p. 182] it follows that

$$\int_0^1 |Q_k(y)|^{-\varepsilon/k}\, dy \le c(\varepsilon) \Big(\sum_{l=0}^{k} |a_l^k| \Big)^{-\varepsilon/k}.$$

Then we obtain

$$\int_0^1 h(Q)^{-\varepsilon}\, dy \le c\big(N(a)\big)^{-\varepsilon}.$$

Proof of Proposition 1. We can represent the trigonometric integral (1) as an iterated integral.

Let J_0 be an inner integral, i.e.,

$$J_0(\alpha, y) = \int_0^1 \exp\Big(2\pi i \Big(\sum_{k=[n/2]+1}^{n} Q_k(y) x^k \Big)\Big)\, dx.$$

Using Lemma 1 and Lemma 3, we get

$$|J_0(a)| \le \min\{1, ch(Q)^{-1}\}.$$

Then it follows from Lemma 4 that

$$|J(a)| \le \min\{1, cN(a)^{-\varepsilon}\}.$$

The last inequality gives the estimate

$$|J(a)| \le \frac{c}{1 + N(a)^{\varepsilon}}.$$

It is easy to prove that if δ is a positive number and

$$\frac{M}{M+\delta} < \varepsilon < 1,$$

where

$$M = \frac{n(n+1)(n+2) - ([n/2]([n/2]+1)([n/2]+2))}{3}$$

then the integral

$$\int \frac{da}{(1+N(a)^{\varepsilon})^{M+\delta}}$$

is convergent. Thus, we get $\gamma \le M$ since δ is a positive number.

As above, we define the set

$$\Omega_P = \big\{a : 1.5P^n < |a_0^n| \le (2P)^n,\ |a_l^n| \le \delta P^n,\ \|Q_k(y)\| \le \delta P^k,\ k = [n/2]+1, \dots, n-1\big\}.$$

It is easy to prove that $\Omega_P \bigcap \Omega_{2P} = \varnothing$ and $\mu(\Omega_P) \ge c(n)P^M$, where the number $c(n)$ is positive. Consequently, if P is sufficiently large and the number δ is sufficiently small, then the integral $J(a)$ has a lower estimate

$$|J(a)| \ge \frac{c}{P}$$

whenever $a \in \Omega_P$, where c is a positive number (see Lemma 2).

Let the last estimation hold for every $P \ge 2^{\nu_0}$, where the number ν_0 is a positive integer. As above, the singular integral θ has a lower estimate

$$\theta \ge \sum_{\nu=\nu_0}^{\infty} \int\limits_{\Omega_{2^\nu}} |J(a)|^g \, da \ge \sum_{\nu=\nu_0}^{\infty} c(n)(2^{M-g})^{\nu}.$$

It is clear that if $g \le M$, then the singular integral θ is divergent. This completes the proof.

Let $p(x,a) \in \mathbb{R}[x_1, \dots, x_2]$ be a polynomial, with real coefficients, of the form

$$p(x,a) = \sum_{\nu_1=0}^{n_1} \sum_{\nu_2}^{n_2} \cdots \sum_{\nu_m=0}^{n_m} a_\nu x^\nu,$$

where, as above, ν is a multi-index and x^ν has an associated degree. We define the norm of the polynomial $p(x,a)$ as

$$N(a) = \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \cdots \sum_{\nu_m=0}^{n_m} |a_\nu|.$$

The following lemma is a version of the proposition by Ricci and Stein [8].

Lemma 5. *Let $d = \max\{\nu_1, \nu_2, \dots, \nu_m\}$ and let $\varepsilon < d^{-1}$ be a fixed positive number. There exists a constant A_ε, dependent only on ε and d, such that the following estimation holds*:

$$\int\limits_{\Pi} |p(x,a)|^{-\varepsilon} \, dx \le A_\varepsilon \big(N(a)\big)^{-\varepsilon}.$$

Proof. We will prove the lemma in the two-dimensional case. Let

$$p(x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} a_{kj} x^k y^j, \qquad N(a) = \sum_{k=0}^{m} \sum_{j=0}^{n} |a_{kj}|.$$

We assume that $N(a) = 1$ and

$$\max_{k,j}\{|a_{kj}|\} = |a_{k_0 j_0}| \geq \frac{1}{(m+1)(n+1)}.$$

Reasoning as in [8], we see that

$$\int_0^1 |p(x,y)|^{-\varepsilon}\, dx \leq \frac{B(\varepsilon,m)}{\left|\sum_{j=0}^{n} a_{k_0 j} y^j\right|^{\varepsilon}},$$

where the constant $B(\varepsilon,m)$ depends only on m and ε. It can be shown that

$$\int_0^1\int_0^1 |p(x,y)|^{-\varepsilon}\, dx \leq \big((m+1)(n+1)\big)^{\varepsilon} B(\varepsilon,m)B(\varepsilon,n).$$

In the general case, the lemma can be proved by induction.

With the use of Lemma 5, Theorem 3 can be proved by analogy with Proposition 1.

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