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On the Rationality Problem for Three-Dimensional Algebraic Varieties

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INTRODUCTION

The article contains a review of the methods for proving the nonrationality of algebraic varieties and the contemporary state of the classical problem of characterization of three-dimensional rational varieties. Recall that the irreducible algebraic variety X over the field k is rational if, over the algebraic closure $\overline{k}$ of the field k, it (to be more precise, $\overline{X} = X \otimes k$) is birationally equivalent to the projective space $\mathbb{P}^n_{\overline{k}}$. Rational curves are characterized by a single invariant, namely, the smooth curve X is rational if and only if the genus $g(X)$ is zero. One of the most remarkable results of the classical algebraic geometry is the rationality criterion for algebraic surfaces, i.e., the smooth projective surface X is rational if and only if $q(X) := h^1(X, \mathcal{O}_X) = 0$, $p_2(X) := h^0\big(X, \mathcal{O}_X(2K_X)\big) = 0$, where K_X is the canonical class of divisors on X.

It is clear that on the rational nonsingular complete variety X (say, over a field of characteristic zero) all principal geometric-differential invariants vanish, i.e., $h^0\big(X, (\Omega^p_X)^{\otimes q}\big) = 0$, where Ω^p_X is a bundle of regular p-forms on X, $p = 1, \dots, \dim X$, $q \geq 1$, in particular, $P_m = h^0\big(X, (\Omega^n)^{\otimes m}\big) = h^0\big(X, \mathcal{O}_X(mK_X)\big) = 0$ for all $m \geq 1$, in other words, the Kodaira dimension $\varkappa(X) = -\infty$. The criterion of rationality for curves and surfaces asserts that the vanishing of the principal geometric-differential invariants is also sufficient for the rationality in three cases.

For a long time beginning with the end of the last century, algebraic geometers tried unsuccessfully to find a criterion of rationality for three-dimensional algebraic varieties in terms of the vanishing of certain discrete birational invariants by analogy with the Castelnuovo rationality criterion for surfaces. The problem of rationality is directly connected with the Lüroth problem, namely, whether every unirational variety is rational. Recall that the variety X is unirational if there exists a rational surjective map $\mathbb{P}^n \to X$.

The negative solution of the Lüroth problem of dimension $n \geq 3$ (see [3, 11, 28] and also the review [6]) did not leave any hope for finding a criterion of rationality in terms of geometric-differential invariants alone: all of them vanish both on a unirational variety and on $\mathbb{P}^n$. As is well known, in the cases of curves and surfaces over an algebraically closed field of characteristic zero the criteria of rationality yield the coincidence of classes of unirational and rational varieties.

We can see from the definitions that the negativity of the Kodaira dimension $\varkappa(X) = -\infty$ is a necessary condition for the rationality and/or unirationality of X. The classical problem of the description of the classes of three-dimensional varieties with $\varkappa = -\infty$ up to birational equivalence was solved in the framework of the Mori theory of minimal models (see [40]).

Every nonsingular projective three-dimensional variety Y with $\varkappa(Y) = -\infty$ over the algebraically closed field k of characteristic zero is birationally equivalent to a projective variety with at most $\mathbb{Q}$-factorial terminal singularities (see Definition 1.2, Section 1) and with the morphism $\varphi\colon X \to S$, i.e., an extremal Mori contraction of fibering type (see, e.g., [32, 40]) from the following three classes:

(I) S is a point, X is a $\mathbb{Q}$-Fano variety, i.e., $-K_X$ is ample and $\rho(X)=1$, where $\rho(X):=\operatorname{rk}\operatorname{Pic} X$ is a Picard number,

(II) S is a nonsingular projective curve, $\varphi\colon X \to S$ is a del Pezzo bundle: all fibers are irreducible reduced (possibly singular) Del Pezzo surfaces, i.e., $-K_X$ is relatively φ-ample over S, and the relative Picard number $\rho(X/S) = 1$ (in this case the general fiber X_η is a smooth Del Pezzo surface with $\rho(X_\eta) = 1$ over the residue field $k(\eta)$ of the general point $\eta \in S$);

(III) S is a normal surface with no more than rational quotient singularities, $\varphi\colon X \to S$ is a conic bundle, i.e., the general fiber $X_\eta = \varphi^{-1}(\eta)$ is a smooth conic in $\mathbb{P}^2_{k(\eta)}$, $-K_X$ is relatively ample, and $\rho(X/S) = 1$ (i.e., $\operatorname{Pic} X = \varphi^* \operatorname{Pic} S + \mathbb{Z}$).

Thus the problem of rationality (and unirationality) is devided into three parts, namely, the characterization of rational (unirational) varieties in each of these three classes. Note that if X is a rational or unirational, then S in (II) and (III) is rational, i.e., $S = \mathbb{P}^1$ in (II) and S is a rational surface in (III). Moreover, it is well known (see, e.g., [16, 38, 55]) that for any conic bundle $\psi\colon Z \to T$ over the surface T there exists a so-called standard model, i.e., a conic bundle $\pi\colon X \to S$ with $\operatorname{Pic} X = \pi^* \operatorname{Pic} S + \mathbb{Z}$, and we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, "\pi"'] & Z \arrow[d, "\psi"] \\ S \arrow[r, "\alpha"] & T \end{tikzcd},$$

where α is birational morphism, Φ is a certain birational mapping which induces an isomorphism of the general fibers of the morphisms and ψ, and X, S are nonsingular and projective.

It should also be pointed out that varieties of type (I) constitute a limited number of families [31]. Moreover, for nonsingular three-dimensional Fano varieties there exists a full classification with an accuracy to within the deformations (see [17, 18, 27, 41, 42, 66]). The number of families of varieties of type (II) and, all the more so, of type (III), is infinite, although for varieties of type (II) the discrete invariant $K^2_{X_\eta} = 1, 2, \ldots, 6, 8, 9$, i.e., the degree of the Del Pezzo surface X_η of the general fiber of the morphism φ is bounded. It is clear that the same nonsingular three-dimensional variety, say, $\mathbb{P}^3$, may fall in different classes (I)–(III) with an accuracy to within the birational equivalence, so that when studying the problem of rationality, say, by means of the classical Noether–Fano method of factorization of birational mappings (see Section 1), we must consider all the three types of varieties simultaneously since elementary birational maps (links) do not, in general, preserve the type. From the point of view of rationality problem over the basic field k (i.e., of the problem of existence of a birational mapping $X \dashrightarrow \mathbb{P}^2$ defined over k), the case of rational surfaces over the algebraically closed field k can serve as an example.

Let X be a smooth projective surface defined over a perfect field. We assume that $\varkappa(X) = -\infty$, and then X is birationally equivalent over k to the smooth projective surface F over k (to be more precise, there exists a birational morphism $X \to F$), where F belongs to one of the following two classes (see [21, 39]):

(D) Del Pezzo surfaces F with $\operatorname{Pic}_k F = \mathbb{Z}$, here $K_F^2 = 1, 2, \ldots, 6, 8, 9$;

(C) conic bundles $\pi \colon X \to C$, where C is a smooth projective curve (possibly having no k-points), $\operatorname{Pic}_k F = \pi^* \operatorname{Pic} C + \mathbb{Z}$.

If F is a rational surface, then C is a curve of genus zero, i.e., either it is a smooth conic without k-points or $C = \mathbb{P}^1$. For rational surfaces over the perfect field k the problem of their birational triviality over k can be solved by the classical Noether method of factorization of birational maps into elementary constituents, namely, links. Here is one of the examples of an elementary link: $\delta \colon F \to \mathbb{P}^2$, which is a blow-up of four conjugate over k points in the general position on $\mathbb{P}^2$ through which a single bundle of conics passes. In this case, $F \in$ (C), and $\mathbb{P}^2 \in$ (D). Numerous results obtained in this direction are summed up in the work [23]. In particular, we have the following criterion of birational triviality of the rational surface $F \in (\mathrm{C}) \cup (\mathrm{D})$ over k:

The rational surface $F \in (\mathrm{C}) \cup (\mathrm{D})$ over the perfect field k is birationally trivial over k if and only if the following two conditions are fulfilled:

(i) F contains a k-point,

(ii) $K_F^2 \geq 5$ (i.e., $K_F^2 = 5, 6, 8$ or 9)

(in the case $K_F^2 = 5$ condition (i) is automatically true [57, 37]).

There is no hope of obtaining a similar simple criterion of rationality for three-dimensional varieties over an algebraically closed field. The methods of studying the problems of rationality, which are known today and are described in Sections 1–3, make it possible to get only partial results in this direction.

Section 1 contains a brief exposition of the three main methods of proving the nonrationality of varieties of dimension three and higher: the Noether–Fano method of factorization of birational mappings, items 1.1–1.3, the Clemens–Griffiths method of an intermediate Jacobian, item 1.4, and the Artin–Mumford method of birational invariance of the Brauer–Grothendieck group (in applications, torsions $H^3_{\mathrm{tors}}(X, \mathbb{Z})$ in three-dimensional cohomologies), item 1.5.

In Section 2 the reader can find a review of the available results concerning the rationality and nonrationality of algebraic three-dimensional varieties from families (I)–(III). All three methods are used (the Kollár degeneration method for proving the nonrationality of the general varieties of dimension $n \geq 3$ is presented in Section 3).

The problems of rationality for smooth Fano three-dimensional varieties are considered in 2.1: this problem is solved for them practically completely (Theorem 5) due to the well-known classification [19, 41]; only in the case of Fano varieties $V_{10} \subset \mathbb{P}^7$ the nonrationality is proved only for general varieties of this kind. In this item the first two methods for proving the nonrationality are used since $H^3_{\mathrm{tors}}(X, \mathbb{Z}) = 0$ for smooth Fano varieties.

Item 2.3 contains the results of Pukhlikov [48, 49] (Theorem 7) concerning the nonrationality of smooth three-dimension varieties X from family (II), namely, the Del Pezzo bundles $\varphi\colon X \to \mathbb{P}^1$ of degree $1, 2, 3$ with a sufficiently large number of degenerations. Use is made of the Noether–Fano method of maximal singularities and, in addition, Alekseev's result [1] (Theorem 6) is given in this item which concerns the criterion of rationality of Del Pezzo bundles of degree 4. The method is reduced to Shokurov's result [67] concerning the rationality criterion for standard conic bundles over geometrically ruled surfaces (Shokurov uses the Clemens–Griffiths method).

Item 2.3 is devoted to the study of the rationality problem for standard conic bundles $\pi\colon X \to S$ over the rational surface S. Here the reader can find Sarkisov's result [54] (Theorem 9) and the result obtained by the author [24] (Theorem 10) concerning the rationality criterion for these bundles. The above-mentioned result of Shokurov [67] (Proposition 4) is also given here.

In Section 3 we discuss multidimensional generalizations of the methods given in Section 1, i.e., the Artin–Mumford method, the Noether–Fano method of maximal singularities (the Clemens–Griffiths method is valid only for dimension three). In addition, a brief exposition of Kollar's method for proving the nonrationality of general varieties by means of generation and reduction to the characteristic $p > 0$ is given here.

The first generalization of the Artin–Mumford method was given in [12], where "nonramified cohomologies" of degree higher than two were used as a birational invariant. The nonramified Brauer group used by Artin and Mumford is generated in terms of nonramified cohomologies of degree 2 and was successfully used in [53] (see also [9, 65]) for constructing counterexamples to the classical Noether conjecture. The method of nonramified cohomologies of higher degrees [12] was also used in [51] (Theorem 11) for constructing other examples of multidimensional unirational nonrational fields of functions on varieties. These problems are discussed in items 3.1–3.3.

Item 3.4 includes Sarkisov's results [55] concerning the nonrationality of multidimensional conic bundles (Theorem 12) and the results of Pukhlikov concerning the nonrationality of certain types of Fano varieties (Theorem 13). These results were obtained by the Noether–Fano method.

Finally, items 3.5 and 3.6 are completely devoted to the Kollar's degeneration method [33, 34]. The main results concerning the nonrationality of certain general Fano varieties obtained by means of this method are given in Theorem 14. The statements concerning the nonrationality of certain general conic bundles and fiberings into cubic surfaces are formulated in Theorem 15.

I began to write this review when I was a guest of Johns Hopkins University in April, 1996. I express my deepest gratitude to the Department of Mathematics of this university and to Professor V. V. Shokurov for the invitation and for his hospitality.

1. THREE METHODS OF PROVING NONRATIONALITY

1.1. Method of factorization of birational maps. This method dates back to M. Noether and G. Fano. In general, the idea is the following. Let $\chi\colon X \dashrightarrow X'$ be a birational map of nonsingular projective varieties defined by the linear system $\mathcal{H}$. We assume that the adjunction break condition is fulfilled for X', i.e., the linear system $|D + mK_{X'}|$ is empty for any divisor D and any integer $m > m_0(X)$. Then, under certain conditions, the linear system $\mathcal{H}$ must possess a base set of sufficiently large multiplicity. To be more precise, let

$$
\begin{tikzcd}
X_N \arrow[d, "\sigma_{N,N-1}"'] \arrow[ddddddr, "\varphi"] & \\
\vdots \arrow[d] & \\
X_{i+1} \arrow[d, "\sigma_{i+1,i}"'] & \\
X_i \arrow[d] & \\
\vdots \arrow[d] & \\
X_1 \arrow[d, "\sigma_{1,0}"'] & \\
X = X_0 \arrow[r, dashed] & X'
\end{tikzcd}
\tag{1}
$$

be the resolution of indeterminacies of the map χ (when it exists, say, over a field of characteristic zero in the sense of Hironaka or $\dim \chi \leq 3$ in any characteristic $\neq 2, 3$ in the sense of Abhyankar) by means of the blow-ups, $\sigma_{i+1,i}\colon X_{i+1} \to X_i$, $i = 0, \ldots, N-1$, with nonsingular centers $B_i \subset X_i$, φ being a birational morphism. In what follows, we use the notation from [28, 20]: $E_{i+1} = \sigma_{i+1,i}^{-1}(B_i) \subset X_{i+1}$ are exceptional divisors, $i = 0, \ldots, N-1$; $\sigma_{i,j} := \sigma_{j+1,j} \circ \cdots \circ \sigma_{i,i-1}\colon X_i \to X_j$, $\sigma := \sigma_{1,0} \circ \cdots \circ \sigma_{N,N-1}\colon X_N \to X$.

It is well known that the canonical class K_{X_N} on X_N can be calculated from the relation

$$K_{X_N} = \sigma^* K_X + \sum_{i=1}^{N} \delta_i \sigma_{N,i}^*(E_i), \tag{2}$$

where $\delta_i = \dim X - \dim B_i - 1$. The method of factorization of birational maps is based on the following Noether–Fano inequality.

Lemma 1. *Let H' be a numerically effective divisor on X' (i.e., $H'C' \geq 0$ for any curve $C' \subset X'$) and let us assume that there exists an integer $m_0 \geq 0$ such that the linear system $|H' + mK_{X'}|$ is empty for any integer $m > m_0$. Let $H = \sigma^* \varphi^* H'$ and $\nu_1, \ldots, \nu_N$ be integers such that*

$$\varphi^* H' = \sigma^* H - \sum \nu_i \sigma_{N,i}^*(E_i).$$

Then $\nu_i \geq 0$, $i = 1, \ldots, N$, and, if the linear system $|H + mK_X|$ is nonempty for a certain $m > m_0$, then there exists a "maximal singularity", i.e., B_{i-1}, such that

$$\nu_i > m\delta_i, \tag{3}$$

where δ_i *is the same as in* (2).

Proof. For the proof see, for instance, [20, 27].

If we express the complete preimages $\sigma^*_{N,i}(E_i)$ of the exceptional divisors E_i in terms of the proper preimages $F_i = (\sigma_{N,i})^{-1}_{\text{proper}}(E_i)$:

$$\sigma^*_{N,i}(E_i) = \sum_{j \geq i} r_{ji} F_j,$$

where r_{ji} are certain nonnegative integers, then we can replace inequality (3) by the "strengthened Fano inequality":

$$\sum_{j \leq i} r_{ij}\nu_j > \sum_{j \leq i} r_{ij}\delta_j m. \tag{4}$$

Inequalities of this kind can be used as follows: either the existence of the maximal singularity leads to a contradiction, and therefore the assumed birational mapping $\chi\colon X \dashrightarrow X'$ does not exist, except for an isomorphism (as, for instance, for the smooth quartic $X_4 \subset \mathbb{P}^4$, see [28]), or for every maximal singularity there exists an elementary birational mapping which simplifies the mapping χ, and, in the final analysis, χ can be decomposed into a composition of elementary mappings. For minimal rational surfaces over a perfect field this gives a full description of all birational mappings via their factorization into elementary mappings and even allows us to study the relations between these elementary transformations (see [23]). In particular, the criterion of k-rationality for surfaces, which was formulated in Introduction, can be proved precisely in this way.

In dimension 3 the situation is much more complicated. The most difficult problem is to construct explicitly the elementary mappings for the possible maximal singularities. In addition, the elementary mappings may lead out of the category of nonsingular varieties. The recent progress in the problem of factorization of birational mappings of three-dimensional varieties of types (I)–(III) (see Introduction) is connected with Mori's theory in the framework of the so-called Sarkisov program (see [56, 14]). In order to formulate the results obtained in this direction, we shall need some basic concepts from this theory and its description at least in general (see [13]).

1.2. *Weil's divisor on the variety* X is a formal linear combination $D = \sum d_i D_i$ with integer coefficients of irreducible subvarieties of codimension one $D_i \subset X$; a similar linear combination with rational coefficients is known as Weil's $\mathbb{Q}$-*divisor.* The divisor is *effective* if $\alpha_i \geq 0$. The locally principal Weil divisor (i.e., a divisor locally representable by one equation) is known as a *Cartier divisor.* The classes of Cartier divisors modulo linear equivalence '$\sim$' is Picard's group $\operatorname{Pic} X$ of linear bundles or, what is the same thing, of invertible bundles on X. The Weil $\mathbb{Q}$-divisor is said to be a $\mathbb{Q}$-*Cartier* divisor if mD is a Cartier divisor for a certain integer $m > 0$. The Cartier divisor D on X is numerically effective or, in abbreviated form, nef, if $DC \geq 0$ for any algebraic curve $C \subset X$.

The normal variety X is said to be $\mathbb{Q}$-*factorial* if every Weil divisor on X is a Cartier $\mathbb{Q}$-divisor. It is clear that this is a condition imposed on the character of singularities of X. The minimal model program for the normal projective $\mathbb{Q}$-factorial variety X with the given $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D, in the first place with $D = K_x$, or, more generally, with $D = K_x + B$, where B is an effective Weil $\mathbb{Q}$-divisor, mainly presupposes, with some additional conditions imposed on the pair (X, B), a construction of a sequence of birational transformations $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n = X'$

and divisors D_i on X_i so that $H^0(X_{i-1}, mD_{i-1}) = H^0(X_i, mD_i)$ for all i and integers $m \geq 1$ and $D_n = D'$ nef. We can construct such a sequence proceeding from the numerical properties of D modifying every time the "negative" part of (X, D), i.e., $C \subset X$ with $DC < 0$. It should be recalled that the numerical equivalence of $\mathbb{Q}$-divisors of the $\mathbb{Q}$-factorial variety X is defined as follows: $D_1 \equiv D_2 \iff D_1 C = D_2 C$ for all curves $C \subset X$. Let $N^1(X)$ be a quotient space $\operatorname{Pic} X \otimes R$ modulo numerical equivalence, and then $N^1(X)$ is a finite-dimensional real space of dimension ρ, where $\rho = \operatorname{rk} \operatorname{NS}(X)$ is Picard's number, i.e., the rank of the Noether–Severi group $\operatorname{NS}(X)$. Let $N_1(X)$ be a space dual to $N^1(X)$. By definition, $N_1(X)$ is a space generated by all algebraic curves modulo numerical equivalence, $\dim N_1(X) = \dim N^1(X) = \rho$. The smallest convex cone in $N_1(X)$, which is generated by the classes of effective curves, is denoted by $\operatorname{NE}(X)$. The Kleiman–Mori cone $\overline{\operatorname{NE}}(X)$ is the closure of $\operatorname{NE}(X) \subset N_1(X)$ in the real topology.

Let X be a normal projective $\mathbb{Q}$-factorial variety and $f\colon X \to Y$ be a projective morphism. If H is any ample divisor on Y and $C \subset X$ is a curve, then $f^*HC = 0$ if and only if C is contained in a certain fiber of the morphism f. According to the numerical criterion of Kleiman's abundance, this shows that the subset $F \subset \overline{\operatorname{NE}}(X)$ generated by the classes of curves $[C]$ contracted by f into a point, constitutes a face of the cone $\overline{\operatorname{NE}}(X)$. A one-dimensional face in $\overline{\operatorname{NE}}(X)$ is called an extremal ray.

We shall now briefly describe the algorithm of the Mori theory of minimal models. We work over the field k of characteristic zero. Suppose that beginning with the pair $(X, D) = (X_0, D_0)$ the sequence $X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \dashrightarrow X_{r-1} \overset{\varphi_{r-1}}{\dashrightarrow} X_r$ and the divisors D_1 on X_i have been constructed. If D_r is nef, then X_r is the required minimal model and the algorithm is completed, i.e., $(X', D') = (X_r, D_r)$. Now if D_r is not nef, then, in order to continue the algorithm, we need the cone theorem, i.e., the theorem stating that the cone $\overline{\operatorname{NE}}(X_r)$ is locally finitely generated in $\{z \mid D_r Z < 0\}$. By virtue of this theorem, there exists an extremal ray $R \subset \overline{\operatorname{NE}}(X_r)$ with $D_r R < 0$, and then we need the contraction theorem stating that there exists a contraction morphism $\varphi\colon X_r \to Y$ of the extremal ray onto the normal variety Y such that the curve $C \subset X_r$ is mapped into a point if and only if $[C] \in R$. There exist three possibilities for the morphism φ:

(1) $\dim Y < \dim X_r$, in this case, the algorithm is terminates and $\varphi\colon X_r \to Y$ is called a *D-Mori fiber space*, for instance, three-dimensional varieties from classes (I)–(III), given in Introduction, are of this kind with $D = K_X$ (in [14] they are called *$\mathbb{Q}$-Mori fiber spaces*;

(2) $\varphi\colon X_r \to Y$ is a birational morphism whose exceptional set is a divisor; in this case, the extremal contraction of φ is said to be *divisorial* and the induction process can be continued: $\varphi_r = \varphi$, $X_{r+1} = Y$, $D_{r+1} = \varphi_*(D_r)$;

(3) φ is a small birational morphism, i.e., its exceptional set does not include a divisor; in this case, the morphism φ is called a *flip contraction*. Here the problem is that the image $\varphi_*(D_r)$ of the divisor D_r is not a $\mathbb{Q}$-Cartier divisor, and therefore there is no reason to speak about its numerical effectiveness. In this situation, the induction process is continued with the aid of a new birational operation D-flip:

$$\begin{tikzcd} X_r \arrow[rr, dashed, "\varphi_r"] \arrow[dr, "\varphi"'] & & X_{r-1} \arrow[dl, "\varphi'"] \\ & Y & \end{tikzcd}, \tag{5}$$

here $\varphi_r\colon X_r \dashrightarrow X_{r+1}$ is a birational map which is an isomorphism in codimension 1, $D_{r+1} = \varphi(D_r)$ is a proper image of the divisor D_r, the morphism $\varphi'\colon X_{r+1} \to Y$ is characterized by the following proprieties: φ' is a small birational morphism, D_{r+1} is $\mathbb{Q}$-Cartier, and diagram (5) is commutative relative to φ' — nef.

Thus, if there are theorems (a theorem on a cone, a contraction theorem, a theorem on the existence of flips) which ensure every step of induction program and a theorem of finiteness for a sequence of flips, then the algorithm described above terminates either at a D-minimal model or a D-Mori bundle. The program of minimal models has been completed only in the following cases (see. [36, 13]):

(a) $\dim X \leq 3$, $D = K_X$ and X has no more than terminal or canonical singularities (see Definition 1 below),
(b) $\dim X \leq 3$, $D = K_X + B$, and the pair (X, B) is log-terminal or log-canonical,
(c) X is a toric variety of any dimension and $D = K_X$.

Definition 1. Let $B = \sum b_i B_i$ be a Weil divisor on the normal variety X, where B_i are irreducible divisors and $0 \leq b_i \leq 1$, $b_i \in \mathbb{Q}$ (possibly $B = \varnothing$). We say that the pair (X, B) *is terminal* (*canonical,, log-canonical, log-terminal*), or, in other words, (X, B) has terminal (canonical, etc, resp.) singularities if the divisor $K_X + B$ is $\mathbb{Q}$-Cartier, and, for any birational morphism from the normal variety $f\colon Y \to X$ with the exceptional divisors E_i, we have

$$K_Y + f^{-1}_{\text{proper}}(B) \equiv f^*(K_x + B) + \sum a_i E_i$$

with $a_i > 0$ ($a_i \geq 0$, $a_i > -1$, $a_i \geq -1$, resp.).

The numbers $a_i \in \mathbb{Q}$ are known as discrepancies and depend only on the discrete valuation v_i of the function field $k(X)$ which is associated with E_i and can be calculated on any normal model Z of the field $k(X)$, where the center of the valuation is a divisor. Thus we can determine the discrepancy $a(v, K_X + B)$ for any discrete valuation v of the field $k(X)$ with any center of dimension $< \dim X - 1$.

The following statement, which can be easily proved, plays an important part in the minimal model program.

Proposition 1 (see, e.g., [36]). *Let X be a normal $\mathbb{Q}$-factorial variety, $D = K_X + B$, and assume that $\varphi\colon X \dashrightarrow X'$ is a step in the D-minimal program, i.e., a divisorial contraction or a flip. Then $a(v, K_{X'} + B') \geq a(v, K_X + B)$ for any valuation v with center in X. Moreover, $a(v, K_{X'} + B') > a(v, K_X + B)$ if and only if φ is not an isomorphism in the neighborhood of the center of the valuation in X. In particular, if (X, B) is terminal* (*canonical, log-terminal, log-canonical*)*, then (X', B') is also of the same kind.*

The last assertion means, in particular, that the type of the singularities of X (when $B = \varnothing$) does not deteriorate upon divisorial extremal contractions and flips.

1.3. Following now [13, 14], we shall briefly describe how the Mori theory can be used in the process of studying the problem of factorization of birational mappings between $\mathbb{Q}$-Mori fiber spaces $\varphi\colon X \to S$ and, in particular, the problem of rationality of the latter.

Recall (see Introduction) that a $\mathbb{Q}$-Mori fiber space of dimension three is a $\mathbb{Q}$-factorial three-dimensional projective variety with terminal singularities (see Definition 1) and the morphism

$\varphi\colon X \to S$ onto the normal variety S, $\dim S < 3$, such that $-K_X$ relative to φ is ample and $\rho(X) - \rho(Y) = 1$. We shall study the birational mappings $\Phi\colon X \dashrightarrow X'$ between these $\mathbb{Q}$-Mori bundles

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\ S & S' \end{tikzcd} \tag{6}$$

(the mapping Φ is, in general, not connected with the structural morphisms φ and φ') using the following version of the theory of minimal models. Let $\mathcal{H}$ be a linear system without fixed components on X. The pair $(X, b\mathcal{H})$ is said to be *terminal* (canonical, etc, resp.) if and only if, for any birational morphism $f\colon Z \to X$ from the normal variety Z with the exceptional divisors E_i, we have

$$K_Z + b\mathcal{H}_Z = f^*(K_X + b\mathcal{H}) + \sum a_i E_i$$

with $a_i > 0$ ($a_i \geq 0$, etc., resp.) for all i, where $\mathcal{H}_Z$ ia a proper preimage of the linear system $\mathcal{H}$ on Z.

We choose a sufficiently large integer μ' and a sufficiently ample divisor A' on S' such that the linear system $\mathcal{H}' = |-\mu' K_{X'} + \varphi'^* A'|$ is very ample on X'. Let $\mathcal{H} = \mathcal{H}_X = \Phi^{-1}_{\text{proper}}(\mathcal{H}')$ be its proper preimage on X, and then, by the definition of the $\mathbb{Q}$-Mori fiber space, we have $\mathcal{H}_X \equiv -\mu K_X + \varphi^* A$ for a certain $\mu \in \mathbb{Q}$, $\mu > 0$ and a certain (not necessarily ample) $A \in \operatorname{Pic} S \otimes \mathbb{Q}$. Let $p\colon (Y, \mathcal{H}_Y) \to (X, \mathcal{H}_X)$ be the resolution of the singularities of X and of the base set of the linear system $\mathcal{H}_X$ in (3). Then we have the following diagrams:

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\Phi"] \arrow[d, "\varphi"'] & & X' \arrow[d, "\varphi'"] \\ S & & S' \end{tikzcd} \qquad \begin{gathered} \mathcal{H}_Y = q^*\mathcal{H}', \\ \mathcal{H}' = |-\mu K_{X'} + \varphi'^* A'|, \\ \mathcal{H} \equiv -\mu K_X + \varphi^* A. \end{gathered} \tag{7}$$

The main ingredient in the problem of factorization of birational maps is the Noether-Fano inequality. Here is its version in the framework of the theory of minimal models.

Proposition 2. *In the previous notation we have the following:*

(i) $\mu \geq \mu'$ *and the equality is satisfied only if* Φ *induces the rational mapping* $S \dashrightarrow S'$;

(ii) *if* $K_X + \frac{1}{\mu}\mathcal{H}$ *is canonical and it is* nef, *then* Φ *is an isomorphism which induces the isomorphism* $S \simeq S'$, *in particular*, $\mu = \mu'$.

Outline of the proof. We have

$$K_Y + \frac{1}{\mu}\mathcal{H}_Y = q^*\Big(K_{X'} + \frac{1}{\mu'}\mathcal{H}'\Big) + \sum a'_i E_i = q^*\varphi'^* A' + \sum a'_i E_i,$$

where $a'_i > 0$ since the singularities of X' are terminal. On the other hand,

$$K_Y + \frac{1}{\mu'}\mathcal{H}_Y = p^*\Big(K_X + \frac{1}{\mu'}\mathcal{H}\Big) + \sum a_i E_i,$$

where nothing is known about a_i. Let $C \subset X$ be a general curve contained in the fibers of the morphism φ and $C' \subset X'$ be the image of C in X' and C'' be the image of C in Y. Then

$$\Big(K_X + \frac{1}{\mu'}\mathcal{H}\Big)C = \Big(K_Y + \frac{1}{\mu'}\mathcal{H}_Y\Big)C'' = \varphi'^* A'C' + \sum a_i' E_i C'' \geq 0.$$

Hence $\mu \geq \mu'$ and $\mu = \mu'$ only if C' is contained in the fibers of φ'. This is the proof of (i).

In order to prove (ii), we must invert the preceding reasoning. This gives $\mu = \mu'$. Now we can easily find out that Φ is an isomorphism of codimension 1. More dedicate arguments show that Φ is indeed an isomorphism that preserves the structure of the bundle, for details see [14].

The positive rational number μ in (7) plays the part of a degree. The problem of factorization of the map Φ consists in the decomposition of this map into quite concrete elementary maps which in [14] are called *links*. The algorithms of decomposition consist in finding an elementary link $\Phi_1\colon X/S \dashrightarrow X_1/S_1$ such that $\Phi_1^{-1}\Phi\colon X_1/S_1 \dashrightarrow X/S$ is of degree $\mu_1 < \mu$. In this case, Φ can even be an identity isomorphism just as certain links of type IV below, only the structure of the Mori bundle changes, and in the new structure $\varphi_1\colon X \to S$ the degree μ_1 turns out to be lower than μ. In the case of surfaces, for instance, these are two types of $\mathbb{P}^1$-bundles on the quadric $\mathbb{P}^1 \times \mathbb{P}^1$. In general, in the case of minimal surfaces over a perfect field, the problem of factoring birational maps whose origin was Noether's theorem on the decomposition of Cremona transformations of a plane in a product of quadratic transformations is simple as compared to the three-dimensional case. There are no terminal singularities on surfaces, and therefore the degree μ is always an integer, or an almost integer, the elementary links are explicitly classified (this happened before Mori theory appeared), and, in a finite number of steps, the process of "untwisting" terminates at an isomorphism, see [23]. In particular, we can prove, in passing, the criterion of k-rationality of surfaces formulated in Introduction.

In dimension 3, there does not yet exist an explicit classification of elementary links (their definitions given below are very abstract), and, in addition, elementary links not always diminish μ, and then a different invariant begins to operate, namely, the canonical threshold $c = \max\{b \mid K_X + b\mathcal{H}$ is canonical$\}$ connected with the nature of singularities of X and with the maximal singularities of their linear system $\mathcal{H}_X$. Since c is a rational number too, we must not only find appropriate links for the decomposition of Φ, but also prove the theorem on the finiteness of the untwisting procedure. Nevertheless, we have the following significant result announced by Sarkisov [56] and proved by Corti [14].

Theorem 1 (see [14]). *Let $\Phi\colon X/S \dashrightarrow X'/S'$ be a birational map of $\mathbb{Q}$-Mori fiber spaces, $\dim X = \dim X' \leq 3$. Then there exists a factorization $\Phi = \Phi_n \cdots \Phi_1$:*

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi_1"] \arrow[d, "\varphi"'] & \cdots X_i \arrow[r, dashed, "\Phi_i"] \arrow[d, "\varphi_i"'] & X_{i+1} \cdots \arrow[r, dashed, "\Phi_n"] \arrow[d, "\varphi_{i+1}"] & X' \arrow[d, "\varphi'"] \\ S & S_i & S_{i+1} & S' \end{tikzcd},$$

where $\Phi_i\colon X_i/S_i \dashrightarrow X_{i+1}/S_{i+1}$ is an elementary map (link) of the following four types.

Links of type I. *These links are commutative diagrams of the form*

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d] & X' \arrow[dd] \\ X \arrow[d] & \\ S & S' \arrow[l] \end{tikzcd},$$

where $Z \to X$ *is an extremal* (*with* $D = K_Z$) *divisorial Mori contraction and* $Z \dashrightarrow X'$ *is a sequence of flips, flops, and their inverses* (*in the sense of Mori*, *i.e., with* $D = K_Z$); *here* $\rho(S'/S) := \rho(S') - \rho(S) = 1$.

Links of type II. *These are commutative diagrams of the form*

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d] & Z' \arrow[d] \\ X \arrow[d] & X' \arrow[d] \\ S \arrow[r, phantom, "\simeq"] & S' \end{tikzcd},$$

where $Z \to X$ *and* $Z' \to X'$ *are extremal divisorial contractions of Mori and* $Z \dashrightarrow Z'$ *is a Mori sequence of flips, flops, and antiflips* (*i.e., inverses of flips*), *here* $S \simeq S'$.

Links of type III. *These are commutative diagrams of the form*

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dd] & Z \arrow[d] \\ & X \arrow[d] \\ S \arrow[r] & S' \end{tikzcd},$$

where $X \dashrightarrow Z$ *is a Mori sequence of flips, flops, and antiflips*, $Z \to X'$ *is an extremal divisorial contraction of Mori*, $\rho(S/S') = 1$; *these links are inverse to links of type* I.

Links of type IV. *These are commutative diagrams of the form*

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & X' \arrow[d] \\ S \arrow[dr] & S' \arrow[d] \\ & T \end{tikzcd},$$

where $X \dashrightarrow X'$ *is a Mori sequence of flips, flops, and antiflips*, $\rho(S/T) = \rho(S'/T) = 1$; *these links are transitions to a different structure of the* $\mathbb{Q}$*-Mori bundle.*

Outline of the proof. Consider the following two cases.

(a) $K_X + \frac{1}{\mu}\mathcal{H}$ has no canonical singularities. Let $c < \frac{1}{\mu}$ be a canonical threshold for $(X, \mathcal{H})$. Then there exists an extremal blow-up $f\colon Z \to X$ with the exceptional divisor E such that

$$K_Z + c\mathcal{H}_Z = f^*(K_X + c\mathcal{H}),$$

where $\mathcal{H}_Z$ is a proper preimage of the linear system $\mathcal{H}$. The program of minimal models for $K_Z + c\mathcal{H}_Z$ over the base S (its existence was proved in [36]) gives the required link Φ_1 of type I or II. Here and in the sequel we use the Noether–Fano inequality (i.e., Proposition 1).

(b) $K_X + \frac{1}{\mu}\mathcal{H}$ has canonical singularities, and then either $K_X + \frac{1}{\mu}\mathcal{H}$ is nef and Φ is an isomorphism or $K_X + \frac{1}{\mu}\mathcal{H}$ is not nef. In the latter case the program of minimal models for $K_X + \frac{1}{\mu}\mathcal{H}$ gives the required link Φ_1 of type III or IV.

The termination of the factorization procedure can be derived from the results of Kollar [36, ch. 18] and Alekseev who assert that the set of log-canonical thresholds does not possess limit points from below (see [14]).

Remark 1. The preceding theorem indicates one of the techniques which would make it possible to obtain, at least in principle, a description of all rational threefolds. Recall that the necessary condition of rationality $\varkappa = -\infty$ and Mori's theorem show that the algorithm of the K_X-minimal theory realizes any rational variety X in the form of the $\mathbb{Q}$-Mori bundle with a rational base. According to Theorem 1, any one of them can be obtained from $\mathbb{P}^3$ with the aid of a finite sequence of elementary links of types I–IV so that the problem comes up against an explicit classification of these links. It should be pointed out once again that in the case of rational surfaces over the perfect field k this classification of links does exist, and this allows us to characterize k-rational surfaces (see [23]). Theorem 1 states, in particular, that any Cremona transformation of $\mathbb{P}^3$ can be decomposed in a product of elementary links. We can regard this as an analog (rough enough) of the Noether classical theorem on the Cremona transformations of $\mathbb{P}^2$ (as well as a similar theorem of Cantor in the case of nonclosed field, see [23]).

1.4. We shall now briefly describe another important method of studying the problem of rationality of three-dimensional varieties over the field $\mathbb{C}$. This is the method of the intermediate Jacobian of Clemens and Griffiths [11].

Let X be a smooth irreducible three-dimensional projective variety over $\mathbb{C}$. If X is rational, then necessarily $H^1(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = 0$. We shall assume that this is the fact in what follows. Then the Hodge decomposition for three-dimensional cohomologies has the form $H^3(X, \mathbb{C}) = H^{1,2}(X) \oplus H^{2,1}(X)$. The integration of the $(2,1)$-forms along three-dimensional cycles defines the embedding $H_3(X, Z)/(\text{tors}) \hookrightarrow H^{2,1}(X)^*$ in the form of a full lattice of rank $2h^{2,1}$ into the complex vector space $H^{2,1}(X)^*$.

The skew-symmetric integer-valued form of intersection E on $H_3(X, \mathbb{Z})$ is unimodular by Poincaré duality and the Hermitian form H on $H^{2,1}(X)^*$ corresponding to it is positive definite. Consequently, according to the Riemann–Frobenius criterion, the complex torus $J(X) := H^{2,1}(X)^*/\operatorname{Im} H_3(X, \mathbb{Z})$ is a principal by polarized Abelian variety with polarization divisor Θ.

In the previous notation the pair $(J(X), \Theta)$ is called *intermediate Jacobian of the variety X*.

The use of the intermediate Jacobian in the birational theory of three-dimensional varieties is based on the following observation: let $\sigma\colon X' \to X$ be a blow-up of the nonsingular irreducible center (a point or an irreducible curve) $B \subset X$, then

$$J(X') = \begin{cases} J(X) & \text{if } B \text{ is a point} \\ J(X) \oplus \mathrm{Jac}(B) & \text{if } B \text{ is a curve,} \end{cases}$$

where $\mathrm{Jac}(B)$ is the Jacobian of the curve B with a principal polarization defined by the Poincare divisor.

This fact, together with the complete reducibility theorem, which states that every principally polarized Abelian variety (A, Ξ) can be uniquely decomposed into a direct sum of simple varieties, makes it possible to determine the birational invariant of the variety X: it is the sum of simple components in the decomposition of the intermediate Jacobian $J(X, \Theta) = \bigoplus (A_i, \Xi_i)$ which are not Jacobians of curves. This invariant is usually called Griffith's component and is denoted by $J_G(X)$. If $J_G(X) \neq 0$, then X is not rational since $J_G(\mathbb{P}^3) = 0$ and, consequently, $J_G(Y) = 0$ for any rational nonsingular three-dimensional variety Y.

Thus, in order to solve the problem of nonrationality of certain three-dimensional varieties X with $h^1(\mathcal{O}_X) = h^3(X) = 0$, we must know how to distinguish the intermediate Jacobian $J(X, \Theta)$ from the Jacobian of curves. All known methods of this distinguishing are based on the study of geometry, in particular, the study of the singularities of the polarization divisor θ. In this study, the first results concerning the nonrationality were obtained for certain nonsingular Fano three-dimensional varieties. (see Section 2). This method proved to be the most successful for studying the problem of rationality of standard conic bundles over rational surfaces. It turned out that for these bundles the intermediate Jacobian $J(X, \Theta)$ was isomorphic to the so-called Prym variety which was connected with a nonramified double cover of curves and which was encountered in classical literature (see review [7]).

Definition 2. Let $(\widetilde{C}, i)$ be a pair consisting of a reduced complete (possibly reducible) curve $\widetilde{C}$ with no more than ordinary double points and an involution i on $\widetilde{C}$, i.e., $i^2 = \mathrm{id}$, which acts nonidentically on every irreducible component of the curve $\widetilde{C}$. Let $C = \widetilde{C}/i$ be a quotient curve and $\pi \colon \widetilde{C} \to C$ be the corresponding double cover. The involution i induces an involution i^* on the group $\operatorname{Pic} \widetilde{C}$, and π is a norm map $\mathrm{Nm} \colon \operatorname{Pic} \widetilde{C} \to \operatorname{Pic} C$. The connected commutative algebraic group $\Pr(\widetilde{C}, C) := \operatorname{Ker}(\mathrm{Nm})^0 = \operatorname{Ker}(\mathrm{id} + i^*)^0 = (\mathrm{id} - i^*) J(C)$ is known as a generalized prymian of the pair $(\widetilde{C}, i)$ (here '^{0}' denotes connected component of the unit).

The concept of polarization can be easily used in connection with generalized prymians (which, being algebraic groups, are not always Abelian varieties if the curve has singularities). However, under certain conditions, $\Pr(\widetilde{C}, C)$ is an Abelian variety whose polarization is divisible by 2 and becomes principal after the division. This takes place under the following Beauville conditions [5, 67]:

(B) $\operatorname{Sing} \widetilde{C} = \{x \in \widetilde{C} \mid i(x) = x\}$, $\pi(\operatorname{Sing} \widetilde{C}) = \operatorname{Sing} C$,

i.e., the fixed points of the involution i are exactly singular points of $\widetilde{C}$; the second condition means that the involution i preserves every one of the branches (i.e., does not interchange them) at every singular point $x \in \operatorname{Sing} \widetilde{C}$. If $\widetilde{C}$ is nonsingular, then C is also nonsingular and $\pi \colon \widetilde{C} \to C$ is a nonramified double cover.

The following result plays an important part in the study of the problem of rationality of three-dimensional varieties by the method of intermediate Jacobian in the cases where this Jacobian is a

Prym variety.

Theorem 2 (Munford–Beauville–Shokurov, see, e.g., [67]). *Let $(\widetilde{C}, i)$ be a pair consisting of a curve $\widetilde{C}$ of genus $2g-1$, $g \neq 5$, an involution i on $\widetilde{C}$, which satisfies conditions* (B) *and also the following conditions:*

(∗) *for any nontrivial decomposition $\widetilde{C} = \widetilde{C}_1 \cup \widetilde{C}_2$ we have*

$$\#(\widetilde{C}_1 \cap \widetilde{C}_2) \geq 4,$$

and let $C = \widetilde{C}/i$ be a quotient curve, and $P = \mathrm{Pr}(\widetilde{C}_1, C)$ be the corresponding principally polarized Prym variety ($p_a(C) = g$, $\dim P = g-1$) with the polarization divisor Ξ. Then (P, Ξ) is a Jacobian of a certain curve or a sum of Jacobians of curves only in the following cases:

(i) *C is a hyperelliptic curve,*
(ii) *C is a hyperelliptic curve with two glued points,*
(iii) *C is a trigonal curve (i.e., there exists a finite morphism of degree 3),*
(iv) *$\widetilde{C} \to C$ is a double cover of a plane curve of degree 5 corresponding to an even theta-characteristic.*

Remark 2. We shall elucidate the meaning of (iv). The theta-characteristic on the algebraic curve C is a divisor θ such that $2\theta \sim K_C$. The evenness is defined by the evenness of $h^0\big(\mathcal{O}_C(\theta)\big)$. On the plane curve $C \subset \mathbb{P}^2$, $\deg C = 5$, the canonical divisor K_C is cut out by conics, and, consequently, there exists on C an isolated odd theta-characteristic θ_0. Since the nonramified double covers $\widetilde{C} \to C$ are defined by the second-order points on the Jacobian $J(C)$, i.e., by the classes of divisors δ_i with $2\delta_i \sim 0$, it follows that the divisors $\theta_i = \theta_0 + \delta_i$ are, obviously, theta-characteristics, and then it makes sense to say that the covers $\widetilde{C} \to C$ are associated with the theta-characteristic. It turns out (see, e.g., [67]) that the Prym variety $\mathrm{Pr}(\widetilde{C}, C)$ of the double cover of the plane curve C of degree 5 is a Jacobian or is not a Jacobian of a certain curve of genus 5 according as the corresponding theta-characteristic θ_i is even or odd.

Now what concerns condition (∗). It easily follows from condition (B) that for any decomposition $\widetilde{C} = \widetilde{C}_1 \cup \widetilde{C}_2$ the number of intersections $\#(\widetilde{C}_1 \cap \widetilde{C}_2)$ is always even. If $\widetilde{C}_1 \cup \widetilde{C}_2 = \{p, q\}$ and $\widetilde{C}_i'$, $i = 1, 2$, are curves obtained from $\widetilde{C}_i$ by gluing the points p and q, then $\mathrm{Pr}(\widetilde{C}, C) = \mathrm{Pr}(\widetilde{C}_1', C_1) \oplus \mathrm{Pr}(\widetilde{C}_2', c_2)$ (see [67]).

The Prym varieties are used in the three-dimensional birational geometry mainly in the following situation. Let $\pi\colon X \to S$ be a standard conic bundle over the smooth projective rational surfaces S (recall that "standard" means that X is nonsingular and that the surfaces $X_D = \pi^{-1}(D)$ are irreducible for every irreducible curve $D \subset S$, i.e., $\rho(X) - \rho(S) = 1$). Suppose that $C \subset S$ is a discriminant curve (i.e., a curve of degeneration of the morphism π) and $\widetilde{C}$ is a curve which parametrizes the irreducible components of degenerate conics over the points of C. Then, if $C \neq \varnothing$, then it is a reduced curve with normal intersections and it induces a double cover $\widetilde{\pi}\colon \widetilde{C} \to C$ which satisfies conditions (B) (see [5, 8]).

Theorem 3 (see [5, 8, 30]). *In the previous notation, the intermediate Jacobian $J(X)$ is isomorphic, being a principally polarized Abelian variety, to Prym variety $\mathrm{Pr}(\widetilde{C}, C)$.*

For the first time, this was observed by Mumford (see [11], supplement) as applied to the smooth three-dimensional cubic hypersurface $X \subset \mathbb{P}^4$; the projection from the straight line $l \subset X$ represents X' as a standard conic bundle over $\mathbb{P}^2$ with the discriminant curve $C \subset \mathbb{P}^2$ of degree 5, where $X' \to X$ is the blow-up of the line l. The corresponding double cover $\widetilde{\pi}\colon \widetilde{C} \to C$ is associated with an odd theta-characteristic, and, consequently, $J(X) = \mathrm{Pr}(\widetilde{C}, C)$ is not a Jacobian of any curve, and this gives a version of the proof of the nonrationality of X by the Clemens–Griffiths method. Beauville [5] generalized this approach to other standard conic bundles and proved, in particular, the following statement: every conic bundle $\pi\colon X \to \mathbb{P}^2$ with the discriminant curve of degree $C \geq 6$ is nonrational.

1.5. The third method of proving the nonrationality of algebraic varieties of dimension $n \geq 3$ uses a discrete invariant, namely, the Brauer group. The $\mathrm{Br}\, k(X)$ of the field of rational functions on the variety X is, obviously, a birational invariant and the cohomological interpretation of $\mathrm{Br}\, k(X)$ as two-dimensional Galois cohomologies $H^2\big(G, \overline{k(X)}^*\big)$ with coefficients in the multiplicative group $\overline{k(X)}^*$ of the field $\overline{k(X)}$ suggests the consideration, from this point of view, of a similar "global" group of etale cohomologies $H^2_{\mathrm{et}}(X, G_m)$ with coefficients in a bundle of invertible elements G_m of the structure bundle $\mathcal{O}_X$. Grothendieck studied this group in great detail. He established its relationship with the group of classes of Azumai $\mathcal{O}_X$-algebras and called it a (cohomological) *Brauer group* $\mathrm{Br}\, X = H^2_{\mathrm{et}}(X, G_m)$. He also proved its birational invariance for nonsingular complete varieties X over the algebraically closed field k of characteristic zero. It turned out that for smooth complete varieties X over $\mathbb{C}$, on which there are no geometric-differential invariants, in particular, on unirational varieties, the (cohomological) Brauer group $\mathrm{Br}\, X$ coincided with the torsion subgroup $\mathrm{Tors}\, H^3(X, \mathbb{Z}) \subset H^3(X, Z)$.

For the first time, this invariant was used by Artin and Mumford [3] for constructing counterexamples to the problem of Lüroth in any dimension ≥ 3. One more example can be found in [68]. In this case, the birational invariance of $\mathrm{Tors}\, H^3(X, \mathbb{Z})$ can be directly proved with the use of Hironaki's theorem on the resolution of indeterminacies of rational maps. The first computations were carried out for standard conic bundles $\pi\colon X \to S$ over the smooth projective surface S. They were based on the following result.

Proposition 3 (Artin–Mumford [3]). *For the smooth projective surface S over the algebraically closed field k there exists an exact sequence of groups*

$$0 \longrightarrow \mathrm{Br}\, S \longrightarrow \mathrm{Br}\, k(S) \xrightarrow{\alpha} \bigoplus_{\substack{\text{curves}\\ C_i \subset S}} H^1_{\mathrm{et}}(C_i, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} \bigoplus_{\substack{\text{points}\\ \alpha \in \bigcup C_i}} \mu^{-1} \xrightarrow{\gamma} \mu^{-1} \longrightarrow 0, \tag{8}$$

where $\mu^{-1} = \bigcup_n \mu_n^{-1} = \bigcup_n \mathrm{Hom}(\mu_n, \mathbb{Q}/\mathbb{Z})$, μ_n denotes a group of roots of degree n on unity, $\mathrm{Br}\, S = H^2_{\mathrm{et}}(S, G_m)$ is a Brauer–Grothendieck group of the surface S: it consists of elements $a \in \mathrm{Br}\, k(S)$ which are not ramified at all points of S, the homomorphism α puts the class of algebras $a \in \mathrm{Br}\, k(S)$ into correspondence with the collection of its local invariants.

As an illustration of this result, we shall consider the standard conic bundle $\pi\colon X \to S$ over the rational surface S with the discriminant curve $C = \sum C_i$, where C_i are irreducible components. Here $\mathrm{Br}\, S = 0$ since S is a rational surface so that α is an embedding. Let $X_\eta = \pi^{-1}(\eta)$ be the general fiber of π, and then it is associated with the quaternion algebra A_η over the field $k(\eta) = k(S)$. The class $[A_\eta] \in \operatorname{Br} k(S)$ is a second-order element which, according to (8) and the injectivity of α, is uniquely defined by its local invariants. In this case, the local invariants are exactly the double covers $\widetilde{\pi}_i \colon \widetilde{C}_i \to C_i$ defined by the components of the fibers of the disintegrating conics over the points from C and

$$\beta(x) = \begin{cases} \frac{1}{2} & \text{if } \widetilde{C}_i \to C_i \text{ has ramification in } x \in C_i \\ 0 & \text{in other cases.} \end{cases}$$

It is easy to see that the exactness of sequence (8) means, in this case, nothing else but the Beauville's conditions (B) for the standard conic bundle $\pi \colon X \to S$.

Conversely, if $C = \sum C_i$ is a certain reduced curve with normal intersections on the smooth rational surface S and $\widetilde{\pi} \colon \widetilde{C} \to C$ is a double cover satisfying conditions (B), then, using the exact sequence (8), we can uniquely reconstruct the second-order element in the group $\operatorname{Br} k(S)$ with local invariants $\widetilde{C}_i \to C_i$. If it is represented by a certain quaternion algebra over $k(S)$, then there exists a standard conic bundle over S with the discriminant curve C and the corresponding cover $\widetilde{\pi} \colon \widetilde{C} \to C$.

Thus the problem of the existence of standard conic bundles over S with the specified double cover $\widetilde{\pi} \colon \widetilde{C} \to C$ reduces to the question of whether every second-order element in $\operatorname{Br} k(S)$ is represented by a certain quaternion algebra. The answer is positive due to the following statements:

(1) every second-order element is represented by the class of a product of quaternion algebras, which is the corollary of the Merkuriev–Suslin theorem,

(2) the class of a product of quaternion algebras is represented by a quaternion algebra, and this is a consequence of the fact that the field $k(S)$ with the algebraically closed k is the field C_2 in the sense of Serre, and in this case any two quaternion algebras over $k(S)$ have a common quadratic field of decomposition.

Thus we can construct many examples of standard conic bundles X over S from the specified covers $\widetilde{\pi} \colon \widetilde{C} \to C$. This is significant due to the following result of Artin–Mumford [3].

Theorem 4. *Let X/S be a standard conic bundle over the rational surface S with the discriminant curve $C \subset S$. Then $\operatorname{Br} X \simeq (\mathbb{Z}/2\mathbb{Z})^{c-1}$, where c is the number of connected components of the curve C.*

This result shows that in order to construct examples of nonrational three-dimensional varieties, it suffices to define local invariants with a non-connected curve C on a certain rational surface S. Moreover, we can choose local invariants such that X in the obtained conic bundle $X \to S$ is unirational. The reader can find examples of this kind in [3]. If $\operatorname{Tors} H^3(X, \mathbb{Z}) = 0$, then $\operatorname{Tors} H^3(X \times \mathbb{P}^n, \mathbb{Z}) \simeq \operatorname{Tors} H^3(X, \mathbb{Z}) = 0$ and, consequently, $X \times \mathbb{P}^n$ is rational just as X. Consequently, the Artin–Mumford construction allows us to find examples of unirational, but not rational varieties in any dimension ≥ 3.

The Brauer–Grothendieck group, or, what is the same, the "nonramified" Brauer group of a function field was used by Saltman [53] (see also [9, 65]) for constructing counterexamples to Noether's conjecture concerning the rationality of the fields of invariants for finite linear groups acting on vector spaces over an algebraically closed field.

The birational invariance of nonramified cohomologies of higher degrees $i \geq 3$, in particular, the nonramified Witt group of the classes of quadratic forms, was used in [12] and [51] for constructing new examples of unirational nonrational varieties of higher dimensions even with the trivial Brauer–Grothendieck group. For details see Section 3, Subsections 3.1–3.3.

2. THE RESULTS CONCERNING THE RATIONALITY AND NONRATIONALITY OF THREE-DIMENSIONAL VARIETIES

2.1. Let us begin with three-dimensional Fano varieties. Recall that the complete normal $\mathbb{Q}$-factorial variety X with no more than terminal singularities is called a Fano variety if the anticanonical divisor $-K_X$ is ample. The Fano two-dimensional varieties are nonsingular Del Pezzo surfaces, all of them are rational. The first examples of nonrational three-dimensional varieties with trivial geometric-differential invariants were obtained for nonsingular Fano three-dimensional varieties, namely, for the smooth cubic hypersurface $X_3 \subset \mathbb{P}^4$ in [11] and for the smooth quadric $X_4 \subset \mathbb{P}^4$ in [28]. The subsequent thorough study made it possible to obtain a complete classification with an accuracy to within deformations of nonsingular Fano three-dimensional varieties (see [19, 27, 41, 42]) and to find out which of them are rational and which are not (see [20, 29, 59, 2]).

As concerns singular three-dimensional Fano varieties and, first of all, $\mathbb{Q}$-Fano varieties with the Picard number $\rho = 1$ (varieties of type (I) given in Introduction), there does not yet exist any classification for them, and, consequently, the question as to their rationality is open, except for certain examples. Note the following two general results in this direction.

(1) (Kawamata [31]). *The three-dimensional $\mathbb{Q}$-Fano varieties with $\rho = 1$ constitute a finite number of families, or, to put it otherwise, Gorenstein's index* (*i.e., the minimal integer c such that $-cK_X$ is a Cartier divisor*) *and the degree of $-K_X^3$ are bounded.*

(2) (Namikawa [43]). *Every three-dimensional Fano variety with terminal Gorenstein singularities* (*i.e., with isolated* cDV*-points in the terminology of* [52]) *can be smoothed out.*

We shall now formulate the results concerning the rationality and non-rationality of smooth Fano three-dimensional varieties. We do not give here the full classification table of these varieties (see, e.g., [27]) but only indicate classes of nonrational varieties among them. If, in this case, one type of varieties can be obtained from another by means of an obvious birational transformation, say, by a blow-up, then we indicate only one of these types. The comments contain methods of proving nonrationality in each case. The proof of rationality of Fano varieties (they constitute the majority in the classification table) is carried out, as a rule, by means of explicit, relatively simple, constructions. Some of them are given in the comments below.

Theorem 5. *All smooth Fano varieties over an algebraically closed field of characteristic zero are rational, except for the following, which are not rational:*

(1) $V_3 \subset \mathbb{P}^4$, *a cubic hypersurface;*

(2) $V_{2:4} \to \mathbb{P}^3$, *a double cover with ramification in the quartic,*

(3) $V_1 \to Q^* \subset \mathbb{P}^6$, *a double cover of the cone Q^* with the base $F_4 \subset \mathbb{P}^5$, which is a Veronese surface,*

(4) $V_{2:6} \to \mathbb{P}^3$, *a double cover with ramification in the sextic,*

(5) $V_4 \subset \mathbb{P}^4$, *a quartic,*

(5′) $V_4' \to Q \subset \mathbb{P}^4$, *a double cover of a quartic,*

(6) $V_{2\cdot 3} \subset \mathbb{P}^5$, *the intersection of a quartic and a cubic,*
(7) $V_{2\cdot 2\cdot 2} \subset \mathbb{P}^6$, *the intersection of three quartics,*
(8)* $V_{10} \subset \mathbb{P}^7$, *the general section of the Grassmannian* $\mathrm{Gr}(1, \mathbb{P}^4) \subset \mathbb{P}^9$ *by a quadric and two hyperplanes,*
(8′)* $V'_{10} \subset \mathbb{P}^7$, *the general section of the cone* $W_5 \subset \mathbb{P}^7$ *with the base, which is a Fano variety* $V_5 \subset \mathbb{P}^6$, *by a quadric,*
(9) $V_{14} \subset \mathbb{P}^9$, *the section of the Grassmannian* $\mathrm{Gr}(1, \mathbb{P}^5) \subset \mathbb{P}^{14}$ *by a subspace of codimension* 5,
(10) $V_{(2,4)} \to \mathbb{P}^1 \times \mathbb{P}^2$, *a double cover with the ramification divisor of bidegree* $(2,4)$,
(11) $V_{(2,2)} \subset \mathbb{P}^2 \times \mathbb{P}^2$, *a divisor of bidegree* $(2,2)$,
(12) $V_{2:7}$, *a double cover of the Fano variety* $V_7 \subset \mathbb{P}^8$ *with the ramification divisor* $-K_{V_7}$,
(13) $V_{(2,2,2)} \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, *a double cover with ramification in a divisor of tridegree* $(2,2,2)$.

In addition, every smooth variety of type (9) *is birationally equivalent to a certain smooth variety of type* (1), *and vice versa.*

Comments concerning the methods of proving. Note, first of all, that the nonrationality of smooth varieties of types (1), (2), (7), (9)–(13) can be proved by the method of intermediate Jacobian of Clemens–Griffiths and, all the more so, the same method can be used (with the aid of degenerations) to prove the nonrationality of all the other general (in the sense of varieties of moduli) types of Fano varieties indicated in the theorem (see [11, 59, 5, 63, 2]). The nonrationality of smooth varieties of types (3)–(6) has been proved by the method of factorization of birational Noether–Fano maps (see [28, 20, 29]). For the references marked by an asterisk (8)* and (8′)*, the proof of nonrationality is known only for general varieties of these types. We do not yet know any type of the indicated Fano varieties the nonrationality of any smooth variety of which would be proved by both methods. Note that the Artin–Mumford method does not work here since $\mathrm{Tors}\, H^3(X, \mathbb{Z}) = 0$ for smooth Fano varieties.

All varieties whose nonrationality has been proved by the method of an intermediate Jacobian, except for varieties of type (2), can be represented birationally as conic bundles so that their intermediate Jacobian turns out to be a Prym variety (Theorem 3) and its distinction from the Jacobians of curves is established by Theorem 2. For instance, the projection of the cubic $\pi_Z \colon V_3 \dashrightarrow \mathbb{P}^2$ from the straight line $Z \subset V_3$ represents it, after the blow-up of the line $V' \to V$, as a standard conic bundle $\pi \colon V' \to \mathbb{P}^2$ with discriminant curve of degree 5, and the corresponding cover $\widetilde{\pi} \colon \widetilde{C} \to C$ is associated with an odd theta-characteristic (see Mumford's supplement in [11] and also [62]).

The following classical construction shows that the variety $V = V_{2\cdot 2\cdot 2} \subset \mathbb{P}^6$ is birationally by equivalent to a standard conic bundle over $\mathbb{P}^2$ with a degeneration curve of degree 7. We denote by $\Pi \simeq \mathbb{P}^2$ a two-dimensional family (bundle) of quadrics in $\mathbb{P}^6$ which pass through V. We choose a straight line $l \subset V$ (they form a one-dimensional family on V, see [5]) and denote by $P_l \simeq \mathbb{P}^4$ the space of two-dimensional planes in $\mathbb{P}^6$ which pass through l. Let $Q_l(V) = \{(q,p) \in \Pi \times P_l \mid p \subset q\}$. The projection $\pi \colon Q_l(V) \to \Pi \simeq \mathbb{P}^2$ is a standard conic bundle with the degeneration curve $C \subset \mathbb{P}^2$ of degree 7. Let $V_l \to V$ be a blow-up of l and $W = \{(q,x) \in \Pi \times V_l \mid \langle l, x\rangle \subset q\}$, where $\langle l, x\rangle$ is a plane generated by the straight line l and the point x. In the diagram

$$
\begin{tikzcd}
 & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
V_l & & Q_l(V)
\end{tikzcd}
,
$$

where $\alpha(q, x) = x$, $\beta(q, x) = (q, \langle l, x \rangle)$ are morphisms, α and β are birational so that V is birationally equivalent to the conic bundle $Q_l(V)$. In this case, the exceptional divisor E of the blow-up $V_l \to V$ is a ruled rational surface covering the base $\mathbb{P}^2$ of the degree 4 so that V is unirational with the degree of mapping 4. The following types of varieties are standard conic bundles with natural projections:

(10) $V_{(2,4)} \to \mathbb{P}^2$ with a degeneration curve of degree 8,

(11) $V_{(2,2)} \to \mathbb{P}^2$ with a degeneration curve of degree 6,

(12) $V_{2:7} \to \mathbb{P}^2$ with a degeneration curve of degree 7,

(13) $V_{(2,2,2)} \to \mathbb{P}^1 \times \mathbb{P}^1$ with a degeneration curve of bidegree $(4, 4)$.

Note that the nonrationality of these types of varieties immediately follows from the criterion of rationality of standard conic bundles over $S = \mathbb{P}^2$ or $\mathbb{F}_N$ is a geometrically ruled surfaces: $|2K_S + C| \neq \varnothing$, where C is a discriminant curve (see 2.3 below and [2]).

In the original proof by Clemens–Griffiths of the nonrationality of the three-dimensional cubic V_3 the intermediate Jacobian $J(V_3)$ was calculated in terms of a two-dimensional family of straight lines on the Fano $F = F(V_3)$-surface V_3. It turned out that $J(V_3)$ was isomorphic to the Albanese variety $\operatorname{Alb} F$ of the surface F and the polarization divisor Θ on $J(V_3)$ was the image of $F \times F$ under the mapping $(x, y) \mapsto a(x) - a(y)$, where $a\colon F \to \operatorname{Alb} F$ is an Albanese mapping. In this case, the diagonal $\Delta = \{(x, x) \in F \times F\}$ is mapped into a single singular point on the divisor Θ. Hence $\bigl(J(V_3), \Theta\bigr)$ cannot be a Jacobian of a curve since $\dim J(V_3) = 5$, $\dim \operatorname{Sing} \Theta = 0$, and according to the Andreotti–Mayer theorem, the codimension of the set of singularities of the polarization divisor on the Jacobian of a curve does not exceed 4.

As was pointed out, in all cases of the proofs of the nonrationality of Fano varieties by the method of an intermediate Jacobian use is made of Prym varieties, except for the case of varieties of type (2) which are double covers of $\mathbb{P}^3$ with ramification in the quartic. For the first time, the nonrationality of smooth varieties of this type was proved by Tikhomirov in [59]. Let $p\colon V \to \mathbb{P}^3$ be a double cover with ramification in the quartic $B \subset \mathbb{P}^3$ and F be a Fano surface of straight lines (relative to the intersection with $-K_V$) on V, and then the Abel–Jacobian map $\varphi\colon \operatorname{Alb} F \to J(V)$ is an isomorphism. Let Z be a family of curves of genus 3 and degree 6 (relative to $-K_V$), and then the image of the Abel–Jacobian map $Z \to J(V)$ is a polarization divisor Θ on the intermediate Jacobian $J(V)$. It turns out that $\dim \operatorname{Sing} \Theta$ is of codimension 5 in $J(V)$, and, consequently, $(J(V), \Theta)$ is not a Jacobian of any curve according to the Andreotti–Mayer theorem. The generalizations of this result to the case where B has several regular singular points in the general position (it stands to reason that in this case V also has singularities) are considered in [15, 10].

The method of an intermediate Jacobian was also used for proving the nonrationality of Fano general varieties of the types for which it is difficult to carry out direct calculations of $J(X)$ (for instance, $(8)^*$, $(8')^*$).

This idea, namely, Clemen's method of degeneration, consists in the following. Let $f: X \to S$ be a family of projective three-dimensional varieties over the smooth (not necessarily complete) curve S such that $X_S = f^{-1}(s)$ is nonsingular for $s \neq s_0$ and let $X_0 = f^{-1}(x_0)$ possess some admissible singularities, $s_0, s \in S$. For instance, Beauville [5] considered the case where $X \to S$ is a family of Fano varieties and possesses no more than regular double points. Let $\bar{j}: S - s_0 \to A_g$ be a mapping which puts every point $s \in S - s_0$ into correspondence with the intermediate Jacobian $J(X_S)$, where A_g are varieties of dimension g. It extends up to the mapping $\bar{j}: S \to A_g = A_g \cup A_{g-1} \cup \cdots \cup A_0$ into the Sataka compactification $\overline{A}_g$ of the variety A_g. In the case where X_0 has no more than regular double points, the image of $\bar{j}(s_0)$ is the intermediate Jacobian $J(X_0')$, where X_0' is the resolution of X_0.

Let us assume now that $J(X_0')$ is not the sum of Jacobians of curves, and then $J(X_s)$ for the general X_s is not the sum of Jacobians of curves either since the closure $\overline{M}_g \subset \overline{A}_g$ of the variety of moduli of curves under the natural embedding $M_g \hookrightarrow A_g$, which associates the curve with its Jacobian, consists of Jacobians of curves of genus $\leq g$ and their sums. Beauville considers in [5] families for which X_0' is a standard conic bundle and $J(X_0')$, being a Prym variety, is not isomorphic to the sum of Jacobians of the curves.

The author of [4] studies the case of more general degenerations, namely, $X_0 \subset X$, which is a reduced divisor with normal intersections consisting of no more than three irreducible components. The generalized intermediate Jacobian $J(X_0)$ is defined in terms of polarized mixed Hodge structures: it is an extension of the principal polarized Abelian variety (the sum of intermediate Jacobians of nonsingular components) by means of a certain torus polarized in an appropriate way. Assuming that every fiber X_s is a rational variety, the author of [4] proves that the family of intermediate Jacobians $J(X) \to S$ is, as a scheme of groups over S, a polarized fibering of the Jacobians of the degenerate family of curves $C \to S$, where C_S is a nonsingular curve, $s = s_0$. Then, under some additional assumptions imposed on the family $X \to S$, the fiber $J(X_{s_0})$ over the point s_0 must be a polarized generalized Jacobian of the semi-stable curve C_{s_0} with a natural polarization. The nonrationality of the general three-dimensional varieties of some types can be proved by means of a construction of a family with X_0 such that the generalized intermediate Jacobian $J(X_0)$ is not isomorphic to the generalized Jacobian of any semistable curve.

In the work [4] this method is also used for proving the nonrationality of certain general three-dimensional bundles into Del Pezzo surfaces (see 2.2 below).

As was pointed out, it is considerably easier to prove the rationality. Studying the classification tables of nonsingular Fano three-dimensional varieties, we see that in order to consider the problem of rationality, we can divide them into the following three classes:

(1) obviously rational, say, a quadric in $\mathbb{P}^4$, direct products of rational varieties, $\mathbb{P}^1$- or $\mathbb{P}^2$-bundles, blow-ups of rational varieties, etc,
(2) varieties X with $\rho(X) = 1$ and their blow-ups,
(3) varieties which are (birationally) representable as standard conic bundles over $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ (see [2]).

In the case (2), the rationality of varieties can be proved by traditional methods with the aid of projections from points, straight line, conics, a tangent space, a double projection from a straight line. For instance, for the variety $V_{2\cdot 2} \subset \mathbb{P}^5$, which is the intersection of two quadrics, the projection from a straight line is a birational isomorphism $V_{2\cdot 2} \dashrightarrow \mathbb{P}^3$. Another example: for the Fano variety

$V_{12} \subset \mathbb{P}^8$ the double projection from the straight line $Z \subset V_{12}$, i.e., a rational mapping defined by the complete linear system $|H - 2Z|$, where H is a hyperplane section, defines the rational mapping $V_{12} \dashrightarrow \mathbb{P}^1$ whose general fiber is a Del Pezzo surface of degree 5 which is birationally equivalent to $\mathbb{P}^2$ over any field, and therefore V_{12} is birationally equivalent to $\mathbb{P}^1 \times \mathbb{P}^2$, i.e., is rational.

It stands to reason that we use the fact of existence of straight lines on Fano varieties proved by Shokurov; this fact is also used for classifying nonsingular Fano three-dimensional varieties. Using a double projection from a straight line, we can prove the rationality of Fano varieties $V_{2g-2} \subset \mathbb{P}^{g+1}$ with $\operatorname{Pic} V_{2g-2} = \mathbb{Z}(-K_V)$ for $g \geq 9$ (i.e., $g = 9, 10, 12$, no other g exists). For $g = 8$ a double projection represents V_{14} (i.e., varieties of type (9) from Theorem 5) birationally as a standard conic bundle $\pi'\colon V' \to \mathbb{P}^2$ with the degeneration curve $C \subset \mathbb{P}^2$ of degree 5. It is shown in [61] that $\pi'\colon V' \to \mathbb{P}^2$ is birationally equivalent over $\mathbb{P}^2$ to the conic bundle $\pi\colon V \to \mathbb{P}^2$ obtained by a projection from a straight line from a certain nonsingular cubic $V_3 \subset \mathbb{P}^4$; the converse is also true, i.e., every smooth cubic in $\mathbb{P}^4$ is birational in this way to the Fano variety $V_{14} \subset \mathbb{P}^9$. For the first time this fact was established by Fano, he constructed a birational map $V_3 \dashrightarrow V_{14}$ connected with an elliptic curve of degree 5 on V_3. Birational maps connected with a rational curve of degree 4 are indicated in [60] and [58].

Finally, the problem of the rationality of varieties from class (3) can be solved with the use of the criterion of rationality for standard conic bundles (see 2.3 below).

2.2. Let us now consider the rationality problem of Del Pezzo fibrations $\delta\colon X \dashrightarrow \mathbb{P}^1$ of type (II) given in Introduction. Let $F = X_\eta$ be a general fiber of this bundle. By definition, F is a smooth projective Del Pezzo surface over $k(\eta) = k(\mathbb{P}^1)$, i.e., over a field of rational functions, and $\operatorname{Pic}_{R(\eta)} F \simeq \mathbb{Z}$. According to the classification of Del Pezzo surfaces, we have $1 \leq K_F^2 \leq 9$. From the criterion of $k(\eta)$-rationality of rational surfaces (see, e.g., [23]) it follows that F is rational over $k(\eta)$ for $K_F^2 \geq 5$. Consequently, under this condition, X is rational over k. Thus the problem reduces to the investigation of the cases $K_F^2 = 1, 2, 3, 4$.

The few results obtained here only concern the nonsingular (or general) X. The case of $K_F^2 = 4$ with a nonsingular X was studied in [1]. The problem reduces to the rationality criterion of standard conic bundles over geometrically ruled surfaces in the following way. Since the field of functions $k(\mathbb{P}^1)$ over the algebraically closed field k is the field C_1 in the sense of Serre, the general fiber F of the morphism $\delta\colon X \to \mathbb{P}^1$ has a $k(\mathbb{P}^1)$-point, i.e., δ has a section. Since $K_F^2 = 4$, F is unirational over $k(\mathbb{P}^1)$, and, consequently, the $k(\mathbb{P}^1)$-point on F are dense, and, hence, there are sufficiently many sections. Let us choose a sufficiently general section, namely, a curve $Z \subset X$, and then $Z \simeq \mathbb{P}^1$ and there exists a link of type III:

$$\begin{tikzcd}
X' \arrow[r, dashed] \arrow[d, "\sigma"'] & X_1 \arrow[dd, "\pi_1"] \\
X \arrow[d, "\delta"'] & \\
\mathbb{P}^1 & \mathbb{F}_N \arrow[l, "\varphi"']
\end{tikzcd}, \qquad (9)$$

where $\sigma\colon X' \to X$ is a blow-up of Z, $X' \dashrightarrow X_1$ is a sequence of flops with centers in those lines $E_i \subset F_i$ in the fibers F_i of the morphism δ which intersect Z, $\pi_2\colon X_1 \to \mathbb{F}_N$ is a standard conic bundle over the geometrically ruled surfaces $\mathbb{F}_N$, and $\varphi\colon \mathbb{F}_N \to \mathbb{P}^1$ is a natural projection.

Theorem 6 [1]. *Let*

$$\chi(X) = \sum_{i=0}^{6}(-1)^i B_i(X) = 6 - B_3(X)$$

be a topological Euler characteristic, where $B_i(X)$ are Betti numbers. Then, for $\chi(X) \neq 0, -8, -4$ X is not rational and for $\chi(X) = 0, -8$ X is rational. For $\chi(X) = -4$ X is not rational if and only if $J(X)$ is not a Jacobian of a curve: in this case X is birationally equivalent to a smooth three-dimensional cubic in $\mathbb{P}^4$.

Outline of the proof. In diagram (9) we have $B_3(X_1) = B_3(X) = 2\dim J(X) = 2\dim J(X_1) = 2p_0(C) - 2$, where $C \subset \mathbb{F}_N$ is a degeneration curve of the standard conic bundle $\pi_1\colon X_1 \to \mathbb{F}_N$. The curve C is equivalent to $5s_N + \alpha f_N$, where s_N is an exceptional section and f_N is a fiber on $\mathbb{F}_N$, $\alpha \in \mathbb{Z}$ is an integer. Then $2p_0(C) - 2 = -20N + 8\alpha - 10$. The curve C is connected and reduced and $(C - s_N)s_N = 0 \pmod 2$ if s_N is a component of C, and therefore $\alpha \geq 5N$ if $s_N \not\subset C$ and $\alpha \geq 4N + 2$ if $s_N \subset C$. On the other hand, according to the rationality criterion, if $|2K_{F_N} + C| \neq \varnothing$ then X_1 is not rational and $|2K_{F_N} + C| = \varnothing \implies X_1$ is rational, except for the case where X_1 is birationally equivalent to a three-dimensional cubic. Hence, if X' is rational, then the divisor

$$-4s_N - 2(N+2)f_N + 5s_N + \alpha f_N = s_N + (\alpha - 2N - 4)f_N$$

is not effective, i.e., $\alpha < 2N + 4$. Hence, $N = 0$ or 1 and, if $N = 0$, then $\alpha = 2, 3$; the cases $\alpha = 0, 1$ are impossible since the curve C is connected and $p_a(C) \neq 0$ and $\chi(X) = 0, -8$ respectively; if $\alpha = 1$, then $\alpha = 5$ and $\chi(X) = -4$. In the latter case, $C \cap s_1 = \varnothing$ and X_1 becomes a standard conic bundle over $\mathbb{P}^2$ (see, e.g., [25]) with a degeneration curve of degree 5.

In the other cases, namely, $K_F^2 = 1, 2$ and 3, the problem of rationality was not completely investigated even for nonsingular X.

In [22, 48, 49] the Noether–Fano method is used for proving the nonrationality of nonsingular varieties of these types under certain conditions formulated below. In [4] the Clemens degeneration method is used to prove the nonrationality of some general varieties with $K_F^2 = 3$. The Kollar degeneration method [34] also makes it possible to prove the nonrationality of the general varieties with $K_F^2 = 2$ and 3.

Here are Pukhlikov's results concerning nonrationality. Let $\delta\colon X \to \mathbb{P}^1$ be a del Pezzo bundle with a nonsingular X, a smooth general fiber F, $K_F^2 = 1, 2$ or 3, all fibers be irreducible and reduced, $\operatorname{Pic} X = \delta^* \operatorname{Pic} \mathbb{P}^1 + \mathbb{Z}(-K_X)$. We assume that the following conditions are fulfilled:

(∗) for any $m \in \mathbb{Z}$ the 1-cycle $m(-K_X)^2 - f$ is not effective;

(∗∗) in the case $K_F^2 = 3$ every singular fiber $F_t = \delta^{-1}(t)$ has only one nondegenerate quadratic singularity.

Theorem 7 [49]. *Assume that $\delta\colon X \to \mathbb{P}^1$ satisfies the preceding conditions, $K_F^2 = 1, 2, 3$ and $\delta'\colon X' \to \mathbb{P}^1$ be an arbitrary fibering into rational surfaces with a normal projective X' and a smooth general fiber F'.*

Then any birational mapping $\chi\colon X \dashrightarrow X'$ (provided that it exists) preserves the Del Pezzo structure of the Del Pezzo bundle, i.e., $\chi^{-1}(|F'|) = |F|$. In other words, there exists an isomorphism *of the base $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $\delta' \cdot \chi = \varphi \cdot \delta$. In particular, X is not rational and not birationally equivalent to a conic bundle.*

The proof uses the Noether–Fano method of factorization of birational maps. The assumption concerning the existence of the birational mapping $\chi\colon X \dashrightarrow X'$ which does not preserve the structure of the bundle leads to a contradiction (under assumptions $(*)$ and $(**)$ that we have made) with the Noether–Fano Lemma 1.

Note that condition $(*)$ means that the bundle $\delta\colon X \to \mathbb{P}^1$ is sufficiently "twisted" with respect to the base. We shall clarify this by way of an example of fibering into cubic surfaces, i.e., in the case $K_F^2 = 3$. The relatively ample bundle $\mathcal{O}(-K_X)$ is very ample on each fiber and defines the embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) + \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3)$ is a locally free bundle of rank 4 on $\mathbb{P}^1$, $0 \le a_1 \le a_2 \le a_3$, coinciding, with an accuracy to within the multiplication by an invertible bundle, with $\delta_* \mathcal{O}_X(-K_X)$. Let H be the class of the tautological invertible bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and L be the class of a fiber of projection $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. Let $X \sim 3H + nL$ be a divisor on $\mathbb{P}(\mathcal{E})$. Then condition $(*)$ is fulfilled for $5n + 3(a_1 + a_2 + a_3) \ge 12$. In particular, if $a_1 = a_2 = a_3 = 0$, i.e., $X \subset \mathbb{P}^3 \times \mathbb{P}^1$ is divisor of degree $(3, n)$, then $n \ge 3$. Since in this case $n \ge 1$ always and the variety X is obviously rational for $n = 1$, only the case $n = 2$ remains open.

Note that in [4] the nonrationality of the general variety of bidegree $(3, n)$ in $\mathbb{P}^3 \times \mathbb{P}^1$ for $n \ge 2$ is proved. A similar result with $n \ge 3$ in this case was also obtained by Kollar [33] by a different degeneration method (see Section 3). The result referring to the general double cover $X \to \mathbb{P}^2 \times \mathbb{P}^1$ with ramification in a divisor of bidegree $(4, 2r)$ was also obtained in [4], namely, the general variety X is not rational for $r \ge 2$; here $K_F^2 = 2$.

The last statement of the theorem to the point that X is not birationally equivalent to any conic bundle follows from the fact that the general fiber F_η for $K_F^2 = 1, 2, 3$ is not birationally equivalent to any conic bundle on $\mathbb{P}^1$ over the field $k(\eta) \simeq k(\mathbb{P}^1)$ since $\operatorname{Pic} F_\eta = \mathbb{Z}(-K_{F_\eta})$ (see, e.g., [23]).

2.3. Let us consider now the question concerning the rationality of standard conic bundles $\pi\colon X \to S$ over the rational surface S with the discriminant curve $C \subset S$. Here X, S are nonsingular, C is a reduced curve with normal intersections, $\operatorname{Pic} X = \pi^* \operatorname{Pic} S + \mathbb{Z}$. We assume that C is connected, otherwise X would be nonrational according to Artin and Mumford (Theorem 4). If $C = \varnothing$, then X is rational since then $\pi\colon X \to S$ is a locally trivial $\mathbb{P}^1$-bundle over S. This follows from the fact that the Brauer group $\operatorname{Br} S$ is trivial since S is rational.

Let $\widetilde{C}$ be a curve which parametrizes the straight lines in fibers $\pi\colon X \to S$, i.e., the components of the disintegrating conics. Then π induces a double cover $\widetilde{\pi}\colon \widetilde{C} \to C$ which satisfies Beauville's conditions (B). Conditions (B) mean that $\widetilde{\pi}\colon \widetilde{C} \to C$ is a natural generalization of the concept of a nontrivial nonramified double cover to the case of a reduced curve with normal intersections. We can see now, for instance, that for the standard conic bundle $\pi\colon X \to S$ with the connected curve C we have $p_a(C) \ne 0$ since there are no nontrivial "nonramified" covers over a curve of genus zero.

The following hypothetical criterion of rationality for the standard conic bundle $\pi\colon X \to S$ over the rational surface S with the discriminant connected curve $C \subset S$ is proposed in [26] and [25].

Conjecture 1. (i) *If X is rational, then $|2K_S + C| = \varnothing$.*

(ii) *Conversely, if $|2K_S + C| = \varnothing$, then either X is rational or $\pi\colon X \to S$ becomes* (*with a sequence of links of types* II *and* III) *a standard conic bundle $\pi_0\colon X_0 \to \mathbb{P}^2$ obtained by straight line of a certain three-dimensional cubic.*

Statements (ii) of this conjecture are proved in [24] in the following more exact sense.

Theorem 8. *In the previous notation, if* $|2K_S+C|=\varnothing$, *then there exist a birational morphism* $\alpha\colon S\to S'$, *a standard conic bundle* $\pi'\colon X'\to S'$ *with a discriminant curve* $C'=\alpha(C)$, *and a birational map* $\varphi\colon X\dashrightarrow X'$ *such that the diagram*

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ S & \xrightarrow{\alpha} & S' \end{array}$$

is commutative, φ *is a sequence of links of type* II *or* III, *and one of the following two conditions is fulfilled:*

(a) $S'=\mathbb{F}_N$ *is a geometrically ruled surface and* $C'f_N\le 3$, *where* f_N *is the class of the fiber* $\mathbb{F}_N\to\mathbb{P}^1$: *in this case* X' *and* X *are rational*;

(b) $S'=\mathbb{P}^2$ *and* $\deg C'\le 5$; *in this case* X' *and* X *are rational, except for the case where* $\deg C'=5$ *and* X' *is birationally equivalent over* $\mathbb{P}^2$ *to the standard bundle* $\pi_0\colon X_0\to\mathbb{P}^2$ *obtained by means of the projection from the straight line of a certain three-dimensional cubic.*

Outline of the proof. If $|2K_S+C|=\varnothing$, then it is easy to verify that the divisor $2K_S+C$ on the rational surface S is not nef by using the fact that K_S+C is effective since $p_a(C)>0$. Now, from the fact that the curve C is reduced, connected, and $(C-Z)Z\ge 2$ for any irreducible component $Z\subset C$ it follows immediately that $2K_S+C$ is nonnegative, being a linear function on the positive part $\overline{\mathrm{NE}}_K=\{z\in\overline{\mathrm{NE}}(S)\mid zK_S\ge 0\}$ of the Mori cone $\overline{\mathrm{NE}}(S)$. Consequently, there exists an extremal ray $R\subset\overline{\mathrm{NE}}(S)$ such that $(2K_S+C)R<0$.

If R is the class of a line on $S=\mathbb{P}^2$, then we get $CR\le 5$. In the case where $CR\le 4$, the rationality of X can be established by a direct calculation: the pencil of lines $|L-x|$ which passes through a sufficiently general point $x\in C$ intersects C at most three times, and then the surface $F_L=\pi^{-1}(L)$ over the general element of the bundle is a rational surface over the field $k(\mathbb{P}^1)$ and we can use the criterion of rationality of surfaces over a nonclosed perfect field. The last statement in (b) can be proved by the method of an intermediate Jacobian (for details see [26]).

If R is the class of the fiber on the ruled surface $S=\mathbb{F}_N$, then $(2K_S+C)R<0$ and $CR\le 3$ and the rationality of X can also be derived from the rationality of F_L, where $L=|f|$ is a bundle defining the morphism $\mathbb{F}_N\to\mathbb{P}^1$.

If, finally, $R=R_+[E]$ is the class of the (-1)-curve E on S, then $(2K_S+C)R<0$, and then $CE\le 2$, the composition of links of type II and links of type III with the contraction $\alpha_1\colon S\to S_1$ is extended to the commutative diagram

$$\begin{array}{ccc} X & \overset{\varphi_1}{\dashrightarrow} & X_1 \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ S & \xrightarrow{\alpha_1} & S_1 \end{array},$$

where $\pi_1\colon X_1\to S_1$ is a standard conic bundle over S_1 with the discriminant curve $C_1=\alpha_1(C)$. Then we use induction on the Picard number $\rho(S)=\rho(S_1)+1$.

As to statement (i), it has not yet been proved in the general case. However, the following results have been obtained in this direction

Proposition 4 (see [26]). *Let X be rational and assume that there exists a birational mapping $\chi\colon X \dashrightarrow \mathbb{P}^3$ which transfers the fibers $X_S = \pi^{-1}(s)$, $s \in S$, into conics in $\mathbb{P}^3$, i.e., which is defined by the linear system $|-K_X + \pi^*A|$, where $A \in \operatorname{Pic} S$ is a class of divisors. Then $|2K_S + C| = \varnothing$.*

Idea of the proof. The general surface T_η in the proper preimage $\chi^{-1}_{\text{proper}}(|H|)$ of the linear system of planes $|H|$ in $\mathbb{P}^3$ is rational and is a double cover of S. Consequently, an involution of the double cover acts on T_η. The classical theory of action of finite groups on rational surfaces makes it possible to find an invariant bundle of rational or elliptic curves whose image on S is a bundle of rational curves L. The general surface $F_L = \pi^{-1}(L)$ is a conic bundle over $\mathbb{P}^1$ and possesses a three-dimensional linear system of rational or elliptic curves which is cut out by the linear system $|-K_X + \pi^*A| = \chi^{-1}_{\text{proper}}(|H|)$. If a three-dimensional linear system of rational curves is cut out, then $K^2_{F_L} \geq 5$ and, consequently, $LC = 8 - K^2_{F_L} \leq 3$. If the curves are elliptic, then $K^2_{F_L} \geq 3$ and $LC \leq 5$. Then it immediately follows that $|2K_S + C| = \varnothing$ (see [26]).

The analysis of the proof of Theorem 8 shows that if $|2K_S + C| = \varnothing$ and if X is birationally equivalent to a cubic in $\mathbb{P}^4$, then there exists a birational map $\chi\colon X \dashrightarrow \mathbb{P}^3$ just as in the hypothesis of Proposition 4. Thus Conjecture 1 proves to be equivalent to the following classical conjecture.

Conjecture 2 (Cantor, 1901). *Any two-dimensional family (congruence of index 1) of rational curves in $\mathbb{P}^3$ such that only one curve of the family passes through the general point can be mapped by birational transformations of $\mathbb{P}^3$ into a two-dimensional family of conics or a bundle of straight lines.*

Note that a similar problem for a bundle of rational curves in the plane $\mathbb{P}^2$ over the perfect field k can be solved as follows. Resolving the base points of the bundle and contracting (-1)-curves in the fibers, we obtain a standard conic bundle $\pi\colon F \to \mathbb{P}^1$. Since F is k-rational, it follows that $K^2_F \geq 5$ (according to the criterion of k-rationality of surfaces). Next, in each of the possible cases $K^2_F = 5, 6, 8$ the birational k-mapping $F \dashrightarrow \mathbb{P}^2$, which maps the fibers F_t into conics, is explicitly indicated (see., e.g., [23]).

Note one more important result referring to condition (i) of Conjecture 1.

Proposition 5 (Shokurov [67]). *Let $\pi\colon X \to S$ be a standard conic bundle over $S = \mathbb{F}_N$ or $\mathbb{P}^2$ with the discriminant curve $C \subset S$. Assume that X is rational, and then $|2K_S + C| = \varnothing$.*

Thus the criterion of rationality is valid in these cases (i.e., when $S = \mathbb{F}_N$ or $\mathbb{P}^2$).

Idea of the proof. Since X is rational, $J(X) = \operatorname{Pr}(\widetilde{C}, C)$ is the sum of Jacobians of curves, in particular, it may be trivial. Then, according to Theorem 2, the curve must be trigonal, hyperelliptic, or a plane curve of degree 5. The case $S = \mathbb{P}^2$ has been thoroughly studied (see [5, 8]): X is rational if $\deg C \leq 4$, X is not rational if $\deg C \geq 6$, in the case $\deg C = 5$ X is rational if and only if $\widetilde{C} \to C$ corresponding to an even theta-characteristic.

Let us consider the case $S = \mathbb{F}_N \supset C$. If $p_a(C) = 1$, then the direct computations give $|2K_S + C| = \varnothing$. In the general case of the trigonal or hyperelliptic curve C we can prove that the embedding $C \hookrightarrow \mathbb{F}_N$ can be only standard, i.e., the corresponding special linear series on C is cut out of a bundle of straight lines on $|f_N|$, and then it is clear that $|2K_S + C| = \varnothing$. The following arguments are used in the proof given in [67].

The adjoint linear system $|K_S + C|$ is free on $S = \mathbb{F}_N$, except for certain cases, and defines the morphism $\varphi_{K_S+C}\colon \mathbb{F}_N \to \mathbb{P}^{p_a(C)-1}$ which coincides on C with the canonical embedding $\varphi_{K_C}\colon C \to \mathbb{P}^{p_a(C)-1}$. Now, since $\mathbb{F}_N$ are $\mathbb{P}^1$-bundles over $\mathbb{P}^1$, either φ_{K_S+C} is a birational map, the image is normal and is cut out by quadrics, or it is a morphism onto a curve of genus zero. In the first case, C is a trigonal curve and the image $\varphi_{K_S+C}(\mathbb{F}_N)$ is a surface which is the intersection of quadrics in $\mathbb{P}^{p_a(C)-1}$ that pass through the canonical image of the curve C.

According to the classical Noether–Enriques theorem, this surface is swept out by 3-chords of the canonical curve $\varphi_{K_C}(C) \subset \mathbb{P}^{p_a(C)-1}$. We infer from the exact sequence

$$0 \longrightarrow H^0\big(S, \mathcal{O}(2K_S + C)\big) \longrightarrow H^0\big(S, \mathcal{O}(2K_S + C)\big) \longrightarrow H^0\big(C, \mathcal{O}(2K_S + C)\big) \longrightarrow 0$$

that $|2K_S + C| = \varnothing$.

If C is a hyperelliptic curve, then the only case where $\varphi_{K_S+C}(C) = \varphi_{K_C}(C)$ is a normal rational curve in $\mathbb{P}^{p_a(C)-1}$ is possible, i.e., $|K_S + C| = |(p_a(C) - 1)f_N|$, and, consequently, $|2K_S + C| = \varnothing$.

The most significant results in the study of the problem of rationality of conic bundles were obtained by the method of Noether–Fano factorization of birational maps. Sarkisov proved in [54, 55] the following theorem.

Theorem 9. *Let $\pi\colon X \to S$ be a standard conic bundle over the rational surface S with the discriminant curve C and*

$$\begin{array}{ccc} X & \stackrel{\chi}{\dashrightarrow} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ S & & S' \end{array} \tag{10}$$

be a birational mapping onto any Mori-fibered space $\pi'\colon X' \to S'$. Then, if $|4K_S + C| \neq \varnothing$, then χ can be closed into a commutative diagram, i.e., there exists a birational mapping $\psi\colon S \dashrightarrow S'$ that makes diagram (10) *commutative.*

This theorem immediately yields the following corollary.

Corollary. *Under the conditions of Theorem* 9 *X is not rational.*

This fact allows us to construct examples of nonrational conic bundles with an intermediate Jacobian, which is isomorphic to the product of Jacobians of curves, or even the intermediate Jacobian is zero. We know (Theorem 8 and Proposition 4) that we cannot construct bundles of this kind over $\mathbb{P}^2$ or $\mathbb{F}_N$ since in these cases the criterion of rationality is defined precisely by the property of the intermediate Jacobian to be or not to be a product of Jacobians of curves. Moreover, if, on a certain smooth rational surface S, there exists a hyperelliptic, trigonal, or even elliptic curve $C \subset S$, then, with the use of any nontrivial nonramified cover $|4K_S + C| \neq \varnothing$, $\widetilde{C} \to C$, we can construct a standard conic bundle $\pi\colon X \to C$ with the discriminant curve C and the corresponding cover $\widetilde{C} \to C$ (see Proposition 2 and Theorems 2 and 3) which will be nonrational according to the preceding corollary. We can obtain a simple example as follows.

Let $C' \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 12$ with $\frac{1}{2}(d-1)(d-2) - 1$ regular double points, $\sigma\colon S \to \mathbb{P}^2$ be a blow-up of all these points, and C be the proper preimage of the curve C'.

Then C is a smooth irreducible curve of genus 1 and the divisor $4K_S + C \sim (d-12)\sigma^* l + 2\sum E_i$ is effective, where l is the class of a line on $\mathbb{P}^2$ and E_i are the exceptional curves of the morphism σ. Thus, the standard conic bundle $\pi\colon X \to S$ which corresponds to any one of the three nontrivial double etale covers $\widetilde{C} \to C$ is not rational.

Note that Conjecture 1 yields the nonrationality of X constructed in this way for the curve C' of degree 6 with 9 double points. This is the first and, may be, the most difficult case of verifying Conjecture 1.

Let us consider the recent results [24] in the scheme of the proof of Conjecture 1 (i) with the use of the Mori theory in the framework of the so-called Sarkisov program [14]. The main idea is to generalize the outline of the proof of the criterion of k-rationality of surfaces to the three-dimensional case by studying birational mappings between all $\mathbb{Q}$-Mori fibered spaces $\varphi\colon X \to S$ by the Noether–Fano method of factorization with the use of Theorem 1.

The main difficulty is that in contrast to surfaces, we have to operate here in the category of singular varieties: X can have $\mathbb{Q}$-factorial terminal singularities or S-log-terminal singularities (i.e., quotient singularities in the case of $\dim S = 2$). Therefore such basic concepts as the degree of a linear system and the multiplicities of base sets inherent in the classical Noether–Fano theory must be replaced by suitable concepts in the framework of the program of minimal models. As was mentioned in 1.3, the coefficient $\mu \in \mathbb{Q}$ is the substitution for the degree of the linear system $\mathcal{H} = |-\mu K_X + \varphi^* A|$ on X and the substitution of the multiplicity of the maximal singularity is the inverse of the canonical threshold $c := \max\{b \in \mathbb{Q} \mid K_X + b\mathcal{H} \text{ is canonical}\}$.

Let $\varphi\colon X \to S$ be a $\mathbb{Q}$-Mori conic bundle, i.e., X has no more than $\mathbb{Q}$-factorial terminal singularities and S has quotient singularities, and let φ be the extremal contraction in the sense of Mori, in particular, $\rho(X) - \rho(S) = 1$. The analysis of the examples shows that the S singularities are much better than arbitrary quotient singularities. This makes it possible to formulate the following proposition.

Conjecture 3. *For any $\mathbb{Q}$-Mori conic bundle $\varphi\colon X \to S$ the singular points on S are no more than double rational (i.e., Du Val) singularities of type A_n.*

It is well known that if X is Gorenstein, then S is nonsingular. If Gorenstein's index of X does not exceed 6, then Conjecture 3 is valid [44]. In addition, Conjecture 3 can be derived from the general conjecture of M. Reid on the existence of good divisors (locally) in the anticanonical linear system $|-K_X|$ (see [44]).

The following result was obtained in [24] in the direction of proving the criterion of rationality for standard conic bundles.

Theorem 10. *Let $\pi\colon X \to S$ be a standard conic bundle with the discriminant curve $C \subset S$. Assume that X is rational, Conjecture* 3 *is valid, and $p_a(C) > 18$. Then statement* (i) *of Conjecture* 1 *holds true.*

Thus the criterion of rationality (Conjecture 1 and, consequently, Cantor Conjecture 2) is valid modulo Conjecture 3 if $p_a(C) > 18$.

Outline of the proof. By the hypothesis, there exists a birational mapping $\chi\colon X \dashrightarrow \mathbb{P}^3$. Let $\mathcal{H} \equiv -\mu K_X + \pi^* A$, which is a linear system without fixed components, be the proper preimage $\chi^{-1}_{\text{proper}}(|\mathcal{O}(1)|)$ of the linear system of planes in $\mathbb{P}^3$. We can choose χ with the least possible μ. Here $\mu \in \frac{1}{2}\mathbb{Z}$ since $\mathcal{H}X_s = -\mu K_X X_s = 2\mu$ for the fiber $X_s = \pi^{-1}(s)$, $s \in S$. Recall that for $\mu = 1$

the criterion of rationality has been proved. Let us make an induction assumption: the theorem is valid for all standard conic bundles and birational maps χ' with $\mu' < \mu$.

Following [14], we say that $\mathcal{H}$ possesses a maximal singularity if $c < 1/\mu$, where c is a canonical threshold for $(X, \mathcal{H})$. We find from Proposition 2 that either $\mathcal{H}$ possesses a maximal singularity or the divisor $A \in \operatorname{Pic} S$ is not numerically effective. Following the algorithm for the proof of Theorem 1 (see [14]), we use links of types I and II to untwist all maximal singularities, and then, with the aid of the sequence of links of type III, the pair $(X, \mathcal{H})$ is mapped birationally, with the preservation of the conic bundle, into the pair $(X', \mathcal{H}')$ with the following properties:

(1) $\pi' \colon X' \to S'$ is a $\mathbb{Q}$-Mori conic bundle where either $\rho(S') = 1$ or $\rho(S') = 2$, and there exists a morphism $\alpha' \colon S' \to \mathbb{P}^1$ with the general fiber $\mathbb{P}^1$,

(2) the proper image $\mathcal{H}'$ of the linear system $\mathcal{H}$ has the form $\mathcal{H}' \equiv -\mu K_X + \pi'^*(A')$ where the divisor A' is not numerically effective and $\mathcal{H}'$ has no maximal singularities,

(3) the discriminant curve $C' \subset S'$ is a birational image of the curve $C \subset S$ and $h^1(\mathcal{O}_{C'}) \geq h^1(\mathcal{O}_C) = p_a(C)$.

Let us now use the following, purely computational, fact (see [24]).

Lemma 2. *Let $\varphi \colon X \to S$ be a $\mathbb{Q}$-Mori conic bundle and $\mathcal{H} \equiv -\mu K_X + \varphi^* A$ be a linear system on X without fixed components. Then*

$$\Gamma := \varphi_*(\mathcal{H}^2) \equiv 4\mu A - \mu^2(4K_S + C)$$

is an effective $\mathbb{Q}$-divisor on S, where $C \subset S$ is a discriminant curve (Weil divisor on S).

Let us apply this lemma to $\pi' \colon X' \to S'$, C', $\mathcal{H}'$ and A'. In the case of $\rho(S') = 1$, the $\mathbb{Q}$-divisors Γ', A', $K_{S'}$ and C are proportional over $\mathbb{Q}$ and, since A' is not nef, it follows that $A' < 0$, and, consequently, $C' < -4K_{S'}$ since $\Gamma' \geq 0$.

Here and only here we use the assumption of Conjecture 3. Assume that it is valid. Then S' is a Del Pezzo surface with canonical Gorenstein singularities (i.e., double rational points) and $\rho(S') = 1$. There exist only 28 types of surfaces of this kind, with $\mathbb{P}^2$ among them, and 15 types of surfaces with singularities of type A_n. Inasmuch as $C' < -4K_S$, it is clear that for C' we have limited possibilities, in particular, $h^1(\mathcal{O}_{C'}) \leq 45$. Now we shall use the fact that the intermediate Jacobians is the product of Jacobians of curves and, consequently, the curve C is hyperelliptic or trigonal (at least if $p_a(C) > 6$). This allows us to improve the estimate of the genus $h^1(\mathcal{O}_{C'}) \leq 18$. Thus, if $h^1(\mathcal{O}_{C'} > 18$, then this case cannot be realized (certainly, modulo Conjecture 3).

Let us consider a different case where $\rho(S') = 2$ and $\alpha \colon S' \to \mathbb{P}^1$ is a ruled surface with isomorphic quotient singularities. We denote by L' the general fiber of the morphism α, and then $L' \simeq \mathbb{P}^1$ over the field $K = k(\mathbb{P}^1)$. Then the algorithm for untwisting the map χ shows that $A'L' < 0$, and, consequently, it follows from

$$0 \leq \Gamma' L' = 4\mu A'L' - \mu^2(4K_{S'} + C')L'$$

that $C'L' < 8$.

Now, if $C'L' \leq 3$, then the preimage of the bundle $|L'|$ on S is a bundle of rational curves L with $CL \leq 3$, and this proves assertion (i) of Conjecture 1 in this case.

Under the condition $p_a(C) > 18$, the cases $C'L' = 4, 5, 6, 7$ can be reduced to a contradiction as follows. Since the curve C is hyperelliptic or trigonal, i.e., possesses a linear series of degree 2 or 3, i.e., $\beta\colon C \to \mathbb{P}^1$, $\deg\beta = 2$ or 3. On the other hand, the bundle L' lifted to S defines the morphism $\alpha\colon C \to \mathbb{P}^1$, $\deg = C'L'$. Consequently, there exists a morphism $\alpha \times \beta\colon C \to \mathbb{P}^1 \times \mathbb{P}^1$. If $\deg\alpha = 5$ or 7, then this map is birational and, hence, $p_a(C) \leq p_a(\widetilde{C}) \leq 12$.

This trick does not work in the cases $C'L' = 6$ and 4 since the map $C \to \widetilde{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ may not be birational (i.e., may be factored via a mapping defined by a hyperelliptic or a trigonal linear series). Therefore, in the cases $C'L' = 4$ and 6, we shall find the link Φ of type IV which will reconstruct the conic bundle $\pi'\colon X' \to S'$ into a different conic bundle $\pi''\colon X'' \to S''$, this being a map over $\mathbb{P}^1$:

$$\begin{tikzcd}
X' \arrow[rr, dashed, "\Phi"] \arrow[d, "\pi'"'] & & X'' \arrow[d, "\pi''"] \\
S' \arrow[dr, "\alpha"] & & S'' \arrow[dl, "\beta"'] \\
& \mathbb{P}^1 &
\end{tikzcd} .$$

The proper image of the linear system $\mathcal{H}' \equiv -\mu' K_{X'} + \pi'^* A'$ on X', where $\mu' = \mu$ is represented as $\mathcal{H}' \equiv -\mu K_X + \pi''^* A''$, with $\mu'' < \mu = \mu$. The link Φ can be constructed as follows. Since $A'L' < 0$, there exists a link of type III or IV according to [14]. Its restriction to the general fiber $F' = {\pi'}^{-1}(L')$ must be a link of type III or IV on a smooth surface, fibered into conics, $F' \to \mathbb{P}^1$ over the field $k(\mathbb{P}^1)$. It is easy to verify that $-K_{F'}$ is ample in the cases $K_{F'}^2 = 4$ and 2 as well.

We know from the theory of rational surfaces over a perfect field (see, e.g., [21]) that in these cases there exists a different structure of conic bundles with the fiber

$$f'' = \begin{cases} -K_{F'} - f' & \text{if } K_{F'}^2 = 4 \\ -2K_{F'} - f' & \text{if } K_{F'}^2 = 2. \end{cases}$$

The transition from one structure to another is continued up to the link Φ of type IV, as it is shown in the diagram, and in this case

$$\mu'' = \begin{cases} \mu + A'L' & \text{if } C'L' = 4 \\ \mu + 2A'L' & \text{if } C'L' = 6. \end{cases}$$

Since $\mu'' < \mu$, assertion (i) of Conjecture 1 is valid, by the induction assumption, for any standard model of conic bundle $\pi''\colon X'' \to S''$. This means that there exists a bundle of rational curves, different from L'', which cuts out of $C'' \subset S''$ a hyperelliptic or trigonal series. Now we can find, by a direct computation, that $p_a(\widetilde{C}'') \leq 10$, where $\widetilde{C}''$ is a discriminant curve of any standard model for $\pi''\colon X'' \to S''$. Next, from the considerations connected with the nature of variation of the intermediate Jacobian upon the transition from this standard model to the original one, $\pi\colon X \to S$, we infer that $p_a(C) = p_a(\widetilde{C}'') \leq 10$.

Note that these considerations also work in the case $C'L' = 7$. Moreover, in this case and in the case $C'L' = 6$ the restriction of the link Φ to the general fiber F' coincides with the involution of Bertini and Geiser respectively. Any one of these involutions is continued up to the birational involution X' and induces a birational isomorphism of two structures of conic bundles, $\pi'\colon X' \to S'$ and $\pi''\colon X'' \to S''$. Then the composition with the map χ gives a new birational map $X \dashrightarrow \mathbb{P}^3$

with the coefficient in $-K_X$, which is smaller than μ, and this contradicts the choice of χ with the least μ.

Remark 3. In order to carry out the full proof of the criterion of rationality for standard conic bundles by this method, we shall obviously need a characterization of rational $\mathbb{Q}$-Fano varieties and Del Pezzo bundles as the model situation of surfaces over a perfect field shows.

3. MULTIDIMENSIONAL GENERALIZATIONS. KOLLÀR DEGENERATION METHOD

3.1. As was already pointed out, the first examples of nonrational varieties with trivially geometric-differential invariants in all dimensions $n \geq 3$ were given by Artin and Mumford [3]: these are varieties of the form $X \times \mathbb{P}^n$, where X is a smooth three-dimensional varieties $\operatorname{Tors} H^3(X, \mathbb{Z}) = 0$ which can even be chosen unirational and even in the form of a standard conic bundle over a rational surface (see also [68]).

In Saltman's work [53] the birational invariance of Brauer group was used to construct counterexamples to Noether's problem concerning the rationality of fields of invariants for infinite linear groups acting on vector spaces over an algebraically closed basic field. In terms of geometry, this problem can be formulated as follows. Let G be a finite group acting by linear transformations on the projective space $\mathbb{P}^n$. The question is whether the factor $X = \mathbb{P}^n/G$ is rational. It is clear that X is unirational.

In order to prove the nonrationality of some varieties X of this kind, it suffices to find a nonzero element from the Brauer group $\operatorname{Br}\big(k(X)\big)$ on the field of functions $k(X)$, i.e., the field of invariants $k(\mathbb{P}^n)^G$, which is "nonramified" in all discrete valuations of the field $k(X)$. It is remarkable that examples of these X do exist, although in a sufficiently large dimension.

Saltman's reasoning was simplified by Bogomolov [9] and Shafarevich [65]. In their interpretation a birational invariant is a nonramified group of Faddeev–Brauer $\Phi\operatorname{Br} k(X)$ (in the notation of [65]), which coincides with Schur's multiplicator subgroup $H^2(G, k^*) = H^2(G, \mu)$ (where μ is a subgroup of all roots of unity) of the group G consisting of elements which disintegrate upon a restriction imposed on all commutative subgroups $H \subset G$. Thus the birational invariant is completely defined by the group G itself and does not depend on the choice of an exact linear representation. Noncommutative p-groups of the period p provide the simplest examples of the group G with the nontrivial subgroup $\Phi\operatorname{Br} k(X) \subset H^2(G, \mu)$ (see [65]).

3.2. For the first time the nonramified cohomologies of degree 3 were used by Colliot–Thelene and Ojanguren [12] for constructing new examples of unirational but nonrational varieties even with a trivial Brauer group. Their arguments were developed and generalized in [51] for constructing a new series of examples of nonrational unirational function fields with the use of nonramified cohomologies of degrees 3 and 4 as birational invariants.

Following [51], we shall briefly describe the method of constructing these examples. Let k be an algebraically closed field of characteristic zero and let $K \supset k$ be a function field over k (i.e., it is finitely generated over k). Recall that two functional fields L and M over k are said to be *stably isomorphic* if there exist independent variables $x_1, \ldots, x_m$, $y_1, \ldots, y_m$ such that the fields of rational functions $L(x_1, \ldots, x_l)$ and $M(y_1, \ldots, y_m)$ are isomorphic over k. The functional field L is stably rational if it is stably isomorphic to the basic field k.

Let $N(k)$ be a set of all subrings of the discrete valuation of rank 1 $A \subset K$, $A \supset k$, with the quotient field K and let us denote by k_A the residue field A. Suppose that μ_n denotes, as usual, a group of roots of degree n of 1. Then, for the Galois group of cohomologies (or, what is the same, for etale cohomologies of fields) the map of the residue is defined for any $A \in N(k)$:

$$\partial_A \colon H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(k_A, \mu_n^{\otimes j-1}), \qquad i \geq 1, \quad j \in \mathbb{Z}.$$

The Brauer group $\mathrm{Br}(K) = H^2(K, \overline{K}^*)$ is periodic and its subgroup of n-torsions is isomorphic to $H^2(K, \mu_n)$, where $\overline{K}^*$ is an algebraic closure of K. The residue map for the Brauer group will also be denoted by ∂_A:

$$\partial_A \colon \mathrm{Br}(K) \to H^1(k_A, \mathbb{Q}/\mathbb{Z}).$$

Definition. *Subgroups of nonramified cohomologies* are defined as follows:

$$H^i_{nr}(K, \mu_n^{\otimes j}) = \bigcap_{A \in N(K)} \mathrm{Ker}\big(H^i(K, \mu_n^{\otimes j}) \xrightarrow{\partial_A} H^{i-1}(k_A, \mu_n^{\otimes j-1})\big).$$

Similarly, for *nonramified Brauer groups we have*

$$\mathrm{Br}_{nr}(K) = \bigcap_{A \in N(K)} \mathrm{Ker}\big(\mathrm{Br}(K) \xrightarrow{\partial_A} H^1(k_A, \mathbb{Q}/\mathbb{Z})\big).$$

Note the following fact from [12] (see also [9]): if the function fields K and L over k are stably isomorphic, then

$$H^i_{nr}(K, \mu_n^{\otimes j}) \xrightarrow{\sim} H^i_{nr}(L, \mu_n^{\otimes j}),$$

in particular, $H^i_{nr}(K, \mu_n^{\otimes j}) = 0$ for stably rational fields K.

Similarly, the Artin–Mumford invariant [3], which is a nonramified Brauer group $\mathrm{Br}_{nr}(K)$, depends only on the class of stably isomorphic functional fields and its subgroups of n-torsions $\mathrm{Br}_{nr}(K)_{(n)}$ can be calculated as $H^2_{nr}(K, \mu_n)$.

Since k is algebraically closed, i.e., $\operatorname{char} k = 0$, it follows that $\mu_n \subset k^* \subset K^*$ for every n, and, according to Kummer's theory, $K^*/K^{*n} \simeq H^1(K, \mu_n)$. We choose a prime number $p \neq 2$, a finite-dimensional vector space U over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and an embedding $\varphi^1 \colon U^\vee \to H^1(K, \mu_p)$, where $U^\vee$ denotes a space, which is dual to U. According to Kummer's theory, we can regard U as a quotient Galois group $\mathrm{Gal}(\overline{K}/K)$.

We set

$$H(K)_n = \sum_{i \geq 0} H^i(K, \mu_n^{\otimes i}),$$

and then U is a multiplication which turns $H(K)_n$ into an anticommutative ring. The embedding $\varphi^1 \colon U^\vee \to H^1(K, \mu_p)$ induces the homomorphism

$$\varphi^i \colon (\wedge^i U)^\vee \to H^i(K, \mu_p^{\otimes i}).$$

Let $S^i = (\operatorname{Ker} \varphi^i)^\perp \subset \wedge^i U$, then we have an embedding

$$\widehat{\varphi}^i \colon \mathrm{Hom}(S^i, \mathbb{F}_p) \hookrightarrow H^i(K, \mu_p^{\otimes i}).$$

We denote by $S^i_{\mathrm{dec}} \subset S^i$ a subgroup of decomposable elements of the form $u \wedge v \in S^i$, $u \in U$, $v \in \wedge^{i-1} U$.

Theorem 11 (see [51]. *In the previous notation let* $f \in \mathrm{Hom}(S^i, \mathbb{F}_p)$ *be an element such that* $f|S^i_{\mathrm{dec}} = 0$, *and then* $\widehat{\varphi}^i(f) \in H^i_{nr}(K, \mu_p^{\otimes i})$. *If* $S^i_{\mathrm{dec}} \neq S^i$, *then* $H^i_{nr}(K, \mu_p^{\otimes i}) \neq 0$ *and the field* K *is* (*stably*) *nonrational.*

This theorem allows us to construct examples of unirational functional field K over k with nontrivial $H^i_{nr}(K, \mu_p^{\otimes i})$ for $i = 2, 3, 4$ and even fields such that $H^3_{nr}(K, \mu_p^{\otimes 3}) \neq 0$ and $\mathrm{Br}_{nr}(K) = 0$.

Let $F = k(X_1, \dots, X_n) \subset k(T_1, \dots, T_n) = L$ be a subfield, where $X_i = T_i^p$, $i = 1, \dots, n$, and let U be the $\mathbb{F}_p$-vector space of dimension n with the basis $u_1, \dots, u_n$ so that $U = \mathrm{Gal}(L/F)$ and there exists an embedding $u_i^\vee\ U^\vee \hookrightarrow H^1(F, \mu_p)$, which puts any class $u_i^\vee\ \chi_i$ in $F^*/F^{*p} \simeq H^1(F, \mu_p)$ into correspondence with the class $U^\vee \hookrightarrow H^1(F, \mu_p)$. Then the morphism $(\wedge^i U)^\vee \to H^i(F, \mu_p^{\otimes i})$ is also an embedding.

In order to construct the required examples, a certain function field $K \supset F$ is chosen in [51] . The composite mapping $\varphi^1_K \colon U^\vee \to H^1(F, \mu_p) \to H^1(K, \mu_p)$ induces the morphism $\varphi^i_K \colon (\wedge^i U)^\vee \to H^i(K, \mu_p^{\otimes i})$. The field $K \supset F$ must be chosen such that the following requirements should be satisfied:

(a) K is unirational over k (recall that we seek examples of nonrational unirational function fields),

(b) for a certain subspace $S \subset \wedge^i U$ the kernel of the map $\varphi^i_K \colon (\wedge^i U)^\vee \to H^i(K, \mu_p^{\otimes i})$ coincides with $S^\perp$,

(c) $S \neq S_{\mathrm{dec}}$.

Now it follows, according to Theorem 11, that K is stably nonrational.

We choose the field $K \supset F$ such that the kernel of the map $H^i(F, \mu_p^{\otimes i}) \to H^i(K, \mu_p^{\otimes i})$ lies in the image of the injection $(\wedge^i U)^\vee \to H^i(F, \mu_p^{\otimes i})$ and therefore defines the subspace $S^\perp \subset (\wedge^i U)^\vee$. Then, for $S = (S^\perp)^\perp$, condition (c) must also be satisfied.

3.3. Examples.

Examples of fields K with $H^2_{nr}(K, \mu_p^{\otimes 2}) \neq 0$. Let A be a central simple algebra over F, $[A]$ be its class in the Brauer group $\mathrm{Br}(F)$, and let Y_A be the corresponding Severi–Brauer variety. Suppose that $s_1, \dots, s_m$ is a collection of generating elements of the subspace $S^\perp \subset \wedge^2 U^\vee$ and $S_1, \dots, S_m$ are central simple algebras for the certain subspace $S \subset \wedge^2 U$ let $s_1, \dots, s_m$ and $s_1, \dots, s_m$ in $H^2(F, \mu_p)$. (Here we identify $H^i(F, \mu_p) \simeq H^i(F, \mu_p^{\otimes i})$ since $\mu_p \subset k \subset F$.)

We choose as K the field of rational functions $F(Y_{S_1} \times \cdots \times Y_{S_m})$ on the product of the corresponding Severi–Brauer variety. According to Amitzur's theory,

$$\mathrm{Ker}\big(H^2(F, \mu_p) \to F(Y_{S_1} \times \cdots \times Y_{S_m})\big) = \langle [S_1], \dots, [S_m] \rangle,$$

i.e., $\mathrm{Ker}\, \varphi^i_K$ coincides with $S^\perp$.

Let us verify that the field K is unirational over k. By construction, the images of the elements $s_1, \dots, s_m$ in the group $H^2(F, \mu_p^{\otimes 2})$ come from $H^2(\mathrm{Gal}(L/F), \mu_p^{\otimes 2})$ and, consequently, are trivialized over $L = k(T_1, \dots, T_n)$, i.e., the corresponding Severi–Brauer varieties split up over L. Hence, the composite of the fields LK is a field of rational functions over L, and, hence, over k as well. This means that conditions (a) and (b) are fulfilled.

It remains to find subspaces $S \subset \wedge^2 U$ such that $S \neq S_{\rm dec}$. The first case where it becomes possible is $n = 4$. In this case, the subspaces $S \subset \wedge^2 U$ with $S \neq S_{\rm dec}$ were described by Bogomolov in [9] for $p \neq 2$: in this case, elements of the form $u \wedge v$, $u, v \in U$ are isomorphic vectors of the form

$$q \ \wedge^2 U \to \wedge^4 U, \qquad u \mapsto u \wedge v,$$

and $S \neq S_{\rm dec}$ if and only if $S = \operatorname{Ker}(q|S) \oplus T$, where $T \neq 0$ and $q|T$ is anisotropic.

Let $\alpha = \dim S$, $\beta = \dim S_{\rm dec}$, and then only the following cases are possible:

$$\alpha = 1, \quad \beta = 0; \qquad \alpha = 2, \quad \beta = 0; \qquad \alpha = 2, \quad \beta = 1; \qquad \alpha = 3, \quad \beta = 1; \qquad \alpha = 3, \quad \beta = 2.$$

The case $\alpha = 1$, $\beta = 0$ was considered by Saltman [53]. The following specific example of realization of the case $\alpha = 3$, $\beta = 2$ is given in [51]:

$$S = \langle u_1 \wedge u_2,\ u_1 \wedge u_4,\ u_1 \wedge u_3 + u_2 \wedge u_4 \rangle,$$

here $q|S$ is represented by the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$,

$$S^\perp = \langle u_3^\vee \wedge u_4^\vee,\ u_2^\vee \wedge u_3^\vee,\ u_1^\vee \wedge u_3^\vee - u_2^\vee \wedge u_4^\vee \rangle,$$

where $u_i^\vee$ is a basis dual to u_i, and

$$K = F\big(Y_{A_\xi(x_3,x_4)} \times Y_{A_\xi(x_2,x_3)} \times Y_{A_\xi(x_1,x_3)\times(x_4,x_2)}\big).$$

Here $A_\xi(a,b)$ is a cyclic central simple algebra over F generated by two elements i and j with the relations

$$i^p = a, \qquad j^p = b, \qquad ij = \xi ij,$$

where $\xi \in \mu_p$ is a primitive root of degree p of 1.

Examples of fields K with $H^3_{nr}(K, \mu_p^{\otimes 3}) \neq 0$, $\mathrm{Br}_{nr}(K) = 0$. Let M be a field of characteristic $\neq p$ which contains pth degree roots of unity. For the cyclic central simple algebra $A_\xi(a,b)$ and any element $c \in M^*$ let Z_A denote the normed variety $N_{rd}(x) = c$, where N_{rd} is a reduced norm in the algebra A.

In order to construct the required examples, use is made in [51] of the following result of Suslin:

$$\operatorname{Ker}\big(H^3(M, \mu_p^{\otimes 2}) \to H^3(M(Z_{A,C}), \mu_p^{\otimes 2})\big) = \langle [A] \cup C \rangle,$$

where C is regarded as an element $H^1(M, \mu_p) \simeq M^*/M^{*p}$.

Let $S \subset \wedge^3 U$ be a certain subspace and suppose that the following conditions are fulfilled:

(H1) we can choose a basis $(s_1, \dots, s_m)$ in $S^\perp$ such that $s_j = v_j \wedge w_j \wedge y_j$, $v_j, w_j, y_j \in U^\vee$, $j = 1, \dots, m$,

(H2) for every $j \in \{1, \dots, m\}$ and any $i \in \{1, \dots, n\}$ no more than one of the elements v_j, w_j, y_j has a nonzero value on u_i.

Let V_j, W_j, Y_j be images of the elements v_j, w_j, y_j in $F^*/F^{*p} = H^1(F, \mu_p)$, $Z_j = Z_{A_\xi}(V_j, W_j), Y_j$. We shall define the required field K as a field of rational functions $F(Z_1 \times \cdots \times Z_m)$. Then the following result is proved in [51]:

> the field K is unirational and $\mathrm{Br}_{nr}(K) = 0$, and, if $S \neq S_{\mathrm{dec}}$, then $H^3_{nr}(K, \mu_p^{\otimes 3}) \neq 0$, and consequently, K is stably nonrational over k.

Here condition (H1) is used in order to employ Suslin's result and condition (H2) is used in order to prove the statement $\mathrm{Br}_{nr}(K) = 0$.

Here is a simple example of $S \neq S_{\mathrm{dec}}$ with $n = 6$ (see [51]):

$$S = \langle u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6 \rangle,$$

$$\begin{aligned} S^\perp = \langle &u_j^\vee \wedge u_l^\vee \wedge u_m^\vee \text{ for } 1 \leq j < l < m \leq 6 \text{ and } (j,l,m) \notin \{(1,2,3),(4,5,6)\},\\ &u_1^\vee \wedge u_2^\vee \wedge u_3^\vee - u_4^\vee \wedge u_5^\vee \wedge u_6^\vee \rangle \\ = \langle &\dots, (u_1^\vee - u_4^\vee) \wedge (u_2^\vee + u_5^\vee) \wedge (u_3^\vee + u_6^\vee) \rangle. \end{aligned}$$

Here conditions (H1) and (H2) are obviously satisfied. For other examples of this kind see [51].

Examples of fields K with $H^4_{nr}(K, \mu_2^{\otimes 4}) \neq 0$. In this case $p = 2$ and the following result of Jacob and Rost concerning quadratic forms is used. Recall that a quadratic form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ is called n-fold Pfistor form $\langle\!\langle a_1, \dots, a_n \rangle\!\rangle$.

Let M be a field of characteristic 2, and let Φ be a 4-fold Pfistor form $\langle\!\langle a_1, \dots, a_n \rangle\!\rangle$ over M, and M be field of rational functions on the quadric $\Phi = 0$. Then

$$\mathrm{Ker}\big(H^4(M, \mathbb{Z}/2\mathbb{Z}) \to H^4(M(\Phi), \mathbb{Z}/2\mathbb{Z})\big) = \{0, (a_1) \cup \cdots \cup (a_4)\}.$$

For the subspace $S \subset \wedge^4 U$ we shall require the fulfillment of the following condition imposed on the basis $(s_1, \dots, s_m)$ of the space $S^\perp$:

(H1) every element s_j is represented as $u_{1,j} \wedge u_{2,j} \wedge u_{3,j} \wedge u_{4,j}$, $u_{i,j} \in U^\vee$, $1 \leq i \leq 4$, $1 \leq j \leq m$.

Let $U_{i,j}$ be images of $u_{i,j}$ in F^*/F^{*2} and $\Phi_j = \langle\!\langle U_{1,j}, \dots, U_{4,j} \rangle\!\rangle$. We define the field K as a functional field $F(\Phi_1)(\Phi_2)\cdots(\Phi_m)$, and then the following result is valid [51]:

> the field K is unirational over k and, if $S \neq S_{\mathrm{dec}}$, then $H^4_{nr}(K, \mu_2^{\otimes 4}) \neq 0$, and, consequently, K is stably nonrational over k.

As is stated in [51], in this case we can also prove that $\mathrm{Br}_{nr}(K) = 0$.

3.4. Let us now consider some generalizations of the method of factorization of Noether–Fano birational maps to varieties of higher dimensions $n \geq 4$. The first generalization of this kind was made by Sarkisov [55] who extended Theorem 9 on the mappings of conic bundles to any dimension. Let $\pi \colon X \to S$ be a standard conic bundle over the rational variety S of dimension $\dim S \geq 3$ with the degeneration divisor $C \subset S$ ("standard" means that X and S are nonsingular and projective, the morphism π is plane, and $\rho(X/S) = 1$). Then C is a reduced divisor with normal intersections and, moreover, elementary transformations may give a standard model where the degeneration divisor will have only simple normal intersection, i.e., all of its irreducible components will be nonsingular and will intersect transversally. The main result is

Theorem 12 [55]. *Let $\pi\colon X \to S \supset C$ be a standard conic bundle and $\chi\colon X \dashrightarrow X'$ be a birational mapping onto a nonsingular complete variety. We assume that there exists on X a free linear system H' such that $|H' + mK_{X'}| = \varnothing$, $m \geq 1$ (say, X' is represented as a bundle with the general fiber, which is a Fano variety). Assume that*

$$|4K_X + C| \neq \varnothing. \tag{11}$$

Then there exists on X' a structure of conic bundle $\pi'\colon X' \to S'$ and there is a birational map $\alpha\colon S \dashrightarrow S'$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \overset{\chi}{\dashrightarrow} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ S & \overset{\alpha}{\dashrightarrow} & S' \end{array},$$

in other words, under condition (11) *any birational mapping preserves the structure of the conic bundle.*

The proof of this theorem is, in fact, the same as in the dimension $\dim X = 3$. This was pointed out by Danilov when he reviewed the work [54]. He noted that the restriction to a three-dimensional case was unessential in it.

It is easy to indicate examples of unirational X of any dimension under condition (11), i.e., counterexamples to the Lüroth problem since it obviously follows from Theorem 12 that X is not rational under condition (11). Since any birational automorphism X preserves the structure of a conic bundle, it follows that for the group of birational automorphisms $\mathrm{Bir}(X)$ there holds an exact sequence

$$1 \longrightarrow \mathrm{Bir}(X_\eta) \longrightarrow \mathrm{Bir}(X) \longrightarrow G \longrightarrow 1,$$

where X_η is the general fiber of the morphism $\pi\colon X \to S$ over the residue field $k(\eta) \simeq k(S)$ of the general point $G \subset \mathrm{Bir}(S)$ and $\eta \in S$ is a certain subgroup, which is trivial in the general case.

Pukhlikov's works [46–48, 50] are devoted to the application of the Noether–Fano method to Fano varieties. In [50] he studied the case of a smooth four-dimensional quintic $X_5 \subset \mathbb{P}^5$ and in [47] he investigated double projective spaces and double quadrics which are Fano varieties with index 1. In addition, in [48] Pukhlikov studied the case of three-dimensional quartic in $\mathbb{P}^4$ with a single regular quadratic singularity. We shall sum up Pukhlikov's results as the following theorem.

Theorem 13. *Let X be a Fano variety of one of the following types:*

(i) *a smooth hypersurface of degree* 5 *in* $\mathbb{P}^5$,

(ii) $\varphi\colon X \to \mathbb{P}^m$ *is a double cover with ramification in the smooth divisor $D_{2m} \subset \mathbb{P}^m$ of degree $2m$, $m \geq 4$,*

(iii) $\psi\colon X \to Q \subset \mathbb{P}^{m+1}$ *is a double cover of the smooth quadric Q with a smooth divisor of ramification $D \subset Q$, which is cut out by a hypersurface of degree $2(m-1)$, $m \geq 4$,*

(iv) $X \to \mathbb{P}^4$ *is a hypersurface of degree* 4 *with a single nondegenerate quadratic singularity.*

Let X' be a nonsingular projective variety $\dim X' = \dim X$, *and let H' be a divisor on X' such that the linear system $|H'|$ is free. Then, if $|H' + nK_{X'}| = \varnothing$, $n \geq 1$, then there is no birational mapping X on X'; and if $|H' + nK_{X'}| = \varnothing$, $n \geq 2$, and $H' = -K_{X'}$ and* $\dim|-K_X| \leq \dim|H'|$, *then every birational map $\chi\colon X \dashrightarrow X'$ is an isomorphism in cases* (i)–(iii) *and a birational automorphism generated, with an accuracy to within the automorphism, by birational involutions, namely, by reflections relative to* 24 *lines passing through the singular points and by an involution of the reflection relative to the singular point (i.e., the involution of the double cover which originates when a projection is performed from this point). To be more precise, the group* $\mathrm{Bir}(X)$ *in case* (iv) *is a semi-direct product of the form*

$$1 \longrightarrow \mathop{*}_{i=0}^{24} \langle r_i \rangle \longrightarrow \mathrm{Bir}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1,$$

where "$*$" means a free product of the indicated birational involutions.

In cases (i)–(iii) $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

It immediately follows from this theorem that the variety X of types (i)–(iv) is not rational. Presumably, a further development of the method will make it possible to obtain a similar result for any smooth hypersurface $X_n \subset \mathbb{P}^n$ of degree $n \geq 6$.[1]

3.5. Kollar proposed in [33–35] a new remarkable method for proving nonrationality, even nonruledness, of general varieties of dimension $n \geq 3$ with trivial holomorphic tensors (of Fano varieties, Del Pezzo bundles, conic bundles). Here we shall briefly expose the idea of the method and formulate the results, referring the reader for details to the above-mentioned original works.

From now on the field k is assumed to be algebraically closed of an arbitrary characteristic. Recall that the irreducible reduced algebraic variety X of dimension n is ruled (uniruled, separably uniruled, resp.) if there exist an $(n-1)$-dimensional variety T and a rational map $u\colon \mathbb{P}^1 \times T \dashrightarrow X$ which is birational (finite at the general point, finite at the general point and separable, resp.).

Lemma 3 (see [35, V.5.1]). *Let X be a smooth proper variety of dimension n, and M be a big invertible bundle on X. Assume that for certain $i > 0$ there exists an embedding $M \to \wedge^i \Omega^1_X$ where Ω^1_X is a bundle of regular differential 1-form. Then X is separably nonruled. If $\operatorname{char} k = p > 0$ and X is uniruled, then p divides the degree of any surjective, finite at the general point, map $u\colon \mathbb{P}^1 \times T \dashrightarrow X$.*

Note (see [35], V.5.1.1) that if $\operatorname{char} k = 0$, then the conditions of the lemma can be satisfied only if $i = n = \dim X$ and X is a variety of the general type.

Kollar's degeneration method consists in the following. For the prime number p, for a smooth variety X over a field of characteristic p, and for an invertible bundle L on X and a nonzero section $s \in H^0(X, L^p)$ we construct a purely nonseparable cyclic p-cover $\pi\colon Y \to X$ with ramification in the divisor of zeros $(s = 0) \subset X$ of the section s. In the characteristic zero, Y is nonsingular if and only if the divisor of ramification is nonsingular and in the characteristic $p > 0$ Y always has singularities, which are connected with the critical points of the section s (in the characteristic $p > 0$ the concept of a critical point of the section $s \in H^0(X, L^p)$ has sense). For instance, the hypersurfaces $y^p - f(x_1, \ldots, x_n) = 0$ has singularities exactly when

$$\frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0,$$

[1] This result (with certain inessential restrictions) was recently obtained by Pukhlikov, see "Birational automorphisms of Fano hypersurfaces," Preprint 97-13, Max-Planck-Institut für Math., 1997.

i.e., at the critical points of the functions $f(x_1, \dots, x_n)$.

If we choose L and $s \in H^0(X, L^p)$ sufficiently general so that all critical points of s would be nondegenerate, then the singularities of Y will be sufficiently simple and easily solvable. Let $\sigma\colon Y' \to Y$ be a resolution. Kollar shows in [33, 35] (practically by means of direct calculations) that there exists an embedding $\sigma^*\pi^*(K_X \otimes L^p) \to \Omega_{Y'}^{n-1}$. It turns out that in many cases we can choose L such that the bundle $K_X \otimes L^p$ will be ample, and then, by Lemma 3, Y' will be separably nonruled, in particular, nonruled. This allows us to construct examples of nonruled Fano varieties of dimension $n \geq 3$ in any characteristic by lifting nonruled cyclic p-covers in characteristic zero and using the following general result concerning the deformation of ruled varieties.

Lemma 4 (see [35, V.5.12]). *Let $f\colon V \to S$ be a proper and flat morphism with irreducible and reduced fibers with irreducible S. Suppose that R is a ring of discrete valuation (not necessarily equally characteristic) and let $g\colon Z \to \operatorname{Spec} R$ be a proper and flat morphism with reduced fibers. Let Z_0 be an irreducible component of the closed fiber $g^{-1}(0)$ and Z_g be the general fiber. Assume that Z_g is birationally equivalent to a certain fiber of the morphism f. Then, if Z_0 is geometrically (i.e., over the closure of a residue field) nonruled, then $X_s = f^{-1}(s)$ is geometrically nonruled for the very general point $s \in S$, i.e., any point $s \in S$ outside of the countable union of closed subsets.*

The important examples of nonruled (and all the more so nonrational) Fano varieties are provided by cyclic covers of projective spaces and hypersurfaces of small degrees in projective spaces.

Theorem 14 (see [35, V.5.13 and V.5.14]). (i) *Let p be a prime number and $s \in H^0\big(\mathbb{P}^n, \mathcal{O}(pd)\big)$ be a very general section over the field $k = \mathbb{C}$. Then the cyclic p-cover $\pi\colon Y \to \mathbb{P}^n$ is a smooth Fano variety which is not ruled and, consequently, not rational for $n \geq 3$ and*

$$\frac{n+1}{p-1} > d > \frac{n+1}{p}.$$

(ii) *Let $X_d \subset \mathbb{P}^{n+1}$ be very general hypersurface of degree d. Then X_d is not ruled if $d \geq 2\lceil (n+3)/3 \rceil$, and birationally nonequivalent to a conic bundle if $d \geq 3\lceil (n+3)/4 \rceil$. Here $\lceil\cdot\rceil$ denotes the upper integer part of the number.*

Outline of the proof. (i) Let $D \subset \mathbb{P}^n$ be a divisor of zeros of the section s. Then $K_Y = \pi^*\big(K_{\mathbb{P}^n} + \frac{p-1}{p}D\big)$, and, consequently, $-K_Y$ is ample (i.e., Y is a Fano variety) if and only if $\frac{p-1}{p}pd < n+1$, i.e., $d < \frac{n+1}{p-1}$.

All cyclic covers of this kind can be parametrized by means of a projectivization of the space $H^0\big(\mathbb{P}^n, \mathcal{O}(pd)\big)$ any one of these covers in the characteristic p is lifted to the characteristic zero. According to Lemma 3, in the characteristic p a sufficiently general cover of this kind is not ruled if $-n-1+pd > 0$, i.e., $d > \frac{n+1}{p}$, and then, according to Lemma 4, its lifting to characteristic zero is not ruled either.

(ii) The first step here is the lifting of the cyclic cover in the characteristic p up to a hypersurface in characteristic zero. Let $T = \operatorname{Spec} R$, where R is a ring of the discrete valuation with a local parameter t, a quotient field K, and a field of residues k; $f, g \in R[x_0, \dots, x_r]$ are homogeneous varieties of degree cd and d, respectively, and $g^c - f$ is not identically zero in $k[x_0, \dots, x_r]$. In the weighted projective space $\mathbb{P}_T(x_0, \dots, x_r, y)$ with weights $(1, \dots, 1, d)$ over T we define the subscheme

$$Z = (y^c - f = 0,\ ty - g = 0) \subset \mathbb{P}_T(x_0, \dots, x_r, y),$$

which is a family of weighted complete intersections over T. The general fiber of this subscheme is isomorphic to the hypersurface $(g^c - t^c f = 0) \subset \mathbb{P}_K^{r+1}$, the closed fiber is isomorphic to the cyclic c-cover over the hypersurface $(g^c - t^c f = 0) \subset \mathbb{P}_K^{r+1}$, the cover being ramified along $(g(c) = 0) \subset \mathbb{P}_k^{r+1}$, $(f(s) = 0)$.

As a consequence of this construction we obtain the following special case of the first assertion from (ii).

(ii′) *Let $X_{pe} \subset \mathbb{P}^{n+1}$ be a very general hypersurface such that $(p+1)e \geq n+3$. Then it is not ruled.*

Proof. Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree e over the field of characteristic p and $s \in H^0\big(Y, \mathcal{O}_Y(pe)\big)$ be a general section. We denote by Z_0 the cyclic p-cover of Y connected with the section s. Then $K_Y \otimes L^p \simeq \mathcal{O}_Y(e - 2 - n + pe)$ is ample if and only if $(p+1) \geq n+3$. By the preceding construction we can lift Z_0 to the hypersurface of degree pe in $\mathbb{P}^{n+1}$ which is nonruled according to the lemma.

Let us continue with the proof of statement (ii). Let X_d be a hypersurface of degree d. If, for the fixed prime number p, we have $d = pe$ and $(p+1)e \geq n+3$, then statement (ii′) works. In the general case, we must use additional degenerations in order to reduce the proof to this case.

For $d' < d$ we shall consider the family of hypersurfaces whose general fiber is a smooth hypersurface of degree d and the special fiber is the union of hypersurfaces X' of degree d' and hyperplanes $d - d'$. If X' is not ruled, then, according to Lemma 4, X is not ruled either, and we obtain the following statement.

(ii″) *Let $X_d \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree d and*

$$d \geq \left\lceil \frac{n+3}{p+1} \right\rceil. \tag{12}$$

Then X_d is not ruled.

Proof. We use the previous technique of degeneration with $d' = pe$ and $e = \lfloor d/p \rfloor$, where $\lfloor \cdot \rfloor$ denotes the lower integer part of the number a. Then inequality (12) is equivalent to the inequality $(p+1)\lfloor d/p \rfloor$ and (ii″) follows from statement (ii′).

Now, in order to obtain the first of assertions (ii), we must use (ii″) with $p = 2$.

The second of assertions (ii) can also be derived from (ii″) as follows. Assume that X_d is birationally equivalent to the conic bundle $X_1 \to Y_1$ with base Y_1. Then, over a certain double covering of the base $Y_\eta \to Y_1$, the conic bundle $X_1 \times_{Y_1} Y_\eta \to Y_\eta$ possesses a rational section and, consequently, is birationally equivalent to $Y_\eta \times \mathbb{P}^1$. Hence, the composite map

$$Y_\eta \times \mathbb{P}^1 \dashrightarrow X_1 \times_{Y_1} Y_\eta \dashrightarrow X_1 \dashrightarrow X$$

defines the ruledness of X of degree 2 so that we cannot take $p = 2$ in (ii″). However, in that case it suffices to take $p = 3$, in order to obtain the required statement.

3.6. The same method can be used for hypersurfaces $X_{c,d} \subset \mathbb{P}^m \times \mathbb{P}^{n+1}$ of bidegree (c, d). The direct application of this method shows that the very general $X_{c,d}$ is not ruled for $c \geq m+3$ and $d \geq \lceil \frac{n+3}{3} \rceil$. A certain refinement of this method, given in [34], allows us to study the problem of rationality for $X_{c,d}$ in the case where the fibers of the projection $X_{c,d} \to \mathbb{P}^m$ are rational. In [33] Kollar considers the cases where fibers are conics or cubic surfaces. To be more precise, everything begins, as usual, with the study of cyclic p-covers in the characteristic p.

Let p be a fixed prime number and $X_{ap,bp} \to \mathbb{P}^m \times \mathbb{P}^n$ be a cyclic p-cover ramified along the very general hypersurface of bidegree (ap, bp). Then, just as in the theorem, $X_{ap,bp}$ is not rational and even not ruled if $ap > m+1$ and $bp > n+1$. A situation is investigated in [33] when $bp = n+1$ and $n = 1$ and 2, i.e., $X_{2a,2} \to \mathbb{P}^m$ is a conic bundle and $X_{3a,3} \to \mathbb{P}^m$ is a fibering into cubic surfaces. Let us formulate the main results.

Theorem 15 (see [33]).

(i) *$X_{2a,2}$ is not ruled if $2a > m+1$,*

(ii) *$X_{3a,3}$ is not ruled if $3a > m+1$,*

(iii) *let $X_{c,2} \subset \mathbb{P}^m \times \mathbb{P}^2$ be a very general hypersurface of bidegree $(c, 2)$, $m \geq 2$; then it is not ruled if $c \geq m+3$,*

(iv) *let $X_{c,3} \subset \mathbb{P}^m \times \mathbb{P}^n$ be a very general hypersurface of bidegree $(c, 3)$, $m > 1$; then it is not ruled if $c \geq m+4$.*

Statement (i) implies nonrationality of the general conic bundle $X_{2a,2} \to \mathbb{P}^m$ with the degeneration divisor $C \subset \mathbb{P}^m$ of degree $4a > 2m+2$. This is somewhat better than Sarkisov's result (Theorem 12) with the estimate of the degree of the degeneration divisor $4a > 4m+4$, although of course, Sarkisov's theorem is valid for any standard conic bundles and not only for the general ones, and over an arbitrary rational base and not only over $\mathbb{P}^n$.

For fiberings into cubic surfaces $X_{3a,3} \to \mathbb{P}^m$ a result similar to (ii) for $m = 1$ was obtained earlier by Bardelli [4] by the method of degeneration of the intermediate Jacobian.

The method of proving Theorem 15 is similar to the method of proving Theorem 14. We first consider the case of cyclic p-covers in the characteristic $p = 2$ or 3 $X_{ap,bp} \to \mathbb{P}^m \times \mathbb{P}^n$ ramified along the very general divisor of bidegree (ap, bp) with $ap > m+1$ and $bp = n+1$, $n = 1$ or 2. The transition from cyclic covers to hypersurfaces can be carried out in a standard manner.

The difference from the preceding cases (Theorem 14) is that the varieties under consideration, namely, conic bundles and families of cubic surfaces, are originally ruled. However, it turns out that under the condition $ap > m+1$ they are ruled if and only if their general fibers (i.e., conics in characteristic 2 and cubic surfaces in characteristic 3) are ruled over functional fields. It only remains to study all possible techniques of defining uniruledness of conics and cubic surfaces over nonclosed imperfect fields (the case of perfect fields was thoroughly studied in classical literature, see, e.g., [37] for cubic surfaces).

The main technique of proving the theorem is the following lemma which generalizes Lemma 3.

Lemma 5 (see [33], 3.1). (i) *Let $f\colon Y \to X$ be a proper surjective morphism of smooth varieties, $\dim Y = n$, and M be a big invertible bundle on X. Assume that for a certain $i > 0$ there exists a nonzero mapping $h\colon f^*M \to \wedge^i \Omega^1_Y$. Let Z be an affine variety of dimension $n-1$ and $\varphi\colon Z\mathbb{P}^1 \to Y$ be a dominant and separable morphism. Then there exists a morphism $\psi\colon Z \to X$ such that the diagram*

$$\begin{array}{ccc} Z \times \mathbb{P}^1 & \xrightarrow{\varphi} & Y \\ \big\downarrow & & \big\downarrow f \\ Z & \xrightarrow{\psi} & X \end{array}$$

is commutative.

(ii) *Let* $F = k(X)$ *be a field of rational functions on* X *and* Y_F *be the general fiber of the morphism* f. *Then there exists a one-to-one correspondence between the following sets:*

$$\left\{\begin{matrix}\textit{degree } d \textit{ of the separable} \\ \textit{uniruledness of } Y\end{matrix}\right\} \leftrightarrows \left\{\begin{matrix}\textit{degree } d \textit{ of the separable uniruledness} \\ \textit{of the general fiber } Y_F \textit{ over } F\end{matrix}\right\}.$$

In particular, Y *is ruled if and only if* Y_F *is ruled over* F.

The following lemma ensures the existence of the required nonzero map under our conditions.

Lemma 6 (see [33], 2.5). *Let* k *be an algebraic closed field of characteristic* $p > 0$, $s \in H^0\bigl(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(a,b)^{\otimes p}\bigr)$ *be the general section, and* $q\colon Y \to \mathbb{P}^m \times \mathbb{P}^n$ *be a smooth projective model of the cyclic* p*-cover* $\mathbb{P}^m \times \mathbb{P}^n$ *with ramification in the divisor of zeros* $(s = 0)$. *Then there exists a nonzero map*

$$q^*\mathcal{O}(ap - m - 1, bp - n - 1) \to \wedge^{m+n-1}\Omega^1_Y \simeq \Omega^{m+n-1}_Y.$$

Finally, the nonruledness of the general conic and the cubic surface over a functional field is established by the following lemma.

Lemma 7. (i) *Let* k *be a field of characteristic* 2 *and* $F = k(x_1, \dots, x_m)$ *be a field of rational functions over* k *in* $m \geq 2$ *variables. Suppose that* d *is a fixed even number and* $a, b, c \in k[x_1, \dots, x_m]$ *are general varieties of degree* d. *Then the conic*

$$C = (y_0^2 = a y_1^2 + b y_1 y_2 + c y_2^2) \subset \mathbb{P}^2_F$$

is not ruled over F. *Moreover,* C *does not possess any uniruledness of an odd degree* (see [33], 4.2).

(ii) *Let* S *be a nonsingular Del Pezzo surface of degree* $K_S^2 = 1, 2$ *or* 3 *over a certain field* F *with* $\operatorname{Pic}_F S \simeq \mathbb{Z}(-K_S)$. *Assume that* S *is smooth at the general point; then it is not ruled and, consequently, not rational over* F (see [33], 5.6).

These statements are well known in the case of a perfect field F, and therefore only the case of an imperfect field F is considered in [33] which is precisely the case that is used to prove the main results.

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