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# Optimization Problem for the Brunn–Minkowski Inequality

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Many of the well-known classical inequalities are related to one or several optimization problems. For instance, the mean arithmetical inequality and the mean geometric inequality correspond to the problem on a rectangle of maximal area with a given perimeter. The solution of the classical isoperimetric problem on a plane domain with a minimal length of boundary and a given area means that the ratio of the square of the boundary length of the domain and the area of this domain is no less than $4\pi$.

In this note, a problem of this kind is proposed for the Brunn–Minkowski inequality

$$\sqrt[n]{V(X+K)} \geq \sqrt[n]{V(X)} + \sqrt[n]{V(K)}, \tag{1}$$

which is known for its applications. Here and in what follows, $X$ and $K$ denote convex closed bounded domains in the $n$-dimensional Euclidean space $E^n$, $V(\cdot)$ is a function of an $n$-dimensional volume, $V(X) > 0$, $V(K) > 0$;

$$X + K = \bigcup_{x \in X} (x + K),$$

where $x + K$ is the translation of the domain $K$ by the vector $x$. An elementary proof of this inequality is given in [2]. We draw the attention to the fact that the equality in (1) takes place if and only if $X$ and $K$ are homothetic with a positive coefficient of homothety.

The question concerning the form of the domain $X$ of a maximal volume with the given so-called *extrinsic Minkowski area*

$$\mu = \mu_+(X, K) = \underline{\lim}_{\alpha \searrow 0+} \frac{V(X + \alpha K) - V(X)}{\alpha} \tag{2}$$

is a generalization of the classical isoperimetric problem for the given domain $K$ (see [1, pp. 88–89]). We have $X = \beta K$ as the required domain, where

$$\beta = \left( \frac{\mu}{nV(K)} \right)^{1/(n-1)}.$$

This follows from the inequality

$$\mu_+(X, K) \geq nV^{(n-1)/n}(X) V^{1/n}(K), \tag{3}$$

which, in turn, is a consequence of (1). Just as in (1), the equality in (3) takes place if and only if $X$ is homothetic to $K$ with a positive coefficient of homothety.

In the sequel, instead of the extrinsic Minkowski area, we shall consider the volume of the set

$$\Gamma_K(X) = \bigcup_{\substack{v\in E^n:(v+K)\cap X=\varnothing \\ V((v+K)\cap X)=0}} (v+K) = \bigcup_{\substack{v\in E^n \\ \varnothing\neq(v+K)\cap X\subset\partial(v+K)\cap\partial X}} (v+K),$$

which is called a *$K$-boundary of the domain $X$*. Here $\partial$ denotes the boundary of the corresponding domain. From the point of view of the natural interpretation, $V\big(\Gamma_K(X)\big)$ is more advantageous than $\mu_+(X,K)$. For instance, for $n=2$, the concept of the $K$-boundary simulates the trace of a peace of chalk when a hand moves in accordance with the rule of calligraphy, the chalk performing a translational movement along the plane of the board. Pay attention to the fact that $\mu_+(X,K)$ in (2) has a physical dimension $n$, and not $n-1$, the former dimension being more natural for it; therefore we should have introduced an additional factor of dimension, e.g., $V^{-1/n}(K)$, dependent on $K$.

The question concerning the form of the domain $X$ of the maximal volume and with a given volume of its $K$-boundary is here an analog of the classical isoperimetric problem. We assume that the domain $K$ is given. In the classical case, $K$ degenerates into a point, and the $(n-1)$-dimensional volume of the boundary $\partial X$ is a priori known.

**Theorem.** *For the given $K$ with $V(K)>0$ and $V=V(X)>0$, the minimum of $V\big(\Gamma_K(X)\big)$ is attained only on the domain $X=X_0$ which is homothetic to the $K$-boundary of the point with a positive coefficient of homothety.*

We shall denote by $\widetilde{K}$ the $K$-boundary of the origin $\overline{0}\in E^n$.

It is a matter of routine to show that

$$\begin{aligned}\widetilde{K} &= \bigcup_{\substack{v\in E^n: \\ \overline{0}\in\partial(v+K)}} (v+K) = \bigcup_{\substack{v\in E^n: \\ \overline{0}\in(v+K)}} (v+K) \\ &= \{v+k \mid -v\in K,\ k\in K\} = \{v+k \mid v\in -K,\ k\in K\} = (-K)+K.\end{aligned}$$

Here we denote by $(-K)$ the domain which is centrally symmetric to $K$. The representation of the $K$-boundary of the point as the sum $\widetilde{K}=(-K)+K$ means, in particular, that $\widetilde{K}$ is also a closed convex bounded domain [3]. The domain $\widetilde{K}$ is centrally symmetric with respect to the origin. This seems to be the only property that $X_0$ inherits from a ball, which is an answer to the classical isoperimetric problem.

In order to prove the theorem, we have to verify the equality

$$X\cup\Gamma_K(X) = X+\widetilde{K}. \tag{4}$$

Indeed, we have from inequality (1) that

$$\sqrt[n]{V(X)+V(\Gamma_K(X))} = \sqrt[n]{V(X\cup\Gamma_K(X))} = \sqrt[n]{V(X+\widetilde{K})} \geq \sqrt[n]{V(X)} + \sqrt[n]{V(\widetilde{K})};$$

moreover, the equality here holds if and only if $X$ and $\widetilde{K}$ are homothetic with a positive coefficient of homothety. Rewriting the last inequality in the form

$$V\big(\Gamma_K(x)\big) \geq \Big(\sqrt[n]{V(X)} + \sqrt[n]{V(\widetilde{K})}\Big)^n - V(X),$$

we obtain the assertion which is formulated in the theorem.

Let us prove relation (4). Since $\overline{0} \in \widetilde{K}$, we have $X \subset X + \widetilde{K}$. If $z \in X \cap (v+K) \subset \partial X \cap \partial(v+K)$, then $v + K \subset z + \widetilde{K}$, $z \in X$, and therefore $\Gamma_K(X) \subset X + \widetilde{K}$. Thus, $X \cup \Gamma_K(X) \subset X + \widetilde{K}$. Conversely, assume that $y \in X + \widetilde{K}$ and $y \notin X$. Let us show that $y \in \Gamma_K(X)$. There exists $v_0 \in E^n$ such that $y \in v_0 + K = K_0$ and $K_0 \cap X \neq \varnothing$. If $V(K_0 \cap X_0) = 0$, then $K_0 \subset \Gamma_K(X)$ and $y \in \Gamma_K(X)$. Let $V(K_0 \cap X) > 0$. Then we denote by $l$ the strictly separating hyperplane between the point $y$ and the closed convex bounded set $X$. Let $v_1 \in E^n$ be such that $y \in v_1 + K = K_1$, and the hyperplane $l$ strictly separates $K_1$ and $X$. Consider the family of sets $K_t = (1-t)v_0 + tv_1 + K$, $0 \leq t \leq 1$. Since $K$ is convex, we have $y \in K_t$ for all $t \in [0,1]$. From the continuity considerations there exists $t_0 \in (0,1)$ such that $K_{t_0}$ intersects $X$ only along the boundary points. For example, let $t_0 = \min\{t \in [0,1] \mid V(K_t \cap X) = 0\}$. Then $K_{t_0} \subset \Gamma_K(X)$, and therefore $y \in \Gamma_K(X)$. Relation (4) is proved.

From the formal point of view, this theorem does not solve an analog of the classical isoperimetric problem which was formulated before this theorem. However, assuming the contrary and using this theorem, we see that for the given $V\big(\Gamma_K(X)\big)$, the maximum of $V(X)$ is attained only for the domain $X = X_0$ which is homothetic to $\widetilde{K}$ with the positive coefficient of homothety $\alpha$. The latter can be found from the equation

$$(\alpha+1)^n - \alpha^n = \frac{V(\Gamma_K(x))}{V(\widetilde{K})}, \qquad \alpha > 0.$$

In conclusion, we use the opportunity to draw attention to the other concept of the $K$-boundary for $X$, which is more natural and which can be represented in the form $\widehat{X} \setminus X$, where $\widehat{X}$ is a closed convex bounded domain of the minimal volume such that $X$ is contained among its interior points and the closure of $\widetilde{X} \setminus X$ is the union of domains resulting from $K$ upon the proper motions of $E^n$. In this case, the images of a certain interior point of $K$ must form a noncontractible set in $E^n \setminus X$. When the answer to the question on the existence and uniqueness of the $K$-boundary in this sense is positive, the solution to the optimization problem from the presented theorem essentially depends on the value of $V = V(X)$. For instance, as $V \to \infty$, we should expect that $X_0$ will tend to take the shape of a ball. Locally, from the point of view of $K$, the boundary concentric spheres for $\widehat{X}_0$ and $X_0$ can be represented as parallel planes with a minimal possible distance between them which is admitted by the dimensions of $K$. However, in the general case, a ball is obviously not an answer. For example, if $n = 2$ and $K$ is a rectangle with sides $\alpha$ and $1/\alpha$, and if $V = 1$, then, according to the theorem which we have already proved, there exists an $X$ with $V(X) = 1$ and $V(\widehat{X} \setminus X) \leq 8$ irrespective of $\alpha$. At the same time, for the disk $Y$ of unit area $V(\widehat{Y} \setminus Y)$ will be arbitrary large with an increase in $\alpha$, larger than the area of a triangle with the base $\alpha$ and the height equal to the radius of the disk $Y$. The type of the domains for $\widehat{X}$ which is presented by the theorem is not a unique alternative to a ball. As a justification, consider $n = 3$, and the rectangular parallelepiped with edges $1/\alpha$, 1, and $\alpha$ as $K$. In this case, the author could not find for $\alpha = 1$ and $\alpha = 5$ a better candidate for $\widehat{K}$ than a ball and $\widetilde{K}$, respectively, in order to define the $K$-boundary of the point $\overline{0} \in E^3$. But to make up for it, for $\alpha = 2$, he found even two different convex domains of the same volume which is smaller than that of the ball and of $\widetilde{K}$. These candidates are cylinders of height $2\alpha$ and $2/\alpha$ with the radii of the bases equal to

$$\sqrt{\frac{1}{\alpha^2} + \frac{1}{4}} \qquad \text{and} \qquad \sqrt{1 + \frac{\alpha^2}{4}}$$

respectively. It is assumed that the domains which result from $K$ upon the motions in $E^3$ and whose union gives $\widehat{K}$ do not contain $\overline{0}$ as an interior point. As $\alpha \to \infty$, there is a possibility to diminish $V(\widetilde{K})$ 14/13 times, but this possibility is of a different nature than for $\alpha = 2$. To this end, we have to replace the parallelepiped $\widetilde{K}$, which is a rectangular prism with height $2\alpha$, by a prism of the same height $2\alpha$ but with an octogonal base. To do this, we slightly enlarge the base rectangle with sides $2/\alpha$ and 2 to obtain a rectangle with sides $2/\alpha + 1/(7\alpha)$ and $2 + 2/\alpha^2$ and then we cut off the right triangles with legs $1/(2\alpha) + 1/(14\alpha)$ and $1/2 + 1/\alpha^2$ from the angles of the enlarged rectangle. A similar truncation at the bases of the obtained octogonal prism leads to an additional diminishing of the number of candidates for the $K$-boundary in this case.

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