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On the Lower Bound of the Number of Variables for the Solvability of the Hilbert–Kamke System in 3-Adic Integers

D. A. Mit'kin

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Let s, n be two natural numbers and let p be a prime number. Denote by $s(n,p)$ the minimal natural number such that for $s \geq s(n,p)$ and for any integers $A_1, \dots, A_n$ the system of equations

$$\sum_{i=1}^{s} \binom{x_i}{j} = A_j, \qquad j = 1, \dots, n,$$

is solvable in integer p-adic numbers $x_1, \dots, x_s$.

Arkhipov [1] established for the first time the greatest lower and least upper bounds of $s(n,2)$. Also he showed that the minimal number of variables that are necessary when realizing the corresponding arithmetical conditions for the positivity of a special series of the Hilbert–Kamke problem is defined by the value of $s(n,2)$. Later, the exact value was also found for $s(n,2)$ (see [2, 3]). We note that in the mentioned estimates of the value $s(n,2)$, a considerable role was played by possibilities of application of induction and special integral polynomials, which take relatively few values modulo the power of a prime number. Arkhipov and Karatsuba [4] succeeded in application of analogous arguments to the Artin problem on the representation of zero by a form in local fields. They proved that for any prime p, there exists an infinite sequence of forms with integer coefficients and with the number of variables increasing as the exponential function of the degree of the form such that they do not nontrivially represent 0 in the field of p-adic numbers.

The exact estimate of $s(n,3)$ is important for the special series of the Hilbert–Kamke problem in prime numbers (see [5, 6]). The author suppose that by analogy with the formula for $s(n,2)$, the value of $s(n,3)$ is recursively defined: $s(n,3) = q(n)$, where

$$q(0) = 0, \qquad q(n) = \begin{cases} 2(3^{n/2} - 1) + 3q\Big(\Big[\dfrac{n}{3}\Big]\Big) & \text{if } n \text{ is even} \\ 3^{(n+1)/2} - 2 + 3q\Big(\Big[\dfrac{n}{3}\Big]\Big) & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

As in the case of $s(n,2)$, the proof of both estimates $s(n,3) \leq q(n)$ and $s(n,3) \geq q(n)$ is expected to be complicated technically in the general case. In the present paper, we prove the inequality $s(n,3) \geq q(n)$ for those n for which all positive numbers of the series

$$n, \Big[\frac{n}{3}\Big], \Big[\frac{n}{3^2}\Big], \dots \tag{2}$$

are simultaneously odd or are simultaneously not divisible by 3. In the proof, we use specific integral polynomials of the even degree l which assume only two values 0 and 1 modulo $3^{l/2}$. The restrictions on n are caused by the found possibilities of application of induction.

Introduce the following notation: $[a]$ is the integral part of a; $\{a\}$ is the fractional part of a; $\nu_p(a)$ is the largest exponent of a prime p, which divides an integer a; $p^k \parallel a$ denotes $p^k \mid a$ or $p^{k+1} \nmid a$;

$$\Delta_h^{(0)} f(x) = f(x), \qquad \Delta_h f(x) = f(x+h) - f(x), \qquad \Delta_h^{(j+1)} f(x) = \Delta_h\big(\Delta_h^{(j)} f(x)\big), \quad j = 0, 1, \ldots;$$

$$\binom{x}{0} = 1, \qquad \binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}, \quad k = 1, 2, \ldots .$$

Lemma 1. *Let $l \geq 2$ be an odd number and let ζ be a primitive cubic root of* 1. *The polynomial*

$$f_l(x) = \sum_{j=1}^{l} a_j \binom{x}{j}, \qquad \text{where} \quad a_j = -\frac{1}{3} \sum_{k=1}^{2} (\zeta^k - 1)^j,$$

is integral and satisfies the congruence

$$f_l(x) \equiv \begin{cases} 0 \pmod{3^{l/2}} & \text{for an integer } x \text{ if } x \equiv 0 \pmod 3 \\ 1 \pmod{3^{l/2}} & \text{if } x \not\equiv 0 \pmod 3. \end{cases}$$

The coefficients of $f_l(x)$ satisfy the conditions

$$3^{(l-2)/2} \parallel a_l, \qquad a_j \equiv 0 \pmod{3^{(l-2)/2 - [(l-j)/2]}}, \quad j = 1, \ldots, l.$$

The **proof** see in [2, Lemma 7].

Lemma 2. *Let n, l, j be natural numbers, $j \leq l \leq n$, and let $\alpha(n,j)$ be defined by one of the formulas*

$$\left[\frac{n-j+1}{2}\right] + 1 \qquad \text{or} \qquad \left[\frac{n-j+1}{3}\right] + 1 + \nu_3\left(\frac{n!}{(j-1)!}\right). \tag{3}$$

Then the inequality

$$\alpha(n,j) + \left[\frac{l-1}{2}\right] - \left[\frac{l-j}{2}\right] \geq \left[\frac{l+1}{2}\right]$$

holds true.

Proof. If $\alpha(n,j) = [(n-j+1)/2] + 1$, then the inequality is obvious. Consider the second case. In the series $1, 2, \ldots, n$, there are $[n/3]$ integers which are divisible by 3, and there are $[(j-1)/3]$ numbers with the same property in the series $1, 2, \ldots, j-1$. Therefore,

$$\nu_3\left(\frac{n!}{(j-1)!}\right) = \left[\frac{n}{3}\right] - \left[\frac{j-1}{3}\right] + \nu_3\left(\frac{[n/3]!}{[(j-1)/3]!}\right) \geq \left[\frac{n}{3}\right] - \left[\frac{j-1}{3}\right].$$

Now it is sufficient to show that

$$\left[\frac{n-j+1}{3}\right] + \left[\frac{n}{3}\right] - \left[\frac{j-1}{3}\right] - \left[\frac{l-j}{2}\right] \geq 0$$

for $j \leq l \leq n$. Since the left-hand side does not increase with increasing l and

$$\left[\frac{n}{3}\right] - \left[\frac{j-1}{3}\right] \geq \left[\frac{n-j+1}{3}\right],$$

it is sufficient to establish the validity of the inequality $2[(n-j+1)/3] \geq [(n-j)/2]$. Representing $n-j$ in the form $n-j = 6k+r$, $k, r \in \mathbb{Z}$, where $k \geq 0$, $0 \leq r \leq 5$, we transform the latter inequality into the form $k + 2[(r+1)/3] \geq [r/2]$, where it can easily be verified. The lemma is proved.

Lemma 3. *Let h be a real number, and let $f(x)$ be a polynomial of degree n with real coefficients. Then the relation*

$$\Delta_{hy} f(x) = \sum_{k=1}^{n} \Delta_h^k f(x) \binom{y}{k}$$

holds true.

The **proof** see in [6, Chap. 1].

Lemma 4. *Let $l \geq 2$ be an even number, let ζ be a primitive cubic root of* 1, *and let*

$$a_j = -\frac{1}{3} \sum_{k=1}^{2} (\zeta^k - 1)^j, \qquad j = 1, 2, \dots .$$

The following equality holds for the polynomial

$$f_l(x) = \sum_{j=1}^{l} a_j \binom{x}{j} :$$

$$\Delta_3 f_l(x) = 3^{l/2} \varphi_l(x), \qquad \textit{where} \quad \varphi_l(x) = c_{l,1} \binom{x}{l-1} + c_{l,2} \binom{x}{l-2}$$

is an integral polynomial, $c_{l,2} = -3^{-l/2} a_{l+1}$, *and* $c_{l,1} = 3 - l/2 + 1 a_l \not\equiv 0 \pmod 3$.

Proof. By Lemma 3, we have

$$\Delta_3 f_l(x) = \sum_{k=1}^{3} \Delta_1^{(k)} f_l(x) \binom{3}{k} = \sum_{k=1}^{3} \binom{3}{k} \sum_{j=k}^{l} a_j \binom{x}{j-k} = \sum_{m=0}^{l-1} \binom{x}{m} \sum_{\substack{1 \leq k \leq 3 \\ k+m \leq l}} \binom{3}{k} a_{m+k}$$

$$= 3a_l \binom{x}{l-1} + (3a_l + 3a_{l-1}) \binom{x}{l-2} = 3a_l \binom{x}{l-1} - a_{l+1} \binom{x}{l-2}$$

since

$$3a_{l-1} + 3a_l + a_{l+1} = -\frac{1}{3} \sum_{k=1}^{2} (\zeta^k - 1)^{l-1} \big(3 + 3(\zeta - 1) + (\zeta - 1)^2\big) = 0.$$

Now, the assertion of the lemma follows from the relations of Lemma 1 for the coefficients a_l and a_{l+1} for even l, i.e.,

$$3^{(l-2)/2} \parallel a_l, \qquad a_{l+1} \equiv 0 \pmod{3^{l/2}}.$$

Lemma 5. *Let n, h be two natural numbers, $n \geq 3$, $u = [n/3]$, $h \leq u$, $\alpha(n,j)$ be defined for a natural $j \leq n$ by one of formulas* (3), *and let*

$$\beta(l) = \min_{j=1,\dots,l} \Big(\alpha(n,j) + \Big[\frac{l-1}{2}\Big] - \Big[\frac{l-j}{2}\Big] \Big).$$

Then, for $l = 3h, 3h-1, 3h-2$ the inequality $\beta(l) - [(l+1)/2] \geq \alpha(u,h)$ holds in all cases, except for the case $\alpha(n,j) = [(n-j+1)/2]+1$, $n = 3u$, $h = u$, $l = 3u$.

Proof. First assume that $\alpha(n,j) = [(n-j+1)/2]+1$. Then it is sufficient to show that

$$\left[\frac{n-j+1}{2}\right] - \left[\frac{l-j}{2}\right] \geq \left[\frac{u-h+1}{2}\right] + 1.$$

The left-hand side of this inequality is greater than or equal to $[(n-l+1)/2]$; this is greater or equal to $[3(u-h)/2]+1$, except for the case $n = 3u$ and $l = 3h$ where $[(n-l+1)2] \geq [3(u-h)/2]$. It is easy to see that $[3(u-h)/2] \geq [(u-h+1)/2]$ for $h < u$. The case $h = u$ is obvious.

Consider now the case of the second definition of $\alpha(n,j)$. It is sufficient to show that for any $j = 1, 2, \ldots, 3h$, the inequality

$$\alpha(n,j) - 1 - \left[\frac{3h-j}{2}\right] \geq \alpha(u,h)$$

holds. Since $\alpha(n,j)$ does not decrease when n increases, we can restrict ourselves by the proof of the last inequality with $\alpha(3u,j)$ instead of $\alpha(n,j)$, i.e., it is sufficient to establish that the following inequality holds for $1 \leq j \leq 3h$:

$$\left[\frac{3u-j+1}{3}\right] + \nu_3\left(\frac{(3u)!}{(j-1)!}\right) - \left[\frac{3h-j}{2}\right] \geq \left[\frac{u-h+1}{3}\right] + 1 + \nu_3\left(\frac{u!}{(h-1)!}\right).$$

Here, by virtue of the relations

$$\nu_3\left(\frac{(3u)!}{(j-1)!}\right) = u - \left[\frac{j-1}{3}\right] + \nu_3\left(\frac{u!}{[(j-1)/3]!}\right), \qquad \left[\frac{j-1}{3}\right] \leq h-1,$$

it is sufficient to restrict ourselves by the proof of the inequality

$$\left[\frac{3u-j+1}{3}\right] + u - \left[\frac{j-1}{3}\right] - \left[\frac{3h-j}{2}\right] \geq \left[\frac{u-h+1}{3}\right] + 1.$$

However, we note that it is necessary to verify this inequality only for $j = 3h$, $3h-1$, $3h-2$ since its left-hand side does not increase when j increases by 3. For the indicated j, the left-hand side of the last inequality is equal to $2(u-h)+1$, and obviously, $2(u-h)+1 \geq [(u-h+1)/3]+1$. The lemma is proved.

Lemma 6. *For integers $g_{k,l}$, which are defined by the equality*

$$g_{k,l} = \Delta_3^{(k-1)}\binom{0}{l-1}$$

for natural k and l, the following relations are satisfied:

(1) $g_{k,l} = 0$ *if* $l > 3k-2$,

(2) $g_{k,l} = 1$ *if* $l = 3k-2$,

(3) $\nu_3(g_{k,l}) \geq \min\limits_{r \leq [(l-k)/2]}\left(k-1-r+\nu_3\left(\frac{(k-1)!}{r!(k-1-r)!}\right)\right)$ *if* $l < 3k-2$.

Proof. By Lemma 3, we have

$$g_{k,l} = \sum_{1\le i_1,\dots,i_{k-1}\le 3} \Delta_1^{(i_1+\dots+i_{k-1})}\binom{0}{l-1}\binom{3}{i_1}\cdots\binom{3}{i_{k-1}}$$

$$= \sum_{\substack{1\le i_1,\dots,i_{k-1}\le 3\\ i_1+\dots+i_{k-1}=l-1}} \binom{3}{i_1}\cdots\binom{3}{i_{k-1}} = \sum_{r=0}^{k-1}\binom{k-1}{r}3^{k-1-r}\sum_{\substack{1\le i_1,\dots,i_{k-1-r}\le 2\\ i_1+\dots+i_{k-1-r}=l-1-3r}} 1$$

$$= \sum_{r=0,1,\dots,k-1}\binom{k-1}{r}\binom{k-1-r}{l-k-2r}3^{k-1-r}.$$

The interval of summation with respect to r is not empty if $l+1-2k \le (l-k)/2$, i.e., in the case $l \le 3k-2$; moreover, for $l = 3k-2$, the variable r of summation assumes only one value $r = k-1$, and $g_{k,l} = 1$. For $l < 3k-2$ we have $\nu_3(g_{k,l}) \ge 1$; a more exact lower estimate of $\nu_3(g_{k,l})$ can obviously be written as in item (3) of the lemma.

Lemma 7. *Let k, l, u, and h be natural numbers, $h \le u$, $l \le 3h$, $g_{k,l} = \Delta_3^{(k-1)}\binom{0}{l-1}$, and let $\alpha(n,j)$ be defined by one of formulas* (3) *for natural j and n, where $j \le n$. Then the following inequalities hold*:

$$\nu_3(g_{k,l}) \ge \alpha(u,h) \qquad \textit{if} \quad k > u$$

and

$$\nu_3(g_{k,l}) + \alpha(u,k) \ge \alpha(u,h) \qquad \textit{if} \quad h < k \le u.$$

Proof. First let $\alpha(n,j) = [(n-j+1)/2]+1$. Since, by Lemma 6,

$$\nu_3(g_{k,l}) \ge k-1-\left[\frac{l-k}{2}\right],$$

in order to prove the first inequality of the lemma, it is sufficient to verify that for $k > 0$ the inequality

$$k-1-\left[\frac{l-k}{2}\right] \ge \left[\frac{u-h+1}{2}\right]+1$$

holds true. In turn, it is sufficient for this that

$$u-\left[\frac{3h-u-1}{2}\right] \ge \left[\frac{u-h+1}{2}\right]+1;$$

this can easily be verified. To prove the second inequality, it is sufficient to verify that

$$k-1-\left[\frac{l-k}{2}\right]+\left[\frac{u-k+1}{2}\right] \ge \left[\frac{u-h+1}{2}\right]$$

for $h < k \le u$ and $l \le 3h$. In fact, here the left-hand side is greater than or equal to

$$k-1-\left[\frac{3h-k}{2}\right]+\left[\frac{u-k+1}{2}\right] \ge k-1+\left[\frac{u-3h+1}{2}\right] \ge \left[\frac{u-h+1}{2}\right].$$

Consider now the case where $\alpha(n,j)$ is defined by the second formula. Apply the lower estimate of $\nu_3(g_{k,l})$ from item (3) of Lemma 6. In both inequalities of the lemma, it is sufficient to use the value

$$G(k,l,r) = k - 1 - r + \nu_3\left(\frac{(k-1)!}{r!(k-1-r)!}\right)$$

for an arbitrary integer r with the condition $0 \le r \le [(l-k)/2]$ instead of $\nu_3(g_{k,l})$. Since

$$\nu_3(m!) = \left[\frac{m}{3}\right] + \left[\frac{m}{3^2}\right] + \cdots \le \left[\frac{m-1}{2}\right]$$

for a natural m, we have

$$m - \nu_3(m!) \ge m - \left[\frac{m-1}{2}\right] = \left[\frac{m}{2}\right] + 1,$$

this implies that

$$G(k,l,r) \ge \left[\frac{k-1-r}{2}\right] + 1 + \nu_3\left(\frac{(k-1)!}{r!}\right).$$

Let $k > u$. Then it is sufficient to show that

$$\left[\frac{k-1-r}{2}\right] + \nu_3\left(\frac{(k-1)!}{r!}\right) \ge \left[\frac{u-h+1}{3}\right] + \nu_3\left(\frac{u!}{(h-1)!}\right).$$

This inequality holds since

$$k - 1 \ge u, \qquad r \le \left[\frac{3h-u-1}{2}\right] \le \left[\frac{2h-1}{2}\right] = h - 1,$$

and then $k - 1 - r \ge u - h + 1$, while $(k-1)!/r!$ is divisible by $u!/(h-1)!$.

In the case $h < k \le u$, it is sufficient to show that

$$\left[\frac{k-1-r}{2}\right] + \nu_3\left(\frac{(k-1)!}{r!}\right) + \left[\frac{u-k+1}{3}\right] + \nu_3\left(\frac{u!}{(k-1)!}\right) \ge \left[\frac{u-h+1}{3}\right] + \nu_3\left(\frac{u!}{(h-1)!}\right).$$

Since $r \le [(3h-k)/2] \le h - 1$, it is sufficient to verify the inequality

$$\left[\frac{k-1-r}{2}\right] + \left[\frac{u-k+1}{3}\right] \ge \left[\frac{u-h+1}{3}\right].$$

The last inequality is obviously true if $k \ge h+2$, or if $k = h+1$, $r < h-1$. In the case $k = h+1$, $r = h-1$, we return to the initial formula for $G(k,l,r)$. It is sufficient to verify the inequality

$$1 + \nu_3(h) + \left[\frac{u-h}{3}\right] + \nu_3\left(\frac{u!}{h!}\right) \ge \left[\frac{u-h+1}{3}\right] + \nu_3\left(\frac{u!}{(h-1)!}\right),$$

this is easily verified. The lemma is proved.

Lemma 8. *Let l, u, h, and k be natural numbers, $g_{u,l} = \Delta_3^{(u-1)}\binom{0}{l-1}$, and let $\alpha(n,j)$ be defined by the formula $[(n-j+1)/2]+1$ for natural j and n, where $j \le n$. Then the following inequalities hold:*

(1) $\nu_3(g_{u,l-2}) \ge \alpha(u,h)$ *if* $l \le 3h$, $h \le u-1$,
(2) $\nu_3(g_{u,l}) \ge \alpha(u,h)$ *if* $l \le 3h$, $h \le u-2$,
(3) $\nu_3(g_{k,l}+1 \ge \alpha(u,h)$ *if* $l \le 3h$, $h \le u-2$, $k \ge u-1$.

Proof. By Lemma 6, for $\nu_3(g_{u,l-2})$, we have

$$\nu_3(g_{u,l-2}) \ge u-1-\left[\frac{l-2-u}{2}\right] \ge u-1-\left[\frac{3h-2-u}{2}\right] = u-h-\left[\frac{h-u}{2}\right]$$

for $l \le 3h$; this implies the inequality

$$\nu_3(g_{u,l-2}) \ge \left[\frac{u-h+1}{2}\right]+1 \qquad \text{if} \quad h \le u-1.$$

Similarly,

$$\nu_3(g_{u,l}) \ge u-1-\left[\frac{l-u}{2}\right] \ge u-1-\left[\frac{3h-u}{2}\right] = u-1-h-\left[\frac{h-u}{2}\right]$$

for $l \le 3h$; this implies that

$$\nu_3(g_{u,l}) \ge \left[\frac{u-h+1}{2}\right]+1 \qquad \text{if} \quad h \le u-2.$$

Finally, for $l \le 3h$ and $k \ge u-1$, we obtain

$$\nu_3(g_{k,l}) \ge k-1-\left[\frac{l-k}{2}\right] \ge u-2-\left[\frac{3h-u+1}{2}\right] = u-2-h-\left[\frac{h-u+1}{2}\right];$$

this is not less than $[(u-h+1)/2]$ if $h \le u-2$. The lemma is proved.

Lemma 9. *Let ζ be a primitive cubic root of 1. The following formulas hold for the numbers a_j, $j=1,2,\dots$, defined by the equality*

$$a_j = -\frac{1}{3}\sum_{k=1}^{2}(\zeta^k-1)^j :$$

$$a_{j+2} = -3(a_{j+1}+a_j), \tag{4}$$

$$a_j = 3^{[(j-1)/2]}\mu_j, \tag{5}$$

where $\mu_1=1$, $\mu_2=-1$, $\mu_3=0$, $\mu_4=1$, $\mu_5=-1$, $\mu_6=2$, and $\mu_{6m+r}=(-1)^m\mu_r$ for $r=1,\dots,6$, $m=1,2,\dots$.

Proof. Since $\zeta^2+\zeta+1=0$, relation (4) follows from the equality $3+3(\zeta^k-1)+(\zeta^k-1)^2=0$, $k=1,2$. By induction on j, we obtain relation (5).

Theorem. *Let s and n be natural numbers, with n belonging to one of the types of numbers* (A) *or* (B) *depending on the fact that all integers of series* (2) *are simultaneously odd or are simultaneously not divisible by* 3. *Let $\alpha(n,j)$ be defined by the formula*

$$\left[\frac{n-j+1}{3}\right]+1+\nu_3\left(\frac{n!}{(j-1)!}\right)$$

for $j=1,\dots,n$, if n is of type (A), *and by the formula*

$$\left[\frac{n-j+1}{2}\right]+1$$

if n of type (B). *Let further the integer δ assume not more than two values modulo* 3,

$$\delta_j=\begin{cases}\delta & \text{if } j=n \text{ and } n \text{ is an odd number of type (B)}\\ 3\delta & \text{if } j=n-1 \text{ and } n \text{ is an even number of type (B)}\\ 0 & \text{in other cases.}\end{cases}$$

Then there exist integers $A_1,\dots,A_n$ such that the condition $s\ge q(n)$ is necessary for the solvability of the system of congruences of degree n

$$\sum_{i=1}^{s}\binom{x_i}{j}\equiv A_j+\delta_j \pmod{3^{\alpha(n,j)}},\qquad j=1,\dots,n, \tag{6}$$

in integers $x_1,\dots,x_s$ independently of δ, where the function $q(n)$ is recursively defined as in (1).

Proof. We apply the induction on n. For $n=1$, we have $q(1)=1$ and $\alpha(1,1)=1$. Without loss of generality, we assume that δ can take the values which are congruent only to 0 or 1 modulo 3. It is sufficient to choose $A_1=1$ since $A_1+\delta\equiv 0 \pmod 3$, and hence, the condition $s\ge 1$ is necessary.

The number $n=2$ is of type (B), and $\alpha(2,1)=1$, $\alpha(2,2)=2$, and $q(2)=4$. It is sufficient to restrict ourselves by the consideration of the system of congruences

$$\sum_{i=1}^{s}x_i^2\equiv 2A_2+A_1 \pmod 3,$$

$$\sum_{i=1}^{s}\big((x_i-1)^2-1\big)\equiv 2A_2-A_1 \pmod 3$$

instead of system (6). Here, we choose $A_1=0$, $A_2=2$. If k_r is the number of x_i with the condition $x_i\equiv r \pmod 3$, $r=0,1,2$, then it follows from the last congruences that $k_1+k_2\equiv 1 \pmod 3$, $-k_1\equiv 1 \pmod 3$; this implies $k_1\equiv k_2\equiv 2 \pmod 3$ and $s\ge k_1+k_2\ge 4=q(2)$. Thus, the assertion of the theorem is proved for $n=1,2$.

Let $n\ge 3$ and let the assertion of the theorem be true for the system of congruences of degree $<n$. We prove the assertion for n. Let $A_1,\dots,A_n$ be certain integers and let $\delta,s,x_1,\dots,x_s$ be integers which satisfy congruences (6) under the conditions of the theorem. Then the equalities

$$\sum_{i=1}^{s}\binom{x_i}{j}=A_j+\delta_j+3^{\alpha(n,j)}M_j,\qquad j=1,\dots,n, \tag{7}$$

hold for certain integers $M_1, \dots, M_n$. Let $f_l(x)$, $l = 1, 2, \dots$, be polynomials defined in Lemma 1. We set

$$F_l(x) = \begin{cases} f_l(x) & \text{if } l \text{ is even} \\ \Delta_1 f_{l+1}(x) - f_{l+1}(1) & \text{if } l \text{ is odd.} \end{cases}$$

We express the polynomials $F_l(x)$ in terms of the basic integer-valued polynomials $\binom{x}{j}$:

$$F_l(x) = \sum_{j=1}^{l} a_{l,j} \binom{x}{j}, \qquad a_{l,j} \in \mathbb{Z}.$$

Multiplying the jth equality in (7) by $a_{l,j}$ and summing the obtained equalities over j varying from 1 to l for each $l = 1, \dots, n$, we obtain the system of equalities

$$\sum_{i=1}^{s} F_l(x_i) = B_l + \Omega_l + 3^{\beta(l)} C_l, \qquad l = 1, \dots, n, \tag{8}$$

where

$$B_l = \sum_{j=1}^{l} a_{l,j} A_j, \qquad \Omega_l = \sum_{j=1}^{l} a_{l,j} \delta_j, \qquad 3^{\beta(l)} C_l = \sum_{j=1}^{l} 3^{\alpha(n,j)} a_{l,j} M_j,$$

and $\beta(l)$ is the exponent of the highest degree of 3 which divides all the numbers $3^{\alpha(n,j)} a_{l,j}$ for $j = 1, \dots, l$. By the definition of the polynomials $F_l(x)$, we have

$$a_{l,j} = \begin{cases} a_j & \text{if } l \equiv 0 \pmod 2 \\ a_{j+1} & \text{if } l \equiv 1 \pmod 2). \end{cases} \tag{9}$$

for $j = 1, \dots, l$. Hence, by Lemma 1,

$$a_{l,j} \equiv 0 \pmod{3^{[(l-1)/2] - [(l-j)/2]}}, \qquad 3^{[(l-1)/2]} \parallel a_{l,l}, \tag{10}$$

and

$$\beta(l) \geq \min_{j=1,\dots,l} \left(\alpha(n,j) + \left[\frac{l-1}{2} \right] - \left[\frac{l-j}{2} \right] \right).$$

Let us impose the first conditions on the choice of the numbers $A_1, \dots, A_n$, assuming that

$$B_l = \begin{cases} \lambda_1 + 3^{[(l+1)/2]} t_l & \text{for an odd } l \\ \lambda_2 + 3^{[(l+1)/2]} t_l & \text{for an even } l. \end{cases}$$

By induction, with the help of relations (9) and (10), it is easy to verify that 3-integers $A_1, \dots, A_n$ are uniquely determined from the equations

$$\sum_{j=1}^{l} a_{l,j} A_j = B_l, \qquad l = 1, \dots, n,$$

by any integers λ_1, λ_2, and $t_1, \dots, t_n$. The fact that the numbers $A_1, \dots, A_n$ occur to be 3-integers, but not integers, is not essential for congruences. Choose λ_1, λ_2. To this end, we pass to congruences modulo $3^{[(l+1)/2]}$ in equalities (8) for $l = n, n-1$. By Lemma 2, $\beta(l) \geq [(l+1)/2]$, and therefore, we obtain the congruences

$$\sum_{i=1}^{s} F_l(x_i) \equiv B_l + \sum_{j=n-1}^{l} a_{l,j}\delta_j \pmod{3^{[(l+1)/2]}}, \qquad l = n, n-1. \tag{11}$$

The polynomials $F_l(x)$ defined in terms of $f_l(x)$ have the following properties for integral x by virtue of Lemma 1:

$$F_l(x) \equiv \begin{cases} 0 \pmod{3^{l/2}} & \text{if } x \equiv 0 \pmod 3 \\ 1 \pmod{3^{l/2}} & \text{if } x \not\equiv 0 \pmod 3 \end{cases} \qquad \text{for even } l,$$

$$F_l(x) \equiv \begin{cases} 0 \pmod{3^{(l+1)/2}} & \text{if } x \equiv 0 \pmod 3 \\ -1 \pmod{3^{(l+1)/2}} & \text{if } x \equiv 1 \pmod 3 \\ -2 \pmod{3^{(l+1)/2}} & \text{if } x \equiv 2 \pmod 3 \end{cases} \qquad \text{for odd } l.$$

Denote by s_0, s_1, and s_2 the numbers of x_i which are congruent to 0, 1, and 2 modulo 3, respectively. Then relation (11) implies the congruences

$$s_1 + s_2 \equiv \lambda_2 + a_{n,n-1}3\delta \pmod{3^{n/2}},$$
$$-s_1 - 2s_2 \equiv \lambda_1 + a_{n,n-1}3\delta \pmod{3^{n/2}}$$

for even n (having only type (B)), and the congruences

$$s_1 + s_2 \equiv \lambda_2 \pmod{3^{(n-1)/2}},$$
$$-s_1 - 2s_2 \equiv \lambda_1 + a_{n,n}\delta_n \pmod{3^{(n+1)/2}}$$

for odd n. We choose

$$\lambda_2 = 2\Big(q\Big(\Big[\frac{n}{3}\Big]\Big) - 1\Big), \qquad \lambda_1 = \begin{cases} -3\Big(q\Big(\Big[\frac{n}{3}\Big]\Big) - 1\Big) & \text{for even } n \\ -3\Big(q\Big(\Big[\frac{n}{3}\Big]\Big) - 1\Big) - 3^{(n-1)/2}\chi & \text{for odd } n, \end{cases}$$

where χ will be chosen later.

By Lemma 1, $a_j \equiv 0 \pmod{3^{[(j-1)/2]}}$, and therefore, we can define $\mu_j \in \mathbb{Z}$ by the equality $a_j = 3^{[(j-1)/2]}\mu_j$. By the definition of the polynomial $F_n(x)$, we have $a_{n,n-1} = a_{n-1}$, $a_{n-1,n-1} = a_n$, and $a_n \equiv a_{n-1} \equiv 0 \pmod{3^{n/2-1}}$ for even n. For odd n, we have $a_{n,n} = a_{n+1} \equiv 0 \pmod{3^{(n-1)/2}}$. Hence the congruences

$$s_1 \equiv s_2 \equiv q\Big(\Big[\frac{n}{3}\Big]\Big) - 1 \pmod{3^{[n/2]}}$$

hold for even n and for odd n. Represent s_1, s_2 in the form

$$s_r = q\Big(\Big[\frac{n}{3}\Big]\Big) - 1 + 3^{[n/2]}d_r, \qquad d_r \in \mathbb{Z}, \quad r = 1, 2.$$

In the case of odd n, we have the additional congruence

$$d_1 + 2d_2 \equiv \chi - \mu_{n+1}\delta_n \pmod 3.$$

Since δ_n can assume not more than two values modulo 3, χ can be chosen so that

$$d_1 + 2d_2 \not\equiv 0 \pmod 3. \tag{12}$$

Further, we transform equalities (8) into the form which is convenient for the choice of the numbers $t_1, \dots, t_n$. Rewrite equalities (8) into three progressions $r + 3x_{i,r}$, $(r = 0, 1, 2)$, dividing the summation with respect to the numbers x_i:

$$\sum_{r=0}^{2} \sum_{i=1}^{s_r} F_l(r + 3x_{i,r}) = B_l + \Omega_l + 3^{\beta(l)} C_l, \qquad l = 1, \dots, n.$$

Using the finite-difference operator we rewrite this system of equalities

$$\sum_{r=0}^{2} \sum_{i=1}^{s_r} \Delta_{3x_{i,r}} F_l(r) = B_l + \Omega_l - \sum_{r=0}^{2} F_l(r) + 3^{\beta(l)} C_l, \qquad l = 1, \dots, n.$$

Replacing $\Delta_{3x} F_l(r)$ here by the expression

$$\sum_{k=1}^{l} \Delta_3^{(k)} F_l(r) \binom{x}{k},$$

which is equal to it by Lemma 3, we obtain the equalities

$$\sum_{r=0}^{2} \sum_{k=1}^{l} \Delta_3^{(k)} F_l(r) \sum_{i=1}^{s_r} \binom{x_{i,r}}{k} = B_l + \Omega_l - \sum_{r=0}^{2} F_l(r) + 3^{\beta(l)} C_l, \qquad l = 1, \dots, n. \tag{13}$$

It follows from the definition of the polynomials $F_l(x)$ and from Lemma 4 that

$$\Delta_3 F_l(x) = 3^{[(l+1)/2]} \psi_l(x),$$

where

$$\psi_l(x) = c_{l,1} \binom{x}{l-1} + c_{l,2} \binom{x}{l-2}, \qquad c_{l,1}, c_{l,2} \in \mathbb{Z}, \quad c_{l,1} \not\equiv 0 \pmod 3.$$

Denoting

$$X_{k,r} = \sum_{i=1}^{s_r} \binom{x_{i,r}}{k},$$

we rewrite now relations (13) in the form

$$\sum_{r=0}^{2} \sum_{k=1}^{l} \Delta_3^{(k-1)} \psi_l(r) X_{k,r} = 3^{-[(l+1)/2]} \left(B_l + \Omega_l - \sum_{r=0}^{2} F_l(r) + 3^{\beta(l)} C_l \right), \qquad l = 1, \dots, n. \tag{14}$$

Let us simplify the right-hand side of (14). Using the notation introduced above $a_j = 3^{[(j-1)/2]}\mu_j$, where $\mu_j \in \mathbb{Z}$, we write Ω_l in the form $\Omega_l = 3^{[n/2]}\omega_l$, where

$$\omega_l = \begin{cases} \mu_{n-1}\delta & \text{for even } n \text{ and } l = n \\ \mu_n\delta & \text{for even } n \text{ and } l = n-1 \\ \mu_{n+1}\delta_n & \text{for odd } n. \end{cases}$$

Suppose that l is even. Then $F_l(x) = f_l(x)$, and hence, by Lemma 1,

$$F_l(0) = 0, \qquad F_l(1) = a_1 = -\frac{1}{3}\sum_{k=1}^{2}(\zeta^k - 1) = 1,$$

$$F_l(2) = a_2 + 2a_1, \quad a_2 = -\frac{1}{3}\sum_{k=1}^{2}(\zeta^k - 1)^2 = 1.$$

Whence, using the expressions for B_l, s_1, and s_2, we obtain

$$3^{-[(l+1)/2]}\left(B_l + \Omega_l - \sum_{r=0}^{2} F_l(r)\right) = t_l - 3^{-[n/2]-[(l+1)/2]}(d_1 + d_2 - \omega_l)$$

for even l. If l is odd, then $F_l(x) = \Delta_1 f_{l+1}(x) - f_{l+1}(1)$, and hence, by Lemma 1,

$$F_l(0) = 0, \quad F_l(1) = f_{l+1}(2) - 2f_{l+1}(1) = -1, \quad F_l(2) = f_{l+1}(3) - 2f_{l+1}(2) - f_{l+1}(1) = f_{l+1}(3) - 2.$$

Since

$$a_3 = -\frac{1}{3}\sum_{j=1}^{2}(\zeta^j - 1)^3 = 0,$$

we have $f_{l+1}(3) = 3a_2 + 3a_1 = 0$, and therefore, $F_l(2) = -2$. Whence we have

$$3^{-[(l+1)/2]}\left(B_l + \Omega_l - \sum_{r=0}^{2} F_l(r)\right) = t_l + 3^{[n/2]-[(l+1)/2]}\left(d_1 + 2d_2 - 2\chi\left\{\frac{n}{2}\right\} + \omega_l\right)$$

for odd l.

Let us introduce some new notation. Set

$$g_{k,l} = \Delta_3^{(k-1)}\binom{0}{l-1}.$$

Then, by Lemma 3, we have

$$\Delta_3^{(k-1)}\binom{r}{l-1} = \Delta_3^{(k-1)}\binom{0}{l-1} + \Delta_r \circ \Delta_3^{(k-1)}\binom{0}{l-1}$$
$$= g_{k,l} + \binom{r}{1}g_{k,l-1} + \binom{r}{2}g_{k,l-2} = \sum_{\gamma=0}^{2}\binom{r}{\gamma}g_{k,l-\gamma}$$

for $r = 0, 1, 2$. Setting

$$Y_{k,\gamma} = \sum_{\gamma=0}^{2} \binom{r}{\gamma} X_{k,r},$$

$$D_{l,k,\gamma} = c_{l,1} g_{k,l-\gamma} + c_{l,2} g_{k,l-\gamma-1}, \tag{15}$$

$$K_l(d_1, d_2) = \begin{cases} -3^{[n/2]-[(l+1)/2]}(d_1 + d_2 - \omega_l) & \text{if } l \equiv 0 \pmod 2 \\ 3^{[n/2]-[(l+1)/2]}\Big(d_1 + 2d_2 - 2\chi\Big\{\dfrac{n}{2}\Big\} + \omega_l\Big) & \text{if } l \equiv 1 \pmod 2, \end{cases}$$

and restricting ourselves by only $3u$ inequalities, where $u = [n/3]$, in (14), we obtain the system

$$\sum_{\gamma=0}^{2} \sum_{k=1}^{l} D_{l,k,\gamma} Y_{k,\gamma} = t_l + K_l(d_1, d_2) + 3^{\beta(l)-[(l+1)/2]} C_l, \qquad l = 1, \dots, 3u. \tag{16}$$

We shall consider this system as a linear one with respect to the variables $Y_{k,\gamma}$, $k = 1, \dots, u$, $\gamma = 0, 1, 2$. Write the left-hand sides of equations (16) for $l = 3h, 3h-1, 3h-2$ modulo 3. By relation (15) and Lemma 6, these are of the form

$$c_{3h,1} Y_{h,2} + c_{3h,2} Y_{h,1}, \qquad c_{3h-1,1} Y_{h,1} + c_{3h-1,2} Y_{h,0}, \qquad c_{3h-2,1} Y_{h,0} + c_{3h-2,2} Y_{h-1,2},$$

where $c_{l,1} \not\equiv 0 \pmod 3$. Hence the matrix of coefficients has the triangular form modulo 3 with respect to the variables $Y_{k,\gamma}$, $k = 1, \dots, u$, $\gamma = 0, 1, 2$, and we can solve this system with respect to the chosen variables in 3-integer numbers. This solution has the following form:

$$Y_{k,\gamma} = T_{k,\gamma} + L_{k,\gamma} + Q_{k,\gamma}, \qquad k = 1, \dots, u, \quad \gamma = 0, 1, 2,$$

where $T_{k,\gamma}$ are defined by the system

$$\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{l,k,\gamma} T_{k,\gamma} = t_l, \qquad l = 1, \dots, 3u,$$

$L_{k,\gamma} = L_{k,\gamma}(d_1, d_2)$ are defined by the system

$$\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{l,k,\gamma} L_{k,\gamma} = K_l(d_1, d_2), \qquad l = 1, \dots, 3u, \tag{17}$$

and $Q_{k,\gamma}$ are defined by the system

$$\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{l,k,\gamma} Q_{k,\gamma} = 3^{\beta(l)-[(l+1)/2]} C_l - \sum_{\gamma=0}^{2} \sum_{u<k\le l} D_{l,k,\gamma} Y_{k,\gamma}, \qquad l = 1, \dots, 3u.$$

By Lemma 5, relation (15), and by the first inequality of Lemma 7, the numbers $Q_{k,\gamma}$ satisfy the congruences

$$\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{l,k,\gamma} Q_{k,\gamma} \equiv 0 \pmod{3^{\alpha(u,h)}}, \qquad l = 3h, 3h-1, 3h-2,$$

for each $h = 1, \dots, u$. The values $\alpha(u,h)$ do not decrease when h decreases, and therefore, it follows from the presented system of congruences that the congruence

$$Q_{k,\gamma} \equiv 0 \ (\operatorname{mod} 3^{\alpha(u,u)})$$

holds for all $Q_{k,\gamma}$, and then, by induction, we obtain with the help of the second inequality of Lemma 7 that

$$Q_{k,\gamma} \equiv 0 \ (\operatorname{mod} 3^{\alpha(u,h)})$$

for $k \le h$. Thus,

$$Q_{k,\gamma} \equiv 0 \ (\operatorname{mod} 3^{\alpha(u,k)}), \qquad k = 1, \dots, u, \quad \gamma = 0, 1, 2. \tag{18}$$

Since the choice of the numbers $t_1, \dots, t_n$ is at our disposal, the numbers $T_{k,\gamma}$ can also be chosen in an arbitrary way for now. We obtain from (18) the congruences

$$Y_{k,\gamma} \equiv T_{k,\gamma} + L_{k,\gamma}(d_1, d_2) \ (\operatorname{mod} 3^{\alpha(u,k)}), \qquad k = 1, \dots, u, \quad \gamma = 0, 1, 2, \tag{19}$$

where the numbers $T_{k,\gamma}$ can be chosen as arbitrary integers.

1. Let n be an odd number of type (A). In this case, $\delta_n = 0$ and $\omega_l = 0$, and hence, the functions $L_{k,\gamma}(d_1, d_2)$ are fixed. Then we can complete the step of induction without of additional information on the functions $L_{k,\gamma}(d_1, d_2)$. We have from relations (19) and the definition of $T_{k,\gamma}$ that

$$X_{k,r} = \sum_{i=1}^{s_r} \binom{x_{i,r}}{k} \equiv T_k^{(r)} + L_k^{(r)}(d_1, d_2) \ (\operatorname{mod} 3^{\alpha(u,k)}), \qquad k = 1, \dots, u, \quad r = 0, 1, 2, \tag{20}$$

where $T_k^{(r)}$ can be chosen as arbitrary integers, and $L_k^{(r)}(d_1, d_2)$ and $L_{k,\gamma}(d_1, d_2)$ are fixed. It follows from the definition of the type that the number $u = [n/3] \ge 1$ belongs to the same type as the number n. Therefore, by the inductive assumption for u, we choose the numbers $T_k^{(2)}$ in such a way that $s_2 \ge q(u)$ is necessary for the solvability of the system of congruences

$$\sum_{i=1}^{s_2} \binom{x_{i,2}}{k} \equiv T_k^{(2)} + L_k^{(2)}(1, 0) \ (\operatorname{mod} 3^{\alpha(u,k)}), \qquad k = 1, \dots, u.$$

We choose the numbers $T_k^{(1)}$ in such a way that $s_1 \ge q(u)$ is necessary for the solvability of the system of congruences

$$\sum_{i=1}^{s_1} \binom{x_{i,1}}{k} \equiv T_k^{(1)} + L_k^{(1)}(0, 2) \ (\operatorname{mod} 3^{\alpha(u,k)}), \qquad k = 1, \dots, u.$$

Finally, we choose the numbers $T_k^{(0)}$ in such a way that $s_0 \ge q(u)$ is necessary for the solvability of the system of congruences

$$\sum_{i=1}^{s_0} \binom{x_{i,0}}{k} \equiv T_k^{(0)} + L_k^{(0)}(2, 1) \ (\operatorname{mod} 3^{\alpha(u,k)}), \qquad k = 1, \dots, u.$$

Let us return to system (20) with the chosen $T_k^{(r)}$. Since $\delta_n = 0$ and $d_1 + 2d_2 \equiv 1 \pmod 3$, there are the following possibilities in view of relation (12): (d_1, d_2) is equal to $(1,0)$, $(0,2)$, $(2,1)$, or $d_1 + d_2 \geq 4$. In the case $d_1 + d_2 \geq 4$, it follows that

$$s = s_0 + s_1 + s + 2 = 2\big(q(u) - 1\big) + 3^{[n/2]}(d_1 + d_2) \geq q(n).$$

The conditions of the choice of $T_k^{(2)}$ imply that $(d_1, d_2) \neq (1,0)$, and those of the choice of $T_k^{(1)}$ imply that $(d_1, d_2) \neq (0,2)$. Then $(d_1, d_2) = (2,1)$, and it follows from the choice of $T_k^{(0)}$ that $s_0 \geq q(u)$. As a result, we have

$$s \geq 3q(u) - 2 + 3 \times 3^{[n/2]} = q(n).$$

Let n be of type (B). For such n $\alpha(n,j)$ was defined as

$$\left[\frac{n-j+1}{2}\right] + 1.$$

The following two cases are possible: $n = 3u + 2$ or $n = 3u + 1$, with u belonging to type (B).

2. Let $n = 3u + 2$. Then n and u are of the same parity. In the case of even n, we have

$$\left[\frac{n}{2}\right] - \left[\frac{l+1}{2}\right] \geq \left[\frac{3u}{2}\right] + 1 - \left[\frac{3h+1}{2}\right] \geq \left[\frac{u-h+1}{2}\right] + 1 = \alpha(u,h),$$
$$K_l \equiv 0 \pmod{3^{\alpha(u,h)}},$$

for $l = 3h, 3h-1, 3h-2$, $h = 1, \ldots, u$, and hence, the congruences as those for $Q_{k,\gamma}$ hold for the numbers $L_{k,\gamma}$:

$$L_{k,\gamma} \equiv 0 \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \ldots, u, \quad r = 0,1,2,$$

this implies $Y_{k,\gamma} \equiv T_{k,\gamma} \pmod{3^{\alpha(u,k)}}$, $k = 1, \ldots, u$, $r = 0,1,2$, where the numbers $T_{k,\gamma}$ can be chosen as arbitrary integers. By the definition of $Y_{k,\gamma}$, we obtain the congruences

$$X_{k,r} = \sum_{i=1}^{s_r} \binom{x_{i,r}}{k} \equiv T_k^{(r)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \ldots, u, \quad r = 0,1,2,$$

where $T_k^{(r)}$ can be chosen as arbitrary integers. On the basis of the inductior assumption, for each $r = 0,1,2$, we choose the numbers $T_k^{(r)}$ in such a way that $s_r \geq q(u)$ is necessary for the solvability of the system of congruences

$$\sum_{i=1}^{s_r} \binom{x_{i,r}}{k} \equiv T_k^{(r)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \ldots, u.$$

As a result of this choice we have $d_1 \geq 1$, $d_2 \geq 1$, and

$$s = s_0 + s_1 + s_2 = s_0 + 2\big(q(u) - 1\big) + 3^{n/2}(d_1 + d_2) \geq q(n).$$

3. The inequality

$$\left[\frac{n}{2}\right] - \left[\frac{l+1}{2}\right] \geq \left[\frac{u-h+1}{2}\right] + 1 = \alpha(u,h)$$

holds for odd $n = 3u + 2$, in all the cases where $l = 3h, 3h-1, 3h-2$ $(h = 1, \dots, u)$, except for the case where $l = 3h$ and $h = u$. Hence $K_l(d_1, d_2) \equiv 0 \pmod{3^{\alpha(u,h)}}$ for $l = 3h, 3h-1, 3h-2$, $h = 1, \dots, u$, except for $l = 3h$ and $h = u$ where

$$K_l(d_1, d_2) \equiv d_1 + 2d_2 - \chi \pmod 3.$$

According to system (17), we have the following system of congruences for the numbers $L_{k,\gamma}$:

$$\sum_{\gamma=0}^{2}\sum_{k=1}^{u} D_{3u,k,\gamma} L_{k,\gamma} \equiv d_1 + 2d_2 - \chi \pmod 3,$$

$$\sum_{\gamma=0}^{2}\sum_{k=1}^{u} D_{l,k,\gamma} L_{k,\gamma} \equiv 0 \pmod{3^{\alpha(u,[(l+2)/3])}}, \qquad l = 1, \dots, 3u-1,$$

by virtue of equality (15), where $c_{l,1} \not\equiv 0 \pmod 3$, Lemma 7, and the first item of Lemma 8, this implies analogously to the congruences for the numbers $Q_{k,\gamma}$ the congruences $L_{k,\gamma} \equiv 0 \pmod{3^{\alpha(u,k)}}$ for all $k = 1, \dots, u$, $r = 0, 1, 2$, except for the number $L_{u,2}$, for which by the congruence $(c_{3u,1})^2 \equiv 1 \pmod 3$, we obtain the congruence

$$L_{u,2} \equiv c_{3u,1}(d_1 + 2d_2 - \chi) \pmod 3.$$

By the definition of the numbers $Y_{k,\gamma}$ and $L_{k,\gamma}$, we obtain from this that

$$X_{u,r} = \sum_{i=1}^{s_r} \binom{x_{i,r}}{u} \equiv T_u^{(r)} + c_{3u,1}(d_1 + 2d_2 - \chi) \pmod 3, \qquad r = 0, 1, 2,$$

$$X_{k,r} = \sum_{i=1}^{s_r} \binom{x_{i,r}}{k} \equiv T_k^{(r)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \dots, u-1, \quad r = 0, 1, 2,$$

where $T_k^{(r)}$, $k = 1, \dots, u$, $r = 0, 1, 2$, can be chosen as arbitrary integers. Since, by relation (12), $d_1 + 2d_2$ can assume not more than two values modulo 3, we have that by the inductive assumption, the numbers $T_k^{(r)}$ can be chosen so that the inequalities $s_r \ge q(u)$, $r = 0, 1, 2$, hold. Then, however, $d_1 \ge 1$ and $d_2 \ge 1$. According to relation (12), (d_1, d_2) can assume the values $(0,1)$, $(1,0)$, $(0,2)$, $(2,0)$, or $d_1 + d_2 \ge 3$. Hence, $d_1 + d_2 \ge 3$ and

$$s = s_0 + 2\big(q(u) - 1\big) + 3^{[n/2]}(d_1 + d_2) \ge q(n).$$

4. Let now $n = 3u + 1$. First, we consider the case of even n. Then u is odd and the inequality

$$\left[\frac{n}{2}\right] - \left[\frac{l+1}{2}\right] \ge \left[\frac{u-h+1}{2}\right] + 1 = \alpha(u,h)$$

holds in all the cases where $l = 3h, 3h-1, 3h-2$, $(h = 1, \dots, u)$, except for the case $l = 3h$, $h = u$. Therefore, $K_l(d_1, d_2) \equiv 0 \pmod{3^{\alpha(u,h)}}$ for $l = 3h, 3h-1, 3h-2$, $h = 1, \dots, u$, except for the case $l = 3h$, $h = u$ where

$$K_l(d_1, d_2) \equiv d_1 + 2d_2 + \mu_n \delta \pmod 3.$$

According to system (17), we have the following system of congruences for the numbers $L_{k,\gamma}$:

$$\sum_{\gamma=0}^{2}\sum_{k=1}^{u} D_{3u,k,\gamma}L_{k,\gamma} \equiv d_1 + 2d_2 + \mu_n\delta \pmod 3,$$

$$\sum_{\gamma=0}^{2}\sum_{k=1}^{u} D_{l,k,\gamma}L_{k,\gamma} \equiv 0 \pmod{3^{\alpha(u,[(l+2)/3])}}, \qquad l = 1, \dots, 3u-1,$$

by virtue of equality (15), where $c_{l,1} \not\equiv 0 \pmod 3$, Lemma 7, and the first item of Lemma 8, this implies analogously to the congruences of the numbers $Q_{k,\gamma}$ the congruences $L_{k,\gamma} \equiv 0 \pmod{3^{\alpha(u,k)}}$ for all $k = 1, \dots, u$, $r = 0, 1, 2$, except for the number $L_{u,2}$ for which, by the congruence $(c_{3u,1})^2 \equiv 1 \pmod 3$, we obtain the congruence

$$L_{u,2} \equiv c_{3u,1}(d_1 + 2d_2 + \mu_n\delta) \pmod 3.$$

Since δ can assume not more than two values modulo 3, we can represent $\mu_n\delta$ in the form $\mu_n\delta = \eta + \rho$, where ρ is fixed and η can assume only the values 0 and 1 modulo 3. By the definitions of the numbers $Y_{k,\gamma}$ and $L_{k,\gamma}$, we obtain from this that

$$\begin{aligned} X_{u,r} &= \sum_{i=1}^{s_r}\binom{x_{i,r}}{u} \equiv T_u^{(r)} + V(d_1 + 2d_2 + \eta) \pmod 3, \qquad r = 0, 1, 2, \\ X_{k,r} &= \sum_{i=1}^{s_r}\binom{x_{i,r}}{k} \equiv T_k^{(r)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \dots, u-1, \quad r = 0, 1, 2, \end{aligned} \tag{21}$$

where $V(m) = c_{3u,1}m$ and the numbers $T_k^{(r)}$ ($k = 1, \dots, u$, $r = 0, 1, 2$) can be chosen as arbitrary integers. By the inductive assumption for u, we choose the number $T_k^{(2)}$ so that $s_2 \geq q(u)$ is necessary for the solvability of the system of congruences

$$\sum_{i=1}^{s_2}\binom{x_{i,2}}{u} \equiv T_u^{(2)} + \sigma_2 \pmod 3,$$

$$\sum_{i=1}^{s_2}\binom{x_{i,2}}{k} \equiv T_k^{(2)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \dots, u-1,$$

independently of the fact that $\sigma_2 = V(1)$ or $\sigma_2 = V(-1)$. We choose the numbers $T_k^{(1)}$ so that $s_1 \geq q(u)$ is necessary for the solvability of the system of congruences

$$\sum_{i=1}^{s_1}\binom{x_{i,1}}{u} \equiv T_u^{(1)} + \sigma_1 \pmod 3,$$

$$\sum_{i=1}^{s_1}\binom{x_{i,1}}{k} \equiv T_k^{(1)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \dots, u-1,$$

independently of the fact that $\sigma_1 = V(0)$ or $\sigma_1 = V(-1)$. Finally, we choose the numbers $T_k^{(0)}$ so that $s_0 \geq q(u)$ is necessary for the solvability of the system of congruences

$$\sum_{i=1}^{s_0} \binom{x_{i,0}}{u} \equiv T_u^{(0)} + \sigma_0 \pmod 3,$$
$$\sum_{i=1}^{s_0} \binom{x_{i,0}}{k} \equiv T_k^{(0)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \dots, u-1,$$

independently on the fact that $\sigma_0 = V(0)$ or $\sigma_0 = V(1)$.

Let us return to system (21) with the chosen $T_k^{(r)}$. If $d_1 + d_2 \geq 3$, then

$$s = s_0 + 2\big(q(u) - 1\big) + 3^{[n/2]}(d_1 + d_2) \geq q(n).$$

Therefore, it is sufficient to restrict ourselves by the consideration of the cases with $d_1 + d_2 \leq 2$. Let $\eta = 0$. If $d_1 + 2d_2 \equiv -1 \pmod 3$, then (d_1, d_2) is equal to $(2, 0)$ or $(0, 1)$. Then however, by the choice of the numbers $T_k^{(2)}$ and $T_k^{(1)}$, the numbers s_2 and s_1 should be not less than $q(u)$; in turn, this implies that $d_2 \geq 1$ and $d_1 \geq 1$. Hence, $d_1 + 2d_2 \equiv 0 \pmod 3$ or $d_1 + 2d_2 \equiv 1 \pmod 3$. In both cases, by the choice of the numbers $T_k^{(0)}$, we have the inequality $s_0 \geq q(u)$. If $d_1 + 2d_2 \equiv 0 \pmod 3$, then (d_1, d_2) is equal to $(0, 0)$ or $(1, 1)$. However, the choice of the numbers $T_k^{(1)}$ implies that $d_1 \geq 1$, and thus, (d_1, d_2) can only be equal to $(1, 1)$. If $d_1 + 2d_1 \equiv 1 \pmod 3$, then (d_1, d_2) is equal to $(1, 0)$ or $(0, 2)$. It follows, however, from the choice of the numbers $T_k^{(2)}$ that $d_2 \geq 1$ and (d_1, d_2) can only be equal to $(0, 2)$. In both cases $d_1 + d_2 = 2$ and

$$s = s_0 + 2\big(q(u) - 1\big) + 3^{[n/2]}(d_1 + d_2) \geq q(n).$$

Let $\eta = 1$. If $d_1 + 2d_2 \equiv 1 \pmod 3$, then (d_1, d_2) is equal to $(1, 0)$ or $(0, 2)$, and $d_1 + 2d_2 + \eta \equiv -1 \pmod 3$. By the choice of the numbers $T_k^{(2)}$ and $T_k^{(1)}$, the numbers s_2 and s_1 in system (21) should be not less than $q(u)$; in turn, this implies that $d_2 \geq 1$ and $d_1 \geq 1$. Hence $d_1 + 2d_2 \equiv 0 \pmod 3$ or $d_1 + 2d_2 \equiv -1 \pmod 3$ and respectively, $d_1 + 2d_2 + \eta \equiv 1 \pmod 3$ or $d_1 + 2d_2 + \eta \equiv 0 \pmod 3$. In both cases, by the choice of the numbers $T_k^{(0)}$, we have the inequality $s_0 \geq q(u)$. In the case $d_1 + 2d_2 + \eta \equiv 1 \pmod 3$, only the value $(1, 1)$ is possible for (d_1, d_2) since, by the choice of the numbers $T_k^{(2)}$ it follows that $d_2 \geq 1$. Analogously, in the case $d_1 + 2d_2 + \eta \equiv 0 \pmod 3$, only the value $(2, 0)$ is possible for (d_1, d_2) since, by the choice of the numbers $T_k^{(1)}$, it follows that $d_1 \geq 1$. In both cases, the equality $d_1 + d_2 = 2$ holds, and hence,

$$s = s_0 + 2\big(q(u) - 1\big) + 3^{[n/2]}(d_1 + d_2) \geq q(n).$$

5. It remains to consider the induction step for odd $n = 3u + 1$. Then u is even, and the inequality

$$\left[\frac{n}{2}\right] - \left[\frac{l+1}{2}\right] \geq \left[\frac{u-h+1}{2}\right] + 1 = \alpha(u, h)$$

holds for $l = 3h, 3h-1, 3h-2$, $h = 1, \dots, u$, except for the cases where $l = 3h$ with $h = u$ and $h = u-1$, and $l = 3h-1$ with $h = u$. Hence the congruence

$$K_l(d_1, d_2) \equiv 0 \pmod{3^{\alpha(u,h)}}$$

holds for $l = 3h, 3h-1, 3h-2$ $(h = 1, \dots, u)$, except for the indicated cases where the following equalities hold:

$$K_{3u}(d_1, d_2) = -(d_1 + d_2), \qquad K_{3u-1}(d_1, d_2) = d_1 + 2d_2 - \chi, \qquad K_{3u-3}(d_1, d_2) = 3(d_1 + 2d_2 - \chi),$$

According to system (17), we have from this the following system of congruences for the numbers $L_{k,\gamma}$:

$$\begin{gathered}
\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{3u,k,\gamma} L_{k,\gamma} \equiv -(d_1 + d_2) \pmod 3, \\
\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{3u-1,k,\gamma} L_{k,\gamma} \equiv d_1 + 2d_2 - \chi \pmod 3, \\
\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{3u-2,k,\gamma} L_{k,\gamma} \equiv 0 \pmod 3, \\
\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{3u-3,k,\gamma} L_{k,\gamma} \equiv 3(d_1 + 2d_2 - \chi) \pmod{3^2}, \\
\sum_{\gamma=0}^{2} \sum_{k=1}^{u} D_{l,k,\gamma} L_{k,\gamma} \equiv 0 \pmod{3^{\alpha(u,[(l+2)/3])}}, \qquad l = 1, \dots, 3u-4,
\end{gathered}$$

Using representation (15), where $g_{k,l} = \Delta_3^{(k-1)}\binom{0}{l-1}$, and the bounds of $\nu_3(g_{k,l})$ from Lemmas 6 and 8, we reduce this system to the form

$$\begin{gathered}
c_{3u,1} L_{u,2} + c_{3u,2} L_{u,1} \equiv -(d_1 + d_2) \pmod 3, \\
c_{3u-1,1} L_{u,1} + c_{3u-1,2} L_{u,0} \equiv d_1 + 2d_2 - \chi \pmod 3, \\
c_{3u-2,1} L_{u,0} + c_{3u-2,2} L_{u-1,2} \equiv 0 \pmod 3, \\
c_{3u-3,1} g_{u,3u-4} L_{u,1} + (c_{3u-3,1} g_{u,3u-3} + c_{3u-3,2} g_{u,3u-4}) L_{u,0} \\
\quad + (c_{3u-3,1} + c_{3u-3,2} g_{u-1,3u-6}) L_{u-1,2} + c_{3u-3,2} L_{u-1,1} \equiv 3(d_1 + 2d_2 - \chi) \pmod{3^2}, \\
c_{3u-4,1} g_{u,3u-4} L_{u,0} + c_{3u-4,1} g_{u-1,3u-6} L_{u-1,2} + (c_{3u-4,1} + c_{3u-4,2} g_{u-1,3u-6}) L_{u-1,1} \\
\quad + c_{3u-4,2} L_{u-1,0} \equiv 0 \pmod{3^2}, \\
c_{3u-5,1} g_{u-1,3u-6} L_{u-1,1} + (c_{3u-5,1} + c_{3u-5,2} g_{u-1,3u-6}) L_{u-1,0} + c_{3u-5,2} L_{u-2,2} \equiv 0 \pmod{3^2}, \\
\sum_{\gamma=0} \sum_{k=1}^{u-2} D_{l,k,\gamma} L_{k,\gamma} \equiv 0 \pmod{3^{\alpha(u,[(l+2)/3])}}, \qquad l = 1, \dots, 3u-6,
\end{gathered}$$

where $L_{u-2,2} = 0$ if $u = 2$. Considering the obtained system modulo 3, we have with the help of Lemma 6 and the condition $c_{l,1} \not\equiv 0 \pmod 3$ that

$$L_{u,0} \equiv 0 \pmod 3, \quad L_{k,r} \equiv 0 \pmod 3, \qquad k = 1, \dots, u-1, \quad r = 0,1,2. \tag{22}$$

Moreover, the congruences

$$L_{k,r} \equiv 0 \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \dots, u-2, \quad r = 0,1,2, \tag{23}$$

follow from the congruences of the system which correspond to $l = 1, \dots, 3u-6$. Substituting congruences (22) and (23) into the system, we arrive at the congruences

$$c_{3u,1}L_{u,2} + c_{3u,2}L_{u,1} \equiv -(d_1 + d_2) \pmod 3, \qquad c_{3u-1,1}L_{u,1} \equiv d_1 + 2d_2 - \chi \pmod 3, \tag{24}$$

$$L_{u-1,0} \equiv L_{u-1,1} \equiv 0 \pmod{3^2}, \qquad L_{u-1,2} \equiv 3(d_1 + 2d_2 - \chi) - g_{u,3u-4}L_{u,1} \pmod{3^2}. \tag{25}$$

Now we shall need explicit expressions for the numbers $c_{3u,1}$, $c_{3u,2}$, and $c_{3u-1,1}$. According to the definition of the numbers $c_{l,1}$ and $c_{l,2}$, Lemma 3, and the notation of Lemma 9,

$$c_{l,1} = \mu_l, \qquad c_{l,2} = -\mu_{l+1}, \qquad c_{l+1,1} = c_{l,1}, \qquad c_{l+1,2} = c_{l,2}$$

for even l. Since u is even, we obtain from this and Lemma 9

$$c_{3u-1,1} = c_{3u,1} = (-1)^{[u/2]} \times 2, \qquad c_{3u,2} = (-1)^{[u/2]},$$

then we express $L_{u,1}$ and $L_{u,2}$ from congruences (24) as follows:

$$L_{u,1} \equiv -(-1)^{[u/2]}(d_1 + 2d_2 - \chi) \pmod 3, \qquad L_{u,2} \equiv (-1)^{[u/2]}(-d_2 + \chi) \pmod 3.$$

We rewrite the congruence in relations (25) for $L_{u-1,2}$ in the form

$$L_{u-1,2} \equiv 3\lambda(d_1 + 2d_2 - \chi) \pmod{3^2}, \tag{26}$$

where λ is a certain integer.

By the definition of the numbers $Y_{k,\gamma}$ and $L_{k,\gamma}$, we obtain the following congruences from congruences (23)–(26):

$$X_{u,2} \equiv T_u^{(2)} - d_2 \pmod 3, \qquad X_{u,1} \equiv T_u^{(1)} - d_1 \pmod 3, \qquad X_{u,0} \equiv T_u^{(0)} + d_1 + d_2 \pmod 3,$$

$$X_{u-1,r} \equiv T_{u-1}^{(r)} + 3\lambda(d_1 + 2d_2 - \chi) \pmod{3^2}, \qquad r = 0,1,2,$$

$$X_{k,r} \equiv T_k^{(r)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \dots, u-2, \quad r = 0,1,2,$$

where we denote

$$X_{k,r} = \sum_{i=1}^{s_r} \binom{x_{i,r}}{k}.$$

Since n is an odd number of type (B), according to congruence (12), $\eta = \lambda(d_1 + 2d_2 - \chi)$ can have not more than two values modulo 3. By the induction assumption, for an even u of type (B), we choose the numbers $T_k^{(r)}$ for each $r = 0, 1, 2$ so that $s_r \geq q(u)$ is necessary for the solvability of the system of congruences

$$\sum_{i=1}^{s_r} \binom{x_{i,r}}{u} \equiv T_u^{(r)} \pmod 3, \qquad \sum_{i=1}^{s_r} \binom{x_{i,r}}{u-1} \equiv T_{u-1}^{(r)} + 3\eta \pmod{3^2},$$
$$\sum_{i=1}^{s_r} \binom{x_{i,r}}{k} \equiv T_k^{(r)} \pmod{3^{\alpha(u,k)}}, \qquad k = 1, \dots, u-2,$$

independently of η. It is easy to see that the conditions $d_2 \geq 1$ and $d_1 \geq 1$ are necessary for the chosen numbers $T_k^{(r)}$. But since, by relation (12), $d_1 + 2d_2$ is congruent to 1 or -1 modulo 3, we have $d_1 + d_2 \geq 3$, and therefore,

$$s = s_0 + 2\big(q(u) - 1\big) + 3^{[n/2]}(d_1 + d_2) \geq q(n).$$

By the induction principle, the assertion of the theorem is valid for all natural numbers of types (A) and (B).

Corollary. *Suppose that s and n are two natural numbers and n belongs to one of two types* (A) *or* (B) *depending on the fact that all positive numbers of series* (2) *are simultaneously odd or are not simultaneously divisible by* 3. *Then there exist integers $A_1, \dots, A_n$ such that the condition $s \geq q(n)$, where the function $q(n)$ is defined recursively in relations* (1), *is necessary for the solvability of the system of equations of degree n*

$$\sum_{i=1}^{s} \binom{x_j}{j} \equiv A_j, \qquad j = 1, \dots, n,$$

in integer 3*-adic numbers $x_1, \dots, x_s$.*

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