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The Function $\widetilde{\zeta}_{-1}(s)$ and Formulas of the Second Generation for $\pi(x)$ and $R(x)$

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INTRODUCTION

Recall that in the asymptotic formula

$$\pi(x) = \int_2^x \frac{du}{\ln u} + R(x), \tag{1}$$

using the Riemann theory of zeta-function, de la Vallee–Poussin obtained in 1885 the estimate

$$R(x) = O\big(xe^{-c\sqrt{\ln x}}\big). \tag{2}$$

In this case, the Schmidt estimate

$$R(x) = O\big(\sqrt{x}\ln x\big) \tag{3}$$

is a consequence of Riemann's conjecture. However, 100 years passed since that time, but no essential result concerning (2), (3), at least $R(x) = x^{0.999\,999}$, was obtained (see [2, p. 13] and Gel'fond's remarks [1, p. 287]).

This situation is the basis

(a) for a more intensive study of the Riemann zeta-function,
(b) for constructing new *models* in which a proper analog of Riemann's conjecture would be valid and proper formulas for basic functions would hold true.

In [4, 5], a model is studied which is based on a "part of the function $\zeta(s)$"

$$\widetilde{\zeta}(s) = \widetilde{\zeta}(s; P, \beta) = \zeta_1(s) + \chi(s)\zeta_2(s),$$

$$\zeta_1(s) = \prod_{p\le P}\sum_{k=0}^{\beta}\frac{1}{p^{sk}}, \qquad \zeta_2(s) = \prod_{p\le P}\sum_{k=0}^{\beta}\frac{1}{p^{(1-s)k}},$$

$$\chi(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}, \qquad s\ne 2k+1, \quad k = 0,1,2,\dots, \quad s = \sigma + it,$$

$$P = (\ln P_0)^{1-\varepsilon}, \qquad B = [(\ln P_0)^{2\varepsilon/3}], \qquad P_0 = \sqrt{\frac{T}{2\pi}},$$

where ε is an arbitrarily small positive number and p is a prime number.

Let

$$D = D(T,H,K) = \{s : \sigma \in \langle -K, K\rangle,\ t \in \langle T, T+H\rangle\}, \qquad K > 1, \quad T > 0, \quad H \leq \sqrt{T} \tag{4}$$

(the condition $H \leq \sqrt{T}$ is not essential and only express a tendency to localization), and let

$$D(\Delta_0) = \left\{s : \frac{1}{2} + \Delta_0 \leq \sigma \leq K,\ t \in \langle T, T+H\rangle\right\}, \qquad \Delta_0 = \frac{A}{(\ln P_0)^{\varepsilon/2}}. \tag{5}$$

The following theorems were proved in [5].

Theorem A. *For all sufficiently large T*

$$\widetilde{\zeta}(s) \neq 0, \qquad s \in D, \quad \sigma \neq \frac{1}{2},$$

i.e., an analog of Riemann's conjecture is valid for $\widetilde{\zeta}(s)$, $s \in D$, $T \to \infty$

Theorem B. *The function $\pi(x)$, $x \in \langle 2, P\rangle$, is a solution to the integral equation*

$$\ln \widetilde{\zeta}(s) = s\int_2^P \frac{\Phi(x)}{x(x^s-1)}\,dx - \Phi(P)\ln\left(1 - \frac{1}{P^s}\right) + \Omega_1(s), \tag{6}$$

(for any fixed $s \in D(\Delta_0)$), where

$$\Omega_1(s) = \sum_{p\leq P} \ln\left(1 - \frac{1}{p^{s(\beta+1)}}\right) + \ln\left\{1 + \frac{\chi(s)\zeta_2(s)}{\zeta_1(s)}\right\} = O(e^{-A\beta}).$$

Theorem C. *The function $R(x)$, $x \in \langle 2, P\rangle$, is a solution to the perturbed integral equation*

$$\ln \widetilde{\zeta}(s) = s\int_2^P \frac{\Phi(x)}{x(x^s-1)}\,dx - \Phi(P)\ln\left(1 - \frac{1}{P^s}\right) + \Omega_1(s) + \Omega_2(s), \tag{7}$$

where

$$\Omega_2(s) = \sum_{m=0}^{\infty} \frac{1}{m+1}\int_2^P \frac{dx}{x^{(m+1)s}\ln x} = O\left(\frac{\ln T}{T}\right), \qquad s \in D(\Delta_0), \tag{8}$$

is a very small perturbation.

The problems of distribution of zeros of the function $\widetilde{\zeta}(1/2+it)$ were studied in [4].

However, equations (6), (7) for the functions $\pi(x)$, $R(x)$, respectively, have the following properties:

(c) they have a complicated structure, i.e., they contain only the terms $\ln\widetilde{\zeta}(s)$ and $\Omega_1(s)$,

(d) they are not applicable on the critical line $\sigma = 1/2$ (see (5)).

In view of these circumstances, we shall consider a new model in this paper.

1. FUNCTION $\widetilde{\zeta}_{-1}(s)$

Actually, a whole class of functions of the type $\widetilde{\zeta}(s)$ was introduced in [5]. For example, let

$$\widetilde{\zeta}_{-1}(s)=\prod_{p\le P}\Big(1-\frac{1}{p^s}\Big)+\chi(s)\prod_{p\le P}\Big(1-\frac{1}{p^{1-s}}\Big)=\sum_{n<P_0}{}' \frac{\mu(n)}{n^s}+\chi(s)\sum_{n<P_0}{}' \frac{\mu(n)}{n^{1-s}}, \qquad s\neq 2k+1, \qquad k=0,1,2,\dots, \tag{9}$$

where $\mu(n)$ is the Möbius function and

$$P=[(\ln P_0)^{1-\varepsilon}], \qquad P_0=\sqrt{\frac{T}{2\pi}}. \tag{10}$$

Recall that the function $\widetilde{\zeta}(s)$ turned out to be a "part of $\zeta(s)$", i.e., a part of the approximate functional equation for $\zeta(s)$. In this sense, the function $\widetilde{\zeta}_{-1}(s)$ is not a "part of $\zeta(s)$", but, in turn, this function is constructed from the parts of the Euler product

$$\frac{1}{\zeta(s)}=\prod_p\Big(1-\frac{1}{p^s}\Big), \qquad \sigma>1.$$

Hence the function $\widetilde{\zeta}_{-1}(s)$ preserves the relationship with the function $\zeta(s)$.

The following lemma holds true.

Lemma 1. *For all sufficiently large T*

$$\widetilde{\zeta}_{-1}(s)\neq 0, \qquad s\in D, \quad \sigma\neq\frac{1}{2}$$

(see (4)), i.e., an analog of the Riemann conjecture is valid for $\widetilde{\zeta}_{-1}(s)$, $s\in D$, $T\to\infty$.

Proof. The proof of this assertion can be obtained by analogy with the proof of a similar theorem for the function $\widetilde{\zeta}(s)$ given in [5]. However, in view of the independence of $\widetilde{\zeta}_{-1}(s)$ of β this proof is simpler.

In the subsequent sections of this paper, we consider some properties of the model which is based on the function $\widetilde{\zeta}_{-1}(s)$.

2. ZEROS OF THE FUNCTION $\widetilde{\zeta}_{-1}(1/2+it)$ AND THE FORMULA FOR $\pi(x)$

The properties of zeros of the function $\widetilde{\zeta}_{-1}(s)$ on the critical line $\sigma=1/2$, i.e., of zeros of the function

$$e^{i\varphi(t)}\widetilde{\zeta}_{-1}\Big(\frac{1}{2}+it\Big)=\widetilde{Z}_{-1}(t;P), \qquad t\in\langle T,T+H\rangle, \tag{11}$$

where

$$\chi\Big(\frac{1}{2}+it\Big)=e^{-i2\varphi(t)}, \qquad \varphi(t)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+O\Big(\frac{1}{t}\Big), \tag{12}$$

follow now from the formula (see (9), (12), cf. [4, (22)])

$$\widetilde{Z}_{-1}(t;P) = 2B_{-1}(t;P)\cos\{\varphi(t) + C_{-1}(t;P)\}, \tag{13}$$

$$\begin{aligned} B_{-1}(t) &= \prod_{p\le P}\left|1 - \frac{p^{-it}}{\sqrt{p}}\right| = \exp\left(\mathrm{Re}\left\{\sum_{p\le P}\ln\left(1 - \frac{p^{-it}}{\sqrt{p}}\right)\right\}\right), \\ C_{-1}(t) &= \sum_{p\le P}\arg\left(1 - \frac{p^{-it}}{\sqrt{p}}\right) = \mathrm{Im}\left\{\sum_{p\le P}\ln\left(1 - \frac{p^{-it}}{\sqrt{p}}\right)\right\}. \end{aligned} \tag{14}$$

Consequently, the zero $\gamma_\nu = \gamma_\nu^{-1}(P)$ of the function $\widetilde{Z}_{-1}(t,P)$ is defined by the equation (see (13))

$$\varphi(\gamma_\nu) + C_{-1}(\gamma_\nu) = \pi\nu + \frac{\pi}{2}, \qquad \nu = 1,2,\dots . \tag{15}$$

By analogy with case of the function $\widetilde{\zeta}(s)$ (cf. [4, (11)]), we obtain from (15) the following lemma

Lemma 2. *For the subsequent zeros $\rho_\nu = 1/2 + i\gamma_\nu$, $\rho_{\nu+1} = 1/2 + i\gamma_{\nu+1}$ of the function $\widetilde{\zeta}_{-1}(s)$ the asymptotic formula*

$$\gamma_{\nu+1} - \gamma_\nu = \frac{\pi}{\ln P_0} + O\left\{\frac{1}{(\ln P_0)^{3/2}}\right\}, \qquad \gamma_\nu, \gamma_{\nu+1} \in \langle T, T+H\rangle, \tag{16}$$

holds true as $T \to \infty$.

Remark 1. It follows from (16) that the zeros of the function $\widetilde{\zeta}_{-1}(s)$ are distributed locally in a regular way (cf. [4, (13)]). Hence, the zeros of the complicated transcendental equation

$$\sum_{n<P_0}{}' \frac{\mu(n)}{\sqrt{n}}\cos\{\varphi(t) - t\,\ln n\} = 0$$

(see (9), (11)) are distributed locally in a regular way on the interval $\langle T, T+H\rangle$.

Furthermore, the following lemma holds.

Lemma 3. *We have*

$$-\int_2^P \mathrm{Im}\left\{\frac{\rho_\nu}{x^{\rho_\nu} - 1}\right\}\frac{\pi(x)}{x}\,dx + \pi(P)\,\mathrm{Im}\left\{\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right)\right\} = \varphi\left[t_\nu\left(\frac{\pi}{2}\right)\right] - \varphi(\gamma_\nu), \tag{17}$$

where $\gamma_\nu \in \langle T, T+H\rangle$, $H \le \sqrt{T}$, and $\{t_\nu(\pi/2)\}$ is a sequence satisfying the condition

$$\varphi\left[t_\nu\left(\frac{\pi}{2}\right)\right] = \pi\nu + \frac{\pi}{2} \tag{18}$$

(see [3]).

Proof. We write formula (15) in the form

$$\mathrm{Im}\left\{\sum_{p\le P}\ln\left(1 - \frac{1}{p^{\rho_\nu}}\right)\right\} = \varphi\left[t_\nu\left(\frac{\pi}{2}\right)\right] - \varphi(\gamma_\nu) \tag{19}$$

(see (14), (18)). Next from the formula

$$\ln\left(1-\frac{1}{(n+1)^{\rho_\nu}}\right)-\ln\left(1-\frac{1}{n^{\rho_\nu}}\right)=\rho_\nu\int_n^{n+1}\frac{dx}{x(x^{\rho_\nu}-1)} \tag{20}$$

we obtain

$$\begin{aligned}\sum_{p\le P}\ln\left(1-\frac{1}{p^{\rho_\nu}}\right)&=\sum_{n=2}^{P}\{\pi(n)-\pi(n-1)\}\ln\left(1-\frac{1}{n^{\rho_\nu}}\right)\\&=\sum_{n=2}^{P-1}\pi(n)\left\{\ln\left(1-\frac{1}{n^{\rho_\nu}}\right)-\ln\left(1-\frac{1}{(n+1)^{\rho_\nu}}\right)\right\}+\pi(P)\ln\left(1-\frac{1}{P^{\rho_\nu}}\right)\\&=-\rho_\nu\int_2^P\frac{\pi(x)\,dx}{x(x^{\rho_\nu}-1)}+\pi(P)\ln\left(1-\frac{1}{P^{\rho_\nu}}\right).\end{aligned} \tag{21}$$

Now, by virtue of (20), (21), formula (19) yields (17).

Remark 2. Formula (17) is a formula of a new type for $\pi(x)$. It is constructed in accordance with the values of relatively simple functions

$$\operatorname{Im}\left\{\frac{s}{x^s-1}\right\},\qquad \operatorname{Im}\left\{\ln\left(1-\frac{1}{P^s}\right)\right\},\qquad \varphi(t)$$

for terms of the well-defined sequences $\{t_\nu(\pi/2)\}$ and $\{\gamma_\nu\}$.

3. A NEW FORMULA FOR $R(x)$

The following lemma holds.

Lemma 4. *Let*

$$U(x)=\int_2^x\frac{du}{\ln u},\qquad x\in\langle 2,P\rangle. \tag{22}$$

Then we have

$$\begin{aligned}s\int_2^P U(x)\,\frac{dx}{x(x^s-1)}&=U(P)\ln\left(1-\frac{1}{P^s}\right)+\sum_{n=0}^{\infty}\frac{1}{n+1}\int_2^P\frac{dx}{x^{(n+1)s}\ln x}\\&=U(P)\ln\left(1-\frac{1}{P^s}\right)+O\left(\frac{\sqrt{\ln T}}{T}\right)\end{aligned} \tag{23}$$

for $\sigma\in\langle 1/2,K\rangle$ (cf. (8)).

Proof. Let

$$V(x)=V(x,s)=\int_2^x\frac{dv}{v(v^s-1)}, \tag{24}$$

then

$$\int\limits_2^P U(x)\frac{dx}{x(x^s-1)} = U(P)V(P) - \int\limits_2^P \frac{V(x)}{\ln x}\,dx. \tag{25}$$

Since the summation of relations (20) for $\rho_\nu \to s$, $n=2,\dots,P-1$, gives the formula

$$\ln\left(1-\frac{1}{P^s}\right) = s\int\limits_2^P \frac{dx}{x(x^s-1)} + \ln\left(1-\frac{1}{2^s}\right),$$

we have

$$U(P)V(P) = U(P)\int\limits_2^P \frac{dx}{x(x^s-1)} = \frac{U(P)}{s}\left\{\ln\left(1-\frac{1}{P^s}\right) - \ln\left(1-\frac{1}{2^s}\right)\right\}. \tag{26}$$

Furthermore,

$$\begin{aligned} V(x) &= \int\limits_2^x \sum_{n=0}^\infty \frac{1}{v^{(n+1)s+1}}\,dv = \frac{1}{s}\sum_{n=0}^\infty \frac{1}{n+1}\left(\frac{1}{2^{(n+1)s}} - \frac{1}{x^{(n+1)s}}\right) \\ &= -\frac{1}{s}\ln\left(1-\frac{1}{2^s}\right) - \frac{1}{s}\sum_{n=0}^\infty \frac{1}{(n+1)x^{(n+1)s}} \end{aligned}$$

(see (24)) and

$$-s\int\limits_2^P \frac{V(x)}{\ln x}\,dx = U(P)\ln\left(1-\frac{1}{2^s}\right) + \sum_{n=0}^\infty \frac{1}{n+1}\int\limits_2^P \frac{dx}{x^{(n+1)s}\ln x} \tag{27}$$

(see (22)). Hence (see (25)–(27)),

$$s\int\limits_2^P U(x)\,\frac{dx}{x(x^s-1)} = U(P)\ln\left(1-\frac{1}{P^s}\right) + \sum_{n=0}^\infty \frac{1}{n+1}\int\limits_2^P \frac{dx}{x^{(n+1)s}\ln x}. \tag{28}$$

Since

$$\begin{aligned} \int\limits_2^P x^{-(n+1)s}\,\frac{dx}{\ln x} &= \frac{1}{(n+1)s-1}\left\{\frac{1}{2^{(n+1)s-1}\ln 2} - \frac{1}{P^{(n+1)s-1}\ln P} - \int\limits_2^P \frac{dx}{x^{(n+1)s}\ln^2 x}\right\} \\ &= \frac{1}{(n+1)s-1}\{O(1)+O(\sqrt{P})+O(\sqrt{P})\} = O\left\{\frac{\sqrt{P}}{(n+1)T}\right\}, \end{aligned}$$

we have (see (10))

$$\Omega(s,P) = \sum_{n=0}^\infty \frac{1}{n+1}\int\limits_2^P \frac{dx}{x^{(n+1)s}\ln x} = \sum_{n=0}^\infty O\left\{\frac{\sqrt{P}}{(n+1)^2T}\right\} = O\left(\frac{\sqrt{\ln T}}{T}\right). \tag{29}$$

Now, by virtue of (29), (23) follows from (28). This completes the proof of Lemma 4.

Setting $\pi(x) = U(x) + R(x)$ (see (1), (22)) in the expression

$$W = -\int_2^P \frac{\rho_\nu}{x^{\rho_\nu} - 1} \frac{\pi(x)}{x}\,dx + \pi(P)\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right), \tag{30}$$

we find by Lemma 4 ($s = \rho_\nu$) that

$$\begin{aligned} W &= -\int_2^P \frac{\rho_\nu}{x^{\rho_\nu} - 1} \frac{R(x)}{x}\,dx + R(P)\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right) - \rho_\nu \int_2^P \frac{U(x)}{x(x^{\rho_\nu} - 1)}\,dx + U(P)\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right) \\ &= -\int_2^P \frac{\rho_\nu}{x^{\rho_\nu} - 1} \frac{R(x)}{x}\,dx + R(P)\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right) + O\left(\frac{\sqrt{\ln T}}{T}\right). \end{aligned} \tag{31}$$

Thus, we have proved the following theorem (see (17), (30), and (31))

Theorem 1. *We have*

$$-\int_2^P \mathrm{Im}\left\{\frac{\rho_\nu}{x^{\rho_\nu} - 1}\right\} \frac{R(x)}{x}\,dx + R(P)\,\mathrm{Im}\left\{\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right)\right\} + O\left(\frac{\sqrt{\ln T}}{T}\right) = \varphi\left[t_\nu\left(\frac{\pi}{2}\right)\right] - \varphi(\gamma_\nu),$$
$$T \to \infty, \tag{32}$$

for $\gamma_\nu \in \langle T, T+H\rangle$, $H \le \sqrt{T}$.

4. RAPPROCHEMENT OF INTEGRAL EQUATIONS

From the lemma, we obtain

Corollary 1. *The function* $\pi(x)$, $x \in \langle 2, P\rangle$, *is a solution to the integral equation*

$$-\int_2^P \mathrm{Im}\left\{\frac{\rho_\nu}{x^{\rho_\nu} - 1}\right\} \frac{\Phi(x)}{x}\,dx + \Phi(P)\,\mathrm{Im}\left\{\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right)\right\} = \varphi\left[t_\nu\left(\frac{\pi}{2}\right)\right] - \varphi(\gamma_\nu) \tag{33}$$

(*for any fixed* $\gamma_\nu \in \langle T, T+H\rangle$).

Next, we obtain the following corollary from Theorem 1.

Corollary 2. *The function* $R(x)$, $x \in \langle 2, P\rangle$, *is a solution to the perturbed integral equation*

$$-\int_2^P \mathrm{Im}\left\{\frac{\rho_\nu}{x^{\rho_\nu} - 1}\right\} \frac{\Phi(x)}{x}\,dx + \Phi(P)\,\mathrm{Im}\left\{\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right)\right\} + \Omega(\rho_\nu, P) = \varphi\left[t_\nu\left(\frac{\pi}{2}\right)\right] - \varphi(\gamma_\nu) \tag{34}$$

(*for any fixed* $\gamma_\nu \in \langle T, T+H\rangle$), *where*

$$\Omega(\rho_\nu, P) = \sum_{n=0}^{\infty} \frac{1}{n+1} \int_2^P \frac{dx}{x^{(n+1)\rho_\nu}\ln x} = O\left(\frac{\ln T}{T}\right) \tag{35}$$

is a very small perturbation (cf. (8), (29)).

Remark 3. In the model based on the function $\widetilde{\zeta}_{-1}(s)$, the rapprochement of the integral equations for $\pi(x)$ and $R(x)$ (equation (34) is obtained from (33) via a very small perturbation of (35)) can be observed in a "purer" form (compare with (c) in Introduction) than in the model based on the function $\widetilde{\zeta}(s)$.

Recall that the functions $\pi(x)$ and $R(x)$ behave quite differently, namely, $\pi(x) \sim \frac{x}{\ln x}$, $x \to \infty$, and $R(x)$ changes sign infinitely many times as $x \to \infty$ (Littlewood, 1914).

Remark 4. Obviously, there is a certain instability of solutions to the integral equations (33) and (34) which is inherent in the behavior of the functions $\pi(x)$ and $R(x)$ as $x \to \infty$.

Remark 5. We can naturally assume that for any fixed $\gamma_\nu \in \langle T, T+H \rangle$ and for the corresponding $t_\nu(\pi/2)$, there exists a set of solutions to the integral equation (33) of the form

$$\Phi\left[x; \gamma_\nu, t_\nu\left(\frac{\pi}{2}\right), P\right], \qquad x \in \langle 2, P \rangle.$$

However, then the existing solution $\Phi(x) = \pi(x)$, $x \in \langle 2, P \rangle$, to the integral equation (33) must be regarded as a certain "singular solution"; the same is true of equation (34).

Question 1. Does the homogeneous integral equation

$$\int_2^P \operatorname{Im}\left\{\frac{\rho_\nu}{x^{\rho_\nu} - 1}\right\} \frac{\Phi(x)}{x}\, dx = \Phi(P) \operatorname{Im}\left\{\ln\left(1 - \frac{1}{P^{\rho_\nu}}\right)\right\}$$

have a nontrivial solution $\Phi(x) = \Phi(x; \gamma_\nu, P)$?

5. "EXPLICIT FORMULAS" AND "IMPLICIT FORMULAS"

Recall that there exists a class of "explicit formulas" in the theory based on the function $\zeta(s)$. For example, the "explicit formula" of von Mangoldt

$$\psi_0(x) = \frac{\psi(x+0) + \psi(x-0)}{2} = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\ln\left(1 - \frac{1}{x^2}\right), \qquad x > 1,$$

where ρ runs through all zeros of the function $\zeta(s)$ lying in the critical strip, is related to the Chebyshev function $\psi(x)$ (cf. [6, p. 76, 77 and Subsection 7, p. 80]).

In this context, formulas (17) and (32) are called "implicit formulas."

Remark 6. The "implicit formulas" (17), (32), which connect the functions $\pi(x)$ and $R(x)$ with any fixed critical zero $\rho_\nu = 1/2 + i\gamma_\nu$, $\gamma_\nu \in \langle T, T+H \rangle$, of the function $\widetilde{\zeta}_{-1}(s)$ and with the corresponding value of $t_\nu(\pi/2)$ are formulas of a new type and have no analogs in the theory based on the Riemann zeta-function.

It stands to reason that the Riemann formula

$$\Pi_0(x) = \operatorname{li} x - \sum_\rho \operatorname{li} x^\rho + \int_x^\infty \frac{du}{(u^2+1)u \ln u} - \ln 2, \qquad x > 1,$$

where

$$\Pi_0(x) = \frac{\Pi(x+0)+\Pi(x-0)}{2}, \qquad \Pi(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots,$$

connects $\Pi_0(x)$ with all zeros, namely, with an infinite series of zeros, of the function $\zeta(s)$, which lie in the critical strip $\sigma \in (0,1)$ (see [6, p. 81]).

Remark 7. "Implicit formulas" were obtained irrespective of the ordinary operations in a complex domain, namely, the operation of the displacement of the integration line and the residue theorem.

6. GENERALIZATION OF THE FUNDAMENTAL FORMULAS (17) AND (32)

Suppose now that $\{\gamma_{2\nu}(\alpha)\}$ is a sequence satisfying the condition (cf. (15))

$$\varphi[\gamma_{2\nu}(\alpha)] + C_{-1}[\gamma_{2\nu}(\alpha)] = 2\nu\pi + \alpha, \qquad \alpha \in \langle -\pi, \pi \rangle \tag{36}$$

(the existence of $\{\gamma_{2\nu}(\alpha)\}$ is a consequence of the increase of the function $\varphi(t) + C_{-1}(t)$; cf. [4]). Since

$$\left\{\gamma_{2\nu}\left(\frac{\pi}{2}\right)\right\} \cup \left\{\gamma_{2\nu}\left(-\frac{\pi}{2}\right)\right\} = \{\gamma_\nu\},$$

$\{\gamma_{2\nu}(\alpha)\}$ is a generalization of the sequence $\{\gamma_\nu\}$.

Note that in the case $t = \gamma_{2\nu}(\alpha)$, formula (13) yields

$$\widetilde{Z}_{-1}[\gamma_{2\nu}(\alpha); P] = 2B_{-1}[\gamma_{2\nu}(\alpha); P]\cos\alpha. \tag{37}$$

Formula (36) gives the following results (cf. Lemmas 2 and 3 and Theorem 1).

Lemma 5. *For the adjacent terms of the sequence $\{\gamma_{2\nu}(\alpha)\}$ we have the asymptotic formula*

$$\gamma_{2\nu+2}(\alpha) - \gamma_{2\nu}(\alpha) = \frac{2\pi}{\ln P_0} + O\left\{\frac{1}{(\ln P_0)^{3/2}}\right\}, \qquad \gamma_{2\nu}(\alpha), \gamma_{2\nu+2}(\alpha) \in \langle T, T+H \rangle, \quad T \to \infty.$$

Lemma 6. *We have*

$$-\int_2^P \operatorname{Im}\left\{\frac{\rho_{2\nu}(\alpha)}{x^{\rho_{2\nu}(\alpha)} - 1}\right\}\frac{\pi(x)}{x}\,dx + \pi(P)\operatorname{Im}\left\{\ln\left(1 - \frac{1}{P^{\rho_{2\nu}(\alpha)}}\right)\right\} = \varphi[t_{2\nu}(\alpha)] - \varphi[\gamma_{2\nu}(\alpha)], \tag{38}$$

where $\{t_{2\nu}(\alpha)\}$ satisfies the condition

$$\varphi[t_{2\nu}(\alpha)] = 2\nu\pi + \alpha$$

(see [3]), *and* $\rho_{2\nu}(\alpha) = 1/2 + i\gamma_{2\nu}(\alpha)$.

Theorem 2. *We have*

$$\begin{aligned} -\int_2^P \operatorname{Im}\left\{\frac{\rho_{2\nu}(\alpha)}{x^{\rho_{2\nu}(\alpha)} - 1}\right\}\frac{R(x)}{x}\,dx + R(P)\operatorname{Im}\left\{\ln\left(1 - \frac{1}{P^{\rho_{2\nu}(\alpha)}}\right)\right\} \\ + O\left(\frac{\sqrt{T}}{T}\right) = \varphi[t_{2\nu}(\alpha)] - \varphi[\gamma_{2\nu}(\alpha)], \qquad T \to \infty. \end{aligned} \tag{39}$$

In addition, we note that the Corollary 3 follows from (37) (see (14)).

Corollary 3. *For $\alpha \neq \pm\pi/2$, the formula*

$$\ln\left\{\frac{\widetilde{Z}_{-1}[\gamma_{2\nu}(\alpha);P]}{2\cos\alpha}\right\} = -\int_2^P \mathrm{Re}\left\{\frac{\rho_{2\nu}(\alpha)}{x^{\rho_{2\nu}(\alpha)}-1}\right\}\frac{\pi(x)}{x}\,dx + \pi(P)\,\mathrm{Re}\left\{\ln\left(1-\frac{1}{P^{\rho_{2\nu}(\alpha)}}\right)\right\}$$

holds as well as the corresponding formula for the remainder $R(x)$.

7. DISCRETE THEORY OF THE FUNCTIONS $\pi(x)$ AND $R(x)$

Let

$$\langle m+1, m+2) = \bigcup_{n=1}^{N} I_{mn}, \qquad I_{mn} = \langle a_{mn}, b_{mn}), \quad N = 1,2,\dots,$$

and let

$$\langle 2, P) = \bigcup_{m=1}^{P-2}\bigcup_{n=1}^{N} I_{mn} = \bigcup_{k=1}^{K_0} \overline{I}_k, \qquad |\overline{I}_k| = \overline{b}_k - \overline{a}_k, \tag{40}$$

be a subdivision of the interval $\langle 2, P)$ into $K_0 = N(P-2)$ nonoverlapping parts. Note that the formula

$$M_0 \sim \frac{1}{\pi} H \ln P_0, \qquad T \to \infty,$$

holds for the number M_0 of zeros of the the function $\widetilde{\zeta}_{-1}(1/2+it)$, $t \in \langle T, T+H\rangle$, (see (16)). Suppose, for example, that $H = \sqrt{T}$, and let K_0 satisfy the condition

$$K_0 + 1 \leq (1-\varepsilon)\frac{1}{\pi}\sqrt{T}\,\ln P_0,$$

which, in turn, holds for

$$N \leq (1-2\varepsilon)\frac{1}{\pi}\,\frac{\sqrt{T}\,\ln P_0}{P-2} = N_0. \tag{41}$$

Since $\pi(x)$, $x \in \langle 2, P\rangle$ preserves the sign, we have

$$\int_2^P \mathrm{Im}\left\{\frac{\rho_\nu}{x^{\rho_\nu}-1}\right\}\frac{1}{x}\pi(x)\,dx = \int_2^P J(x,\gamma_\nu)\pi(x)\,dx = \sum_{k=1}^{K_0}\int_{\overline{I}_k} J(x,\gamma_\nu)\pi(x)\,dx$$

$$= \sum_{k=1}^{K_0} J(\xi_k^\nu,\gamma_\nu)\int_{\overline{I}_k}\pi(x)\,dx = \sum_{k=1}^{K_0} J(\xi_k^\nu,\gamma_\nu)|\overline{I}_k|\pi_k = -\sum_{k=1}^{K_0}\mathbb{A}(\xi_k^\nu,\gamma_\nu)\pi_k,$$

where

$$\xi_k^\nu = \xi(\overline{I}_k,\gamma_\nu), \quad \pi_k = \pi(m+1), \qquad \overline{I}_k \in \langle m+1, m+2). \tag{42}$$

Suppose, furthermore, that $\{\gamma_{\nu(l)}\}_{l=1}^{K_0+1}$, $\gamma_{\nu(l)} \in \langle T, T+\sqrt{T}\rangle$, is any fixed collection of zeros of the function $\widetilde{\zeta}_{-1}(1/2+it)$. We associate the corresponding collection of formulas (17) with the system of linear equations

$$\sum_{k=1}^{K_0+1}\mathbb{A}[\xi_k^{\nu(l)},\gamma_{\nu(l)}]x_k = \mathbb{B}[\gamma_{\nu(l)}], \qquad l = 1,\dots,K_0+1, \tag{43}$$

where

$$\mathbb{A}[\xi_k^{\nu(l)}, \gamma_{\nu(l)}] = -\frac{|\overline{I}_k|}{\xi_k^{\nu(l)}} \operatorname{Im}\left\{ \frac{\rho_{\nu(l)}}{\left(\xi_k^{\nu(l)}\right)^{\rho_{\nu(l)}} - 1} \right\}, \qquad k = 1, \dots, K_0,$$
$$\mathbb{A}[\xi_{K_0+1}^{\nu(l)}, \gamma_{\nu(l)}] = \mathbb{A}[\gamma_{\nu(l)}] = \operatorname{Im}\left\{ \ln\left(1 - \frac{1}{P^{\rho_{\nu(l)}}}\right) \right\}, \tag{44}$$
$$\mathbb{B}[\gamma_{\nu(l)}] = \varphi\left[t_{\nu(l)}\left(\frac{\pi}{2}\right)\right] - \varphi[\gamma_{\nu(l)}].$$

Therefore the collection of formulas (17) for $\pi(x)$ yields

Corollary 4. *For any fixed integer $N \in \langle 2, N_0 \rangle$ there exists an infinite number of systems of linear equations of the type* (43) (*since, in this case, there exists an infinite number of subdivisions* (40)). *Moreover, each of these systems has an integer-valued solution*

$$\underbrace{\pi(2), \dots, \pi(2)}_{N}, \underbrace{\pi(3), \dots, \pi(3)}_{N}, \dots, \underbrace{\pi(P-1), \dots, \pi(P-1)}_{N}, \pi(P) \tag{45}$$

(see (42)).

This regularity admits the following interpretation. We fix any system (let it be the basic system from the infinite number of systems (43)), and consider each of the remaining systems to be a perturbation of the initial system.

Remark 8. The integer-valued solution (45) to the basic system is an invariant number (a fixed point) with respect to the above-mentioned infinite set of perturbations (transformations) of the basic system.

Remark 9. Thus, the augmented matrix of system (43) which is constructed from the elements of (44) has *arithmetic* properties, i.e., it generates integers.

Let $h[\mathbb{A}(S,N)]$ be the rank of the matrix S (see (43)) for any fixed integer $N \in \langle 2, N_0 \rangle$.

Question 2. Is it true that

$$\min_S \{h[\mathbb{A}(S;N)]\} = \pi(P)?$$

Remark 10. We can put the system of linear equation of the form (43) into correspondence with

(a) the collection of formulas (32) for $R(x)$, but we have to use the second mean value theorem instead of the first one; $R(x)$ decreases on the interval $(m+1, m+2)$,

(b) the systems of generalized formulas (38) and (39) for $\pi(x)$ and $R(x)$ respectively.

8. INDEPENDENCE OF THE THEORY OF FUNCTIONS $\pi(x)$ AND $R(x)$ OF THE ZEROS OF THE FUNCTION $\widetilde{\zeta}_{-1}(s)$

The peculiarities of the model based on the function $\widetilde{\zeta}_{-1}(s)$ are the following facts

(A) An analog of the Riemann conjecture for $\widetilde{\zeta}_{-1}(s)$ holds (see Lemma 1), but this fact was not used (see Remark 7) in the proof of the basic formulas (17), (32), (43) and Corollary 4 for $\pi(x)$ and $R(x)$ on the crirical line. However, formulas (17), (32), and (43) are related to the critical zeros $\rho_\nu = 1/2 + i\gamma_\nu$, $\gamma_\nu \in \langle T, T+H \rangle$ (a finite set) of the function $\widetilde{\zeta}_{-1}(s)$.

(B) Formulas (38), (39), (43′), and Corollary 4′ (43′) and Corollary 4′, which are analogs of (43) and Corollary 4, (cf. Remark 10) are, in turn,

(a) not connected with the analog of the Riemann conjecture for $\widetilde{\zeta}_{-1}(s)$,

(b) not connected even with the critical zeros ρ_ν of the function $\widetilde{\zeta}_{-1}(s)$ (for $\alpha \neq \pm\pi/2$),

(c) connected only with an infinite set (of the power of continuum) of the segments of sequences

$$\{\gamma_{2\nu}(\alpha)\}, \qquad \alpha \in \langle -\pi, \pi\rangle, \quad \alpha \neq \pm\frac{\pi}{2},$$

which lie in the interval $\langle T, T+H\rangle$ of the critical line.

(C) What we said in (B) also holds for the integral equations (33′) and (34′) which are analogs of equations (33) and (34).

Recall that the theory of functions $\pi(x)$ and $R(x)$ based on the (complete) Riemann zeta-function is closely connected with the question concerning the distribution of zeros of the function $\zeta(s)$ in the critical strip (of course, there are no analogs of formulas (38), (39), and (43′) and equations (33′), and (34′) in this theory at all).

In the model based on the function $\widetilde{\zeta}_{-1}(s)$, we have a completely opposite case, namely, by virtue of (A)–(C), we make

Remark 11. The theory of functions $\pi(x)$ and $R(x)$ based of formulas (38), (39), (43′) and on the integral equations (33′) and (34′) is not connected with the zeros of the function $\widetilde{\zeta}_{-1}(s)$ for $\alpha \neq \pm\pi/2$.

Remark made during the correction. Using the formulas from this paper one can prove the orthogonality conditions on the finite closed intervals for the functions $\pi(x)$ and $R(x)$.

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