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On the Measure of Algebraic Independence of the Values of Ramanujan Functions

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INTRODUCTION

Consider the functions

$$P(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) z^n, \qquad Q(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) z^n, \qquad R(z) = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n) z^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$. These function were introduced by Ramanujan [1] in 1916; in particular, he stated that these functions satisfy the system of differential equations

$$\theta P = \frac{1}{12}(P^2 - Q), \qquad \theta Q = \frac{1}{3}(PQ - R), \qquad \theta R = \frac{1}{2}(PR - Q^2), \tag{1}$$

where $\theta = z\frac{d}{dz}$. In 1996, the following theorem was proved in [2, 3].

Theorem. *For any complex q, $0 < |q| < 1$, there are not less than three algebraically independent over $\mathbb{Q}$ numbers among the numbers $q, P(q), Q(q)$, and $R(q)$.*

This general result has a number of important consequences.

First, for $q = e^{-2\pi}$, this result implies that the collection $\pi, e^{\pi}, \Gamma(1/4)$ consists of algebraically independent numbers, and, in particular, the numbers π and e^{π} are algebraically independent. The last assertion is an old problem of the theory of transcendental numbers. In fact, the theorem implies even a more general assertion that the numbers π and $e^{\pi\sqrt{D}}$ are algebraically independent for any natural D.

One more consequence is the algebraic independence of the numbers $J(q), \theta J(q)$, and $\theta^2 J(q)$ for any algebraic q, $0 < |q| < 1$, where

$$J(z) = 1728\frac{Q^3}{Q^3 - R^2} = \frac{1}{z} + 744 + \sum_{n=1}^{\infty} c(n) z^n;$$

the last series yields the Fourier series expansion

$$j(\tau) = J(e^{2\pi i\tau}), \qquad \operatorname{Im}\tau > 0,$$

of the modular function $j(\tau)$. The assertion concerning the algebraic independence of these numbers was conjectured by D. Bertrand [4] in 1977.

Finally, it should be pointed out that as far back as 1851, in the process of constructing the first examples of transcendental numbers, Liouville presented in [5] the series

$$\sum_{n=1}^{\infty} l^{-n^2}, \qquad l \in \mathbb{Z}, \quad l > 1,$$

as an example for which his method allowed one to prove only the irrationality. Later, in a number of papers, the irrationality of these series at the points $l \in \mathbb{Q}$ was proved under certain restrictions imposed on the numerators and denominators of l. No assertions on the transcendence of these sums were known. The transcendence of the Liouville series for any algebraic l, $|l| > 1$, is a consequence of the above-mentioned theorem. This fact was revealed in [6] and [7]. Moreover, the assertion that the numbers

$$\sum_{n=1}^{\infty} q^{n^2}, \quad \sum_{n=1}^{\infty} n^2 q^{n^2}, \quad \sum_{n=1}^{\infty} n^4 q^{n^2}, \qquad q \in \mathbb{A}, \quad 0 < |q| < 1, \tag{2}$$

are algebraically independent holds. Here and in what follows, we denote by $\mathbb{A}$ the field of all algebraic numbers. In the paper by Bertrand [7], the classical algebraic relationships between the Ramanujan functions and the Jacobi theta-functions are written; by the above-indicated theorem, these relations imply the assertion concerning the algebraic independence of values of the theta-functions at an algebraic point. In particular, this means the algebraic independence of numbers (2). See also [8] for a survey of these problems.

The present paper is devoted to qualitative improvements of the above-mentioned results. We introduce some notation. For each polynomial $A(x_1, \dots, x_m) \in \mathbb{Z}[x_1, \dots, x_m]$, we denote by $\deg A$ the degree of the polynomial A with respect to the totality of variables, by $H(A)$ the maximum of moduli of the coefficients of the polynomial A; also, we set $t(A) = \deg A + \ln H(A)$. The following assertion holds.

Theorem 1. *Let $q \in \mathbb{C}$, $0 < |q| < 1$, and let $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$ be such that all numbers $q, P(q), Q(q)$, and $R(q)$ are algebraic over the field $\mathbb{Q}(\omega_1, \omega_2, \omega_3)$. Then there exists a constant γ_1 which depends only on the numbers q and ω_i and is such that for any polynomial $A \in \mathbb{Z}[x_1, x_2, x_3]$, $A \not\equiv 0$, the following inequality holds*:

$$|A(\omega_1, \omega_2, \omega_3)| > \exp(-\gamma_1 S^4 \ln^9 S), \tag{3}$$

where S is an arbitrary number satisfying the inequality

$$S \geq \max\{\ln H(A) + \deg A \times \ln t(A),\ e\}.$$

Note that the exponent 4 cannot be decreased in this estimate. Weaker results were proved in [3] and [9]. In [3], the right-hand side of (3) has the form $\exp\bigl(-\gamma_1 t(A)^4 \ln^{24} t(A)\bigr)$, and in the preprint by Philippon [9], the value $\ln^9 S$ is replaced by a larger value $\ln^{12} S$.

In particular, inequality (3) holds for each of the collections of numbers $\{\pi, e^{\pi}, \Gamma(1/4)\}$ and $\{\pi, e^{\pi\sqrt{3}}, \Gamma(1/3)\}$.

In the case where q is an algebraic number, the estimate of Theorem 1 can be improved.

Theorem 2. *Let q be an algebraic number, $0 < |q| < 1$, and let $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$ be such that all numbers $P(q), Q(q), R(q)$ are algebraic over the field $\mathbb{Q}(\omega_1, \omega_2, \omega_3)$. Then there exists a constant γ_2 which depends only on the numbers q and ω_i and is such that for any polynomial $A \in \mathbb{Z}[x_1, x_2, x_3]$, $A \not\equiv 0$, the following inequality holds*:

$$|A(\omega_1, \omega_2, \omega_3)| > \exp(-\gamma_2 S d^3 \ln^9 S), \tag{4}$$

where S and d are arbitrary numbers satisfying the inequalities

$$S \geq \max\{\ln H(A) + \deg A \times \ln t(A),\ e\}, \qquad d \geq \deg A.$$

The estimate of Theorem 2 is considerably more exact than inequality (3) for polynomials whose coefficients essentially exceed the degree. Thus, if

$$\frac{\ln H(A)}{\ln \ln H(A)} \geq \deg A,$$

then inequality (4) implies

$$|A(\omega_1, \omega_2, \omega_3)| > H(A)^{-\gamma_3 \deg^3 A (\ln \ln H(A))^9}.$$

These inequalities hold for the collections of numbers

$$\{P(q), Q(q), R(q)\}, \qquad \{J(q), \theta J(q), \theta^2 J(q)\}$$

for algebraic q, $0 < |q| < 1$, and also for numbers (2).

The theorem of an estimate of the measure of algebraic independence of the functions $P(z), Q(z)$, and $R(z)$ over $\mathbb{C}(z)$, which is proved in [3], plays an important role in the proof of these results.

Theorem 3 (see [3]). *Let L_1 and L_2 be integers, $L_1 \geq 1$, $L_2 \geq 1$. Then, for any polynomial*

$$A(z, x_1, x_2, x_3) \in \mathbb{C}[z, x_1, x_2, x_3], \qquad A \not\equiv 0, \quad \deg_z A \leq L_1, \quad \deg_{x_i} A \leq L_2,$$

the following inequality holds:

$$\operatorname{ord} A\big(z, P(z), Q(z), R(z)\big) \leq c L_1 L_2^3,$$

where $c = 10^{47}$.

In the statement of this theorem and in what follows in this paper, we denote by $\operatorname{ord} \varphi(z)$ the multiplicity of zero of the function $\varphi(z)$ at the point $z = 0$.

Here are some remarks concerning the structure of this paper. The facts which are presented in Section 1, originate in the general elimination theory. The content of Section 1 consists in an adaptation of the results of this theory to the needs of the theory of transcendental numbers. The corresponding technical results for the polynomial rings $\mathbb{Z}[x_1, \ldots, x_m]$ and $\mathbb{C}[z, x_1, \ldots, x_m]$ were described by the author in [10–15]. A different version of presentation can be found in the paper [16] by Philippon. The aim of Section 1 is the unified presentation of the algebraic foundations of the method, which allow one to apply the obtained technical assertions in the case of integer (Theorem 1) and algebraic (Theorem 2) coefficients of polynomials as well as in the case of coefficients from the field $\mathbb{C}(z)$ (Theorem 3).

In Section 2, we present an algebraic construction of polynomials of the values of Ramanujan functions, which generalizes the corresponding result of [3]. These polynomials are used in Section 3 to prove Theorem 2 and in Section 4 to prove Theorem 1.

1. ALGEBRAIC FOUNDATIONS

Let $\mathbb{K}$ be a field of zero characteristic. In the sequel, we shall repeatedly use the infinitude of $\mathbb{K}$.

In 1922–23 in the papers by Henzelt [17] and Noether [18], a certain polynomial, the Elementarteilerform (denoted by EF for brevity in the sequel), was put into correspondence with each ideal of the ring $\mathbb{K}[x_0, \dots, x_m]$, and a number of important properties of this correspondence was studied. In particular, the following facts were proved:

(a) the EF of a prime ideal is irreducible,
(b) the EF of primary ideal is equal to a certain degree of the EF of the associated prime ideal,
(c) the EF of two different prime ideals are different,
(d) the EF of an unmixed ideal is precisely the product of EF's of its primary components,
(e) the EF of a prime ideal is decomposed over a certain field into a product of linear factors.

A number of properties of the EF were also established in the papers by Krull [19] and van der Waerden [20]. A homogeneous analog of this concept, the Zugeordnete Form, plays an important role in our presentation; this concept was introduced in the paper by Chow and van der Waerden [21] and is called an *associated form* (see [22]). In some cases, this object is also called the *Chow form* or the *Cayley form.*

1.1. An associated form of an ideal. Let $\mathbb{K}[X]$ be a ring of polynomials in the variables $x_0, \dots, x_m$ over $\mathbb{K}$, r be an integer, $1 \le r \le m$, and let u_{ij}, $1 \le i \le r$, $0 \le j \le m$, be variables which are algebraically independent over $K[X]$. We introduce the linear forms

$$L_i(X) = \sum_{j=0}^{m} u_{ij} x_j, \qquad i = 1, \dots, r.$$

Let I be a certain homogeneous ideal of the ring $\mathbb{K}[X]$. By $(I, L_1, \dots, L_r)$ we denote the ideal in the ring

$$\mathbb{K}[X, U] = \mathbb{K}[x_0, \dots, x_m, u_{10}, \dots, u_{rm}] = \mathbb{K}[X][\overline{u}_1, \dots, \overline{u}_r],$$

which is generated by the linear forms $L_1, \dots, L_r$ and by the elements of the ideal $I \subset \mathbb{K}[X]$. By definition, the ideal $\overline{I}(r) \subset \mathbb{K}[U]$ consists of all polynomials G of the ring $\mathbb{K}[U] = \mathbb{K}[u_{10}, \dots, u_{rm}]$ which satisfy the conditions

$$G x_i^M \in (I, L_1, \dots, L_r) \subset \mathbb{K}[X, U], \qquad 0 \le i \le m,$$

for some integer M. Recall that the ideal I of the polynomial ring $\mathbb{K}[X] = \mathbb{K}[x_0, \dots, x_m]$ is said to be *unmixed* if all its primary components are of the same dimension which is equal to that of the ideal I. The *exponent* $\mathfrak{p}$ of the primary ideal I is the minimal natural number n under the condition $\mathfrak{p}^n \subset I$. By the *dimension* $\dim I$ of the homogeneous ideal I we mean the projective dimension of this ideal.

Proposition 1.1. *Let I be an unmixed homogeneous ideal of the ring $\mathbb{K}[X]$, $r = \dim I + 1 \ge 1$. Let $I = I_1 \cap \dots \cap I_s$ be a reduced primary decomposition. We set $\mathfrak{p}_j = \sqrt{I_j}$ for $j = 1, \dots, s$, with k_j as the exponent of the primary ideal I_j.*

Then $\overline{I}(r)$ is the principal ideal in the ring $\mathbb{K}[\overline{u}_1, \dots, \overline{u}_r]$, and if $\overline{\mathfrak{p}}_j(r) = (F_j)$, then the polynomial $F = F_1^{k_1} \cdots F_s^{k_s}$ is the generator of the ideal $\overline{I}(r)$.

Proof. Actually, all the assertions given here are classical. See [10] for a unified presentation of the proof.

The generator F of the principal ideal $\overline{I}(r)$ is called an *associated form* of the ideal I.

1.2. Absolute values and the logarithmic height of a polynomial. Let $\mathcal{M}$ be a set, and let an absolute value $|\cdot|_v$ be given on the field $\mathbb{K}$ for each $v \in \mathcal{M}$, which assumes the values from the set of positive real numbers; moreover, assume that the set of these absolute values satisfies the following product formula:

> for each $a \in \mathbb{K}^*$, there exists only a finite number of elements $v \in \mathcal{M}$ with $|a|_v \neq 1$, and $\prod_{v\in\mathcal{M}} |a|_v = 1$.

In addition, we assume that

> the subset $\mathcal{M}_\infty \subset \mathcal{M}$ consisting of all elements $v \in \mathcal{M}$ for which the corresponding absolute value $|\cdot|_v$ is Archimedean is finite and the restriction of every one of these absolute values to the field $\mathbb{Q} \subset \mathbb{K}$ coincides with an ordinary absolute value.

Everywhere in what follows, the letter ν denotes $\#\mathcal{M}_\infty$ which is the number of elements in the set $\mathcal{M}_\infty$.

It is not assumed that different absolute values correspond to different elements of the set $\mathcal{M}$. Thus, in what follows, the multiplicities in the product formula for Archimedean absolute values are taken into account.

For each $v \in \mathcal{M}_\infty$, there exists an embedding $\sigma\colon \mathbb{K} \to \mathbb{C}$ such that the equality $|a|_v = |\sigma(a)|_{\mathbb{C}}$ holds for any $a \in \mathbb{K}$, where $|\cdot|_{\mathbb{C}}$ is an ordinary absolute value on $\mathbb{C}$ and $\sigma(a)$ is the image of the element $a \in \mathbb{K}$ under the embedding σ (see [23, ch. 6, § 6, Theorem 2]).

For each $v \in \mathcal{M}$ and for the polynomial P with coefficients in the field $\mathbb{K}$,

$$P = \sum_{\overline{\gamma}} a_{\overline{\gamma}} T_1^{\gamma_1} \cdots T_m^{\gamma_m} \in \mathbb{K}[T_1, \dots, T_m], \tag{5}$$

we set

$$|P|_v = \max_{\overline{\gamma}} |a_{\overline{\gamma}}|_v.$$

Lemma 1.1. *Let $|\cdot| = |\cdot|_v$, $v \in \mathcal{M}$, be an absolute value on the field $\mathbb{K}$, and let $P_1, \dots, P_s$ be polynomials in the variables $T_1, \dots, T_m$ with coefficients in $\mathbb{K}$, $P = P_1 \cdots P_s$. Then*

(1) *if $|\cdot|$ is non-Archimedean, then $|P| = |P_1| \cdots |P_s|$,*

(2) *if $|\cdot|$ is Archimedean, then the inequalities*

$$|P| \leq |P_1| \cdots |P_s| (m+1)^{\deg P}, \tag{6}$$

$$|P_1| \cdots |P_s| \leq |P| \exp\left(\sum_{i=1}^{m} \deg_{T_i} P\right) \tag{7}$$

hold.

Proof. For non-Archimedean absolute values this assertion is proved, for example in [24, ch. 3, Proposition 2.1]. Inequality (6) follows from trivial estimates. The proof of (7) see in [25, ch. 3, § 4, Lemma 2].

We define the *logarithmic height* $h(P)$ of each nonzero polynomial (5) by the relation

$$h(P) = \sum_{v \in \mathcal{M}} \ln |P|_v .$$

It is clear that for any nonzero polynomial P the above sum consists of a finite number of nonzero terms, and, thus, the logarithmic height is well defined. In addition, it is easy to verify that the inequality $h(P) \geq 0$ holds, and, moreover, the relation

$$h(\lambda P) = h(P) \tag{8}$$

holds for any nonzero $\lambda \in \mathbb{K}$.

The following example are well known.

Example 1 (the functional case). Let $\mathbb{K} = \Bbbk(z)$, where $\Bbbk$ is an algebraically closed field of zero characteristic. In this case, we set $\mathcal{M} = \Bbbk \cup \{\infty\}$ and define

$$|a|_v = \begin{cases} e^{\deg p - \deg q} & \text{if } v = \infty \\ e^{\operatorname{ord}_v q - \operatorname{ord}_v p} & \text{if } v \in \Bbbk \end{cases}$$

for each $a = p/q$, $p, q \in \Bbbk[z]$, $(p, q) = 1$. All absolute values defined in this way are obviously non-Archimedean, i.e., $\mathcal{M}_\infty = \varnothing$. The product formula holds in this case since the number of roots of the arbitrary polynomial of the ring $\Bbbk[z]$, with due account of the multiplicity, is equal to the degree of this polynomial. It is easy to verify that for any polynomial $P \in \Bbbk[z][T_1, \ldots, T_m]$ with relatively prime coefficient in their totality, the relation

$$h(P) = \ln \max_{\overline{\gamma}} \left| a_{\overline{\gamma}} \right|_\infty = \deg_z P$$

holds. Precisely this case was considered in [3] in the process of proving Theorem 3.

Example 2 (the numerical case). Let $\mathbb{K} \supset \mathbb{Q}$ be a finite extension. We include in the set $\mathcal{M}$ all nontrivial prime ideals of the ring $\mathbb{Z}_{\mathbb{K}}$ and all embeddings of the field $\mathbb{K}$ into the field of complex numbers. Let $a \in \mathbb{K}^*$. For each prime ideal $\mathfrak{n} \subset \mathbb{Z}_{\mathbb{K}}$ we set

$$|a|_{\mathfrak{n}} = N(\mathfrak{n})^{-\operatorname{ord}_{\mathfrak{n}} a},$$

where $\operatorname{ord}_{\mathfrak{n}} a$ is the multiplicity of the prime ideal $\mathfrak{n}$ in the decomposition of the fractional principal ideal $(a) \subset \mathbb{K}$ and $N(\mathfrak{n})$ is the norm of $\mathfrak{n}$. If σ is an embedding of $\mathbb{K}$ into $\mathbb{C}$, then we set

$$|a|_\sigma = |\sigma(a)|_{\mathbb{C}},$$

where $|\cdot|_{\mathbb{C}}$ is an ordinary absolute value on the field $\mathbb{C}$. All absolute embeddings of the first type are non-Archimedean whereas all absolute values of the second type are non-Archimedean. If σ_1 and σ_2 are complex conjugate embeddings, then $|\cdot|_{\sigma_1} = |\cdot|_{\sigma_2}$. In the case under consideration, the number of Archimedean embeddings is equal to the degree of the field $\mathbb{K}$, i.e., $\nu = \#\mathcal{M}_\infty = [\mathbb{K} : \mathbb{Q}]$. See, e.g., [24, ch. 2] for the proof of the product formula.

If the polynomial $P \in \mathbb{K}[X]$ is defined by relation (5), then

$$h(P) = -\ln N(\mathfrak{a}) + \sum_\sigma \ln \max_{\overline{\gamma}} \left|\sigma\left(a_{\overline{\gamma}}\right)\right|_{\mathbb{C}},$$

where $\mathfrak{a}$ is a fractional ideal in the field $\mathbb{K}$ generated by the coefficients of the polynomial P and $N(\mathfrak{a})$ is the norm of $\mathfrak{a}$ (see [24, ch. 3, § 1]). In particular, if $P \in \mathbb{Z}[X]$ and if the coefficients of P are relatively prime in their totality, then

$$h(P) = \nu \ln H(P),$$

where $H(P)$ is the maximum of the moduli of the coefficients of P.

In the process of proving Theorems 1 and 2, we shall use precisely this situation and the field $\mathbb{K} = \mathbb{Q}$ in the case of Theorem 1.

Let us return to the general case. With the use of Lemma 1.1, we can easily prove the following inequalities.

Lemma 1.2. *Let $P_1, \dots, P_s \in \mathbb{K}[T_1, \dots, T_m]$, $P = P_1 \cdots P_s$. Then*

$$h(P) \le h(P_1) + \cdots + h(P_s) + \nu \ln(m+1) \times \deg P, \tag{9}$$

$$h(P_1) + \cdots + h(P_s) \le h(P) + \nu \sum_{i=1}^{m} \deg_{T_i} P. \tag{10}$$

In particular, for $\mathcal{M}_\infty = \varnothing$, the relation

$$h(P) = h(P_1) + \cdots + h(P_s)$$

holds.

Note that if the polynomials $P_1, \dots, P_s$ are homogeneous, then the value m can be replaced by $m-1$ in inequalities (6) and (7); this leads to the same change in (9) and (10). In some cases, we shall also use the following assertion, which is more general but is also a trivial consequence of the above-proved fact: if the polynomials P_i are homogeneous in several independent groups of variables, then the value m in (6) and (9) can be diminished by the number of groups of homogeneous variables, and one term $\deg_{T_i} P$ can be omitted in each group of variables on the right-hand sides of (7) and (10).

1.3. Characteristics of an ideal. Let I be a homogeneous unmixed ideal of the ring $\mathbb{K}[X]$, and let F be the associated form of this ideal. By definition, it follows that the groups of variables $\overline{u}_1, \dots, \overline{u}_r$ enter symmetrically into the polynomial F. In particular, the degrees of the form F in each of these groups are the same.

In what follows, the degree of F in the group of variables $\overline{u}_1$ will be called the *degree of the ideal* I and will be denoted by $\deg I$.

We say that the *logarithmic height of the ideal* I is the logarithmic height of the polynomial F, i.e., we set $h(I) = h(F)$. As was pointed out above, the polynomial F is uniquely defined with an accuracy up to the multipliers from the field $\mathbb{K}$. Therefore, by (8), the value of $h(I)$ is well defined.

Let $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, be a certain fixed absolute value on the field $\mathbb{K}$, and let $\mathcal{K}$ be the completion of the algebraic closure of the completion of $\mathbb{K}$ with respect to $|\cdot|$. The absolute value $|\cdot|$ can be uniquely extended to $\mathcal{K}$ [24, ch. 1, Propositions 2.1 and 2.2]. We fix the absolute value $|\cdot|$ and the field $\mathcal{K}$ up to the end of this section. For each point $\overline{\omega} = (\omega_0, \ldots, \omega_m) \in \mathcal{K}^{m+1}$, we set

$$|\overline{\omega}| = \max_{0 \le j \le m} |\omega_j|.$$

Now, for each homogeneous unmixed ideal $I \subset \mathbb{K}[X]$ and for each nonzero point $\overline{\omega} \in \mathcal{K}^{m+1}$, we define $|I(\overline{\omega})|$ which plays the role of the value of the polynomial at the $\overline{\omega}$ in the subsequent proofs. Let $r = 1 + \dim I$, $\overline{I}(r) = (F)$, and let $S^{(i)} = \|s_{jk}^{(i)}\|$, $1 \le i \le r$, $0 \le j, k \le m$, be skew-symmetric matrices whose elements, except for the skew-symmetry condition $s_{jk}^{(i)} + s_{kj}^{(i)} = 0$, are not connected by any algebraic relations over the ring $\mathbb{K}[X, \overline{u}_1, \ldots, \overline{u}_r]$. In what follows, for each polynomial $E \in \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_r]$, we denote by the letters $\varkappa(E)$ the polynomial in the variables $s_{jk}^{(i)}$, $j < k$, $i = 1, \ldots, r$, with coefficients in the field $\mathcal{K}$ which is obtained as a result of the substitution of the vector $S^{(i)}\overline{\omega}$, $i = 1, \ldots, r$ for the variables $\overline{u}_i$ in E.

We shall now define $|I(\overline{\omega})|$ with the aid of the associated form F of the ideal I in the following way:

$$|I(\overline{\omega})| = |\varkappa(F)| \times |F|^{-1} \times |\overline{\omega}|^{-r \deg I}. \tag{11}$$

It is clear that the value defined in this way does not depend on the choice of the generator F of the ideal $\overline{I}$. Moreover, since F is a homogeneous polynomial in the variables u_{ij} of degree $r \deg I$, the value $|I(\overline{\omega})|$ does not change under the multiplication of the point $\overline{\omega}$ by any nonzero element $\lambda \in \mathcal{K}$, i.e.,

$$|I(\overline{\omega})| = |I(\lambda\overline{\omega})|, \qquad \lambda \in \mathcal{K}^*.$$

In what follows, we shall repeatedly use this property by multiplying the point $\overline{\omega}$ by such an element λ in order to ensure the equality $|\overline{\omega}| = 1$.

Proposition 1.2. *Let I be an unmixed homogeneous ideal of the ring $\mathbb{K}[X]$, $\dim I \ge 0$, $I = I_1 \cap \cdots \cap I_s$ be a reduced primary decomposition, $\mathfrak{p}_j = \sqrt{I_j}$, and let k_j be the exponent of the primary ideal I_j. Let $\overline{\omega} \in \mathcal{K}^{m+1}$, $\overline{\omega} \ne 0$. Then*

(1) $\displaystyle\sum_{j=1}^{s} k_j \deg \mathfrak{p}_j = \deg I,$

(2) $\displaystyle\sum_{j=1}^{s} k_j h(\mathfrak{p}_j) \le h(I) + \nu m^2 \deg I,$

(3) $\displaystyle\sum_{j=1}^{s} k_j \ln|\mathfrak{p}_j(\overline{\omega})| \le \ln|I(\overline{\omega})| + m^3 \deg I.$

If $\mathcal{M}_\infty = \varnothing$, then the equality holds in (2). If $|\cdot|$ is a non-Archimedean absolute value, then the equality also holds in (3), moreover, the term $m^3 \deg I$ on the right-hand side must be omitted.

Proof. Let $r = 1 + \dim I$, $\overline{\mathfrak{p}_j}(r) = (F_j)$, $j = 1, \ldots, s$. According to Proposition 1.1, the polynomial $F = F_1^{k_1} \cdots F_s^{k_s}$ is a generator of the ideal $\overline{I}(r)$. Now equality (1) is obvious, and inequality (2) is obtained with the aid of Lemma 1.2.

Neither of the sides of inequality (3) changes under the substitution of $\lambda\overline{\omega}$ for $\overline{\omega}$, $\lambda \in \mathcal{K}^*$. Therefore, without loss of generality, we can assume that $|\overline{\omega}| = 1$.

For definiteness, we consider the case where $\mathcal{M}_\infty \neq \varnothing$ and $|\cdot|$ is an Archimedean absolute value. In accordance with (6) and (7), we have the inequalities

$$\sum_{j=1}^{s} k_j \ln|F_j| \geq \ln|F| - m^2 \deg I, \qquad \sum_{j=1}^{s} k_j \ln|\varkappa(F_j)| \leq \ln|\varkappa(F)| + m\Big(\frac{m(m+1)}{2} - 1\Big)\deg I. \quad (12)$$

In the first case, we used the fact that F was a homogeneous polynomial of degree $\deg I$ in each group of variables $\overline{u}_i = (u_{i0}, \ldots, u_{im})$, $1 \leq i \leq r$. In the second case, we used the fact that the number of variables $s_{jl}^{(i)}$, $1 \leq i \leq r$, $0 \leq j < l \leq m$, on which the polynomial $\varkappa(F)$ depends was equal to $rm(m+1)/2$, and that $\varkappa(F)$ was a homogeneous polynomial of degree $\deg I$ in each group of variables $s_{jl}^{(i)}$ with a fixed i.

We find from (11), (12), and the first assertion of Lemma 1.2 that

$$\sum_{j=1}^{s} k_j \ln|\mathfrak{p}_j(\overline{\omega})| = \sum_{j=1}^{s} k_j \big(\ln|\varkappa(F_j)| - \ln|F_j|\big) \leq \big(\ln|\varkappa(F)| - \ln|F| + m^3 \deg I\big) \leq \ln|I(\overline{\omega})| + m^3 \deg I.$$

This completes the proof of Proposition 1.2 in the case under consideration. In the other cases, the arguments are similar.

1.4. Relationships between the characteristics of the principal ideal and its generator. Let, as before, $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, be a certain fixed absolute value on the field $\mathbb{K}$, and let $\mathcal{K}$ be an algebraic closure of the completion of $\mathbb{K}$ with respect to $|\cdot|$. Suppose that $\overline{\omega} \in \mathcal{K}^{m+1}$, $\overline{\omega} \neq 0$, and $P \in \mathbb{K}[X]$ is a homogeneous polynomial. In the sequel, we shall use the notation

$$\|P\|_{\overline{\omega}} = |P(\overline{\omega})| \times |P|^{-1} \times |\overline{\omega}|^{-\deg P}.$$

Note that if $|\cdot|$ is a non-Archimedean absolute value, then $\|P\|_{\overline{\omega}} \leq 1$. If $|\cdot|$ is Archimedean, then $\|P\|_{\overline{\omega}} \leq e^{m \deg P}$.

We shall prove the following assertion.

Proposition 1.3. *Let $I = (P)$ be the principal ideal in the ring $\mathbb{K}[X]$ generated by the homogeneous polynomial P, and let $\overline{\omega} \in \mathcal{K}^{m+1}$, $\overline{\omega} \neq 0$. Then*

$$\deg I = \deg P, \quad (13)$$

$$h(I) \leq h(P) + \nu m^2 \deg P, \quad (14)$$

$$\ln|I(\overline{\omega})| \leq \ln\|P\|_{\overline{\omega}} + 2m^2 \deg P. \quad (15)$$

If $\mathcal{M}_\infty = \varnothing$, then we have an equality in (14). *If $|\cdot|$ is a non-Archimedean absolute value, then the term $2m^2 \deg P$ on the right-hand side of* (15) *can be omitted.*

We denote by Δ_j the determinant of the $m \times m$ matrix which is obtained by deleting the column with the number j multiplied by $(-1)^j$ from the matrix

$$\|u_{ik}\|, \qquad i = 1, \ldots, m, \quad k = 0, 1, \ldots, m.$$

Lemma 1.3. *The principal ideal* $\overline{I}(m)$ *is generated by the polynomial*

$$F = P(\Delta_0, \Delta_1, \dots, \Delta_m).$$

Proof. See [12, Lemma 2].

Relation (13) follows immediately from Lemma 1.3. Since $\Delta_j \in \mathbb{Z}[\overline{u}_1, \dots, \overline{u}_m]$, the coefficients of the polynomial $F = P(\Delta_0, \dots, \Delta_m)$ are integer-valued linear combinations of the coefficients of the polynomial P. This implies that the inequality

$$|F|_v \le |P|_v, \qquad v \in \mathcal{M} \setminus \mathcal{M}_\infty, \tag{16}$$

holds for any non-Archimedean absolute value $|\cdot|_v$.

In order to obtain the necessary inequality in the Archimedean case, let us consider some coefficient f of the polynomial F. As was indicated above, the following representation holds:

$$f = \sum_{\overline{\alpha}} a_{\overline{\alpha}} b_{\overline{\alpha}},$$

where $a_{\overline{\alpha}}$ are coefficients of the polynomial P and $b_{\overline{\alpha}}$ are some integers. Since the coefficients of the polynomials Δ_j are equal to ± 1 and each of these polynomial has $m!$ terms, it is easy to see that the modulus of any one of the coefficients $b_{\overline{\alpha}}$ does not exceed $(m!)^{\deg P}$. Taking into account that the above-presented sum for f contains

$$\binom{\deg P + m}{m} \le (\deg P + 1)^m \le e^{m \deg P}$$

terms, we have the estimate

$$|F|_v \le |P|_v (m!)^{\deg P} e^{m \deg P} \le |P|_v e^{m^2 \deg P} \tag{17}$$

for the Archimedean absolute value $|\cdot|_v$. Let $S^{(i)}$, $i = 1, \dots, m$, be skew-symmetric matrices which were defined before Proposition 1.2. In what follows, for each polynomial $E \in \mathbb{K}[\overline{u}_1, \dots, \overline{u}_r]$, we shall denote by $\vartheta(E)$ the polynomial in the variables x_j, $s^{(i)}_{jk}$, $j < k$, $i = 1, \dots, m$, with coefficients in the field $\mathbb{K}$, which is obtained as a result of the substitution of the vector $S^{(i)}\overline{x}$, $i = 1, \dots, m$ for the variables $\overline{u}_i$ in E.

The identity $x_k \Delta_j = x_j \Delta_k + A_1 L_1 + \dots + A_m L_m$, where A_j are polynomials in the ring $\mathbb{K}[\overline{u}_1, \dots, \overline{u}_m]$, implies that $x_k \vartheta(\Delta_j) = x_j \vartheta(\Delta_k)$. Therefore there exists a polynomial $\Lambda \in \mathbb{Z}[X, S^{(1)}, \dots, S^{(m)}]$ such that

$$\vartheta(\Delta_j) = x_j \Lambda, \qquad j = 0, 1, \dots, m. \tag{18}$$

Then we have the following identity with respect to the variables $x_i, s^{(p)}_{ij}$:

$$\vartheta(F) = P\big(\vartheta(\Delta_0), \dots, \vartheta(\Delta_m)\big) = \Lambda^n P(x_0, \dots, x_m) = x_0^{-n} \vartheta(\Delta_0)^n P(x_0, \dots, x_m), \tag{19}$$

where $n = \deg P$. Suppose that the elements of the matrices $S^{(p)}$, $1 \le p \le m$, have the values

$$s^{(p)}_{ij} = \begin{cases} 1 & \text{if } i = p-1,\ j = p \\ -1 & \text{if } i = p,\ j = p-1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that with such a substitution of the values of $s_{ij}^{(p)}$, the polynomial $\vartheta(u_{i,i-1})$ passes into x_i, the polynomial $\vartheta(u_{ii})$ passes into $-x_{i-1}$, and $\vartheta(u_{ij})$ for $j \neq i, i-1$ becomes 0; therefore, the polynomial $\Lambda = \vartheta(\Delta_0)x_0^{-1}$ passes into $(-1)^m x_1 \cdots x_{m-1}$. Substituting now the indicated values of the variables $s_{ij}^{(p)}$ into relation (19) and comparing the coefficients in the same products of the powers of $x_0, \dots, x_m$, on both sides of the obtained relation, we find that each coefficient of the polynomials P can differ from a certain coefficient of the polynomial F only by the sign. But then, for any absolute value $|\cdot|_v$, this implies the inequality $|P|_v \leq |F|_v$ which, together with (16), means that

$$|F|_v = |P|_v, \qquad v \in \mathcal{M} \setminus \mathcal{M}_\infty. \tag{20}$$

With the aid of (17) and (20), we obtain

$$h(I) = h(F) = \sum_{v \in \mathcal{M}} \ln |F|_v \leq \sum_{v \in \mathcal{M}} \ln |P|_v + \sum_{v \in \mathcal{M}_\infty} m^2 \deg P = h(P) + \nu m^2 \deg P.$$

Inequality (14) is proved. As it follows from (20), in the case $\mathcal{M}_\infty = \varnothing$, the last inequality becomes an equality, and this proves (14) in this case.

When proving inequality (15), we can assume without loss of generality that $|\overline{\omega}| = |\omega_0| = 1$. Substituting $x_j = \omega_j$, $j = 0, 1, \dots, m$, into relation (19), we obtain

$$\varkappa(F) = \omega_0^{-n} \varkappa(\Delta_0)^n P(\omega_0, \dots, \omega_m) = \Lambda(\overline{\omega})^n P(\omega_0, \dots, \omega_m). \tag{21}$$

The polynomial Λ can be represented in the form

$$\Lambda = \Lambda(X) = \sum_{\overline{\imath}, \overline{\jmath}} a_{\overline{\imath}, \overline{\jmath}}(X) s_{i_1 j_1}^{(1)} \cdots s_{i_m j_m}^{(m)},$$

where $a_{\overline{\imath}, \overline{\jmath}}(X) \in \mathbb{Z}[X]$, and the summation is carried out over all pairs of vectors $\overline{\imath} = (i_1, \dots, i_m)$, $\overline{\jmath} = (j_1, \dots, j_m)$ such that $i_k < j_k$. Let us estimate from above the value $|a_{\overline{\imath}, \overline{\jmath}}(\overline{\omega})|$ for each admissible pair of vectors $\overline{\imath}$, $\overline{\jmath}$. To this end, we fix the vectors $\overline{\imath}$, $\overline{\jmath}$ such that $i_k < j_k$ and assign the following values to the elements of the matrices $S^{(p)}$:

$$s_{ij}^{(p)} = \begin{cases} 1 & \text{if } i = i_p,\ j = j_p \\ -1 & \text{if } i = j_p,\ j = i_p \\ 0 & \text{otherwise.} \end{cases}$$

Then the polynomial $\varkappa(\Lambda)$ passes into $a_{\overline{\imath}, \overline{\jmath}}(\overline{\omega})$. Now, with the aid of relation (18), we find that the element $\omega_0 a_{\overline{\imath}, \overline{\jmath}}(\overline{\omega})$ can be represented as the result of the substitution into the determinant Δ_0 of the following values for the variables u_{pq}:

$$u_{pq} = \begin{cases} \omega_{j_p} & \text{if } q = i_p \\ -\omega_{i_p} & \text{if } q = j_p \\ 0 & \text{otherwise.} \end{cases}$$

If $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, is an Archimedean absolute value, then we find that $|a_{\overline{\imath}, \overline{\jmath}}(\overline{\omega})| \leq 2^m$, and therefore $|\Lambda(\overline{\omega})| \leq 2^m$ (we consider here $\Lambda(\overline{\omega})$ as a polynomial in the variables $s_{ij}^{(p)}$, $i < j$). Taking into account that the number of terms in the polynomial $\Lambda(\overline{\omega})$ is equal to

$$\left(\frac{m(m+1)}{2}\right)^m,$$

we obtain from (21) the inequality

$$|\varkappa(F)| = |P(\overline{\omega})| \times |\Lambda(\overline{\omega})^n| \le |P(\overline{\omega})| \times |\Lambda(\overline{\omega})|^n \left(\frac{m(m+1)}{2}\right)^{mn}$$
$$\le |P(\overline{\omega})| \left(\frac{m(m+1)}{2}\right)^{mn} 2^{mn} \le |P(\overline{\omega})| e^{2m^2 n},$$

and, with the aid of the inequality $|P| \le |F|$, the estimate

$$\ln|I(\overline{\omega})| = \ln|\varkappa(F)| - \ln|F| \le \big(\ln|P(\overline{\omega})| + 2m^2 \deg P\big) - \ln|P| = \ln\|P\|_{\overline{\omega}} + 2m^2 \deg P,$$

which proves (15) in the case under consideration.

If $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, is a non-Archimedean absolute value, then we proceed by analogy.

1.5. The u-resultant of a polynomial and of the associated form of a prime ideal. Let $\mathfrak{p}$ be a prime ideal of the ring $\mathbb{K}[X]$, and let $Q \in \mathbb{K}[X]$ be a polynomial, $Q \notin \mathfrak{p}$. Then the ideal $\mathfrak{a} = (\mathfrak{p}, Q)$ which is obtained by the addition of the polynomial Q to $\mathfrak{p}$ has the dimension $\dim \mathfrak{p} - 1$. Unfortunately, we cannot say anything about the characteristics $\deg \mathfrak{a}$, $h(\mathfrak{a})$, or $|\mathfrak{a}(\overline{\omega})|$ since the ideal $\mathfrak{a}$ can have in the general case, components of different dimensions, i.e., this ideal is not unmixed. Below we discuss a construction of an unmixed ideal I whose variety of zeros coincides with that of the ideal $\mathfrak{a}$, and, hence, $\dim I = \dim \mathfrak{p} - 1$; we shall also estimate the indicated characteristics of the ideal I in terms of those of the ideal $\mathfrak{p}$ and the polynomial Q.

In what follows, we shall use the notation $\mathbb{K}_p$ for $\mathbb{K}(\overline{u}_1, \dots, \overline{u}_p) = \mathbb{K}(u_{10}, \dots, u_{pm})$, which is the field obtained by the addition of the coefficients u_{ij} of linear forms $L_1, \dots, L_p$ to $\mathbb{K}$, and we set $\mathbb{K}_0 = \mathbb{K}$ in order to unify the notation.

Lemma 1.4. *Let $\mathfrak{p} \subset \mathbb{K}[X]$ be a homogeneous prime ideal, $r = \dim \mathfrak{p} + 1$, $x_0 \notin \mathfrak{p}$, and let the principal ideal $\overline{\mathfrak{p}}(r) \subset \mathbb{K}[\overline{u}_1, \dots, \overline{u}_r]$ be generated by the polynomial F. Then the following assertions hold*:

(1) *there exists a finite normal extension $\mathfrak{K} \supset \mathbb{K}_{r-1}$ such that*

$$F = a \prod_{j=1}^{g} (u_{r0} + \alpha_1^{(j)} u_{r1} + \dots + \alpha_m^{(j)} u_{rm}), \tag{22}$$

where $g = \deg \mathfrak{p} \ge 1$, $a \in \mathbb{K}[\overline{u}_1, \dots, \overline{u}_{r-1}]$ and $\alpha_i^{(j)} \in \mathfrak{K}$,

(2) *each of the points $(1 : \alpha_1^{(j)} : \dots : \alpha_m^{(j)}) \in \mathbb{P}^m_{\mathfrak{K}}$, is a generic zero of the ideal $\mathfrak{p}$,*

(3) *if $\mathcal{L} = \mathbb{K}_{r-1}(\alpha_1^{(1)}, \dots, \alpha_m^{(1)})$, then $[\mathcal{L} : \mathbb{K}_{r-1}] = g$ and there exist embeddings $\sigma_1, \dots, \sigma_g$ of the field $\mathcal{L}$ into $\mathfrak{K}$ over $\mathbb{K}_{r-1}$ such that $\alpha_i^{(j)} = \sigma_j(\alpha_i^{(1)})$, $j = 1, \dots, g$, $i = 1, \dots, m$,*

(4) *for $l = 1, \dots, r-1$, the relations*

$$u_{l0} + \sum_{i=1}^{m} \alpha_i^{(j)} u_{li} = 0, \qquad j = 1, \dots, g,$$

hold true.

Proof. See [11, Lemma 2].

Let $Q \in \mathbb{K}[X]$ be a homogeneous polynomial in the variables $x_0, \ldots, x_m$. Consider the expression

$$G = a^{\deg Q} \prod_{j=1}^{g} Q(1, \alpha_1^{(j)}, \ldots, \alpha_m^{(j)}), \tag{23}$$

where $\deg Q$ is the degree of the polynomial Q in the totality of variables $x_0, \ldots, x_m$.

Lemma 1.5. *Under the above conditions, the following assertions hold:*

(1) $G \in \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_{r-1}]$,

(2) *for the non-Archimedean absolute value* $|\cdot|$ *of the field* $\mathbb{K}(\overline{u}_1, \ldots, \overline{u}_{r-1})$, *the inequality*

$$|G| \le |F|^{\deg Q} |Q|^{\deg \mathfrak{p}}$$

holds; in particular, this inequality holds for the extensions of the absolute values $|\cdot|_v$, $v \in \mathcal{M} \setminus \mathcal{M}_\infty$ *to* $\mathbb{K}(\overline{u}_1, \ldots, \overline{u}_{r-1})$,

(3) *if* $r \ge 2$, *then*

$$\deg_{\overline{u}_j} G \le \deg \mathfrak{p} \, \deg Q, \qquad j = 1, \ldots, r-1.$$

Proof. See [11, Lemma 4].

The polynomial $G \in \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_{r-1}]$ defined by relation (23) is called the *u-resultant* of the associated form F and of the polynomial Q.

Lemma 1.6. *Suppose that* $\mathfrak{p} \subset \mathbb{K}[X]$ *is a prime homogeneous ideal,* $x_0 \notin \mathfrak{p}$, $r = 1 + \dim \mathfrak{p} \ge 2$, F *is the associated form of* $\mathfrak{p}$, $Q \in \mathbb{K}[X]$ *is a homogeneous polynomial,* $Q \notin \mathfrak{p}$, *and let* $\mathfrak{a} = (\mathfrak{p}, Q) \subset \mathbb{K}[X]$. *Then*

(1) *for the u-resultant* G *of the polynomial* Q *and of the form* F, *we have the inclusion* $G \in \overline{\mathfrak{a}}(r-1) \subset \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_{r-1}]$,

(2) *if* $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ *are prime ideals of dimension* $r-2$ *associated with* $\mathfrak{a}$ *and* $\overline{\mathfrak{q}}_j(r-1) = (E_j) \subset \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_{r-1}]$, *then there exist* $w \in \mathbb{K}$, $w \ne 0$, *and natural numbers* $r_1, \ldots, r_s$ *such that the relation* $G = wE_1^{r_1} \cdots E_s^{r_s}$ *holds.*

Proof. See [12, Lemma 6].

1.6. Estimate of the logarithmic height of the u-resultant. We estimate here the value of $h(G)$ for the polynomial G which is defined in Subsection 1.5. For brevity, we introduce the notation $B = \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_{r-1}, \alpha_1^{(1)}, \ldots, \alpha_m^{(g)}]$.

Lemma 1.7. *Let* $\mathcal{K}$ *be an algebraically closed field containing the field* $\mathbb{K}$, *and let the elements* $u'_{ij} \in \mathcal{K}$, $i = 1, \ldots, r-1$, $j = 0, 1, \ldots, m$, *be such that* $a(u'_{ij}) \ne 0$. *Then there exists a homomorphism*

$$\tau \colon B[a^{-1}] \to \mathcal{K},$$

which is identical on $\mathbb{K}$ *and such that* $\tau(u_{ij}) = u'_{ij}$.

Proof. It follows from relation (22) that all elements $a\alpha_i^{(j)}$, $1 \le i \le m$, $1 \le j \le g$, are algebraic integers over the ring $\mathbb{K}[\overline{u}_1, \dots, \overline{u}_{r-1}]$. Therefore the ring $B[a^{-1}]$ is integral over the ring $\mathbb{K}[\overline{u}_1, \dots, \overline{u}_{r-1}, a^{-1}]$. Let us define the homomorphism τ on the ring $\mathbb{K}[\overline{u}_1, \dots, \overline{u}_{r-1}, a^{-1}]$ assuming it to be identical on $\mathbb{K}$, $\tau(u_{ij}) = u'_{ij}$, $\tau(a^{-1}) = a(u'_{ij})^{-1}$. The homomorphism τ can be extended to the ring $B[a^{-1}]$ [23, ch. 5, § 2, Corollary 4 of Theorem 1]. The lemma is proved.

The following lemma is an Archimedean analog of assertion (2) of Lemma 1.5.

Lemma 1.8. *Suppose that* $\mathfrak{p} \subset \mathbb{K}[X]$ *is a homogeneous prime ideal,* $x_0 \notin \mathfrak{p}$, $r = 1 + \dim \mathfrak{p} \ge 1$, F *is the associated form of* $\mathfrak{p}$, $Q \in \mathbb{K}[X]$ *is a homogeneous polynomial, and let* G *be the u-resultant of the form* F *and of the polynomial* Q. *If* $|\cdot|$ *is an Archimedean absolute value of the field* $\mathbb{K}$, *then the following inequality holds*:

$$|G| \le |F|^{\deg Q} |Q|^{\deg \mathfrak{p}} e^{m(r+1) \deg \mathfrak{p} \deg Q}.$$

Proof. Since $|\cdot|$ is an Archimedean absolute value, we can assume, without loss of generality, that $\mathbb{K} \subset \mathbb{C}$ and $|\cdot|$ is the ordinary absolute value on $\mathbb{C}$.

We shall begin with the considering the case $r = 1$. In this case, $\alpha_i^{(j)} \in \mathbb{C}$, and, with the aid of inequality (7) which is applied to (22), we find that

$$|a| \prod_{j=1}^{g} |\overline{\alpha}^{(j)}| \le |F| e^{m \deg \mathfrak{p}}, \tag{24}$$

where $\overline{\alpha}^{(j)} = (1, \alpha_1^{(j)}, \dots, \alpha_m^{(j)})$. Since

$$|Q(\overline{\alpha}^{(j)})| \le |Q|(m+1)^{\deg Q} |\overline{\alpha}^{(j)}|^{\deg Q},$$

we find from (23), with the aid of inequality (24), that

$$|G| \le \left(|a| \prod_{j=1}^{g} |\overline{\alpha}^{(j)}|\right)^{\deg Q} |Q|^{\deg \mathfrak{p}} (m+1)^{\deg \mathfrak{p} \deg Q} \le |F|^{\deg Q} |Q|^{\deg \mathfrak{p}} e^{2m \deg \mathfrak{p} \deg Q};$$

this implies the assertion of Lemma 1.8 for $r = 1$.

Consider the case $r \ge 2$. Let $u'_{ij} \in \mathbb{C}$, $i = 1, \dots, r-1$, $j = 1, \dots, \deg \mathfrak{p}$, be arbitrary numbers such that

$$a(u'_{ij}) \ne 0, \qquad |u'_{ij}| \le 1, \tag{25}$$

and let τ be the homomorphism defined in Lemma 1.7. We denote $\beta_i^{(j)} = \tau(\alpha_i^{(j)}) \in \mathbb{C}$ and $\overline{\beta}^{(j)} = (1, \beta_1^{(j)}, \dots, \beta_m^{(j)}) \in \mathbb{C}^{m+1}$, $j = 1, \dots, \deg \mathfrak{p}$. Since

$$\tau(a) \prod_{j=1}^{g} (u_{r0} + \beta_1^{(j)} u_{r1} + \dots + \beta_m^{(j)} u_{rm}) = F(\overline{u}'_1, \dots, \overline{u}'_{r-1}, \overline{u}_r),$$

we have, by (7), the inequality

$$|\tau(a)|\prod_{j=1}^{g}|\overline{\beta}^{(j)}| \le |F|(m+1)^{(r-1)\deg\mathfrak{p}}e^{m\deg\mathfrak{p}}. \tag{26}$$

Since

$$|Q(\overline{\beta}^{(j)})| \le |Q|(m+1)^{\deg Q}|\overline{\beta}^{(j)}|^{\deg Q},$$

we find from (23), with the aid of inequality (26), that

$$|\tau(G)| \le \left(|\tau(a)|\prod_{j=1}^{g}|\overline{\beta}^{(j)}|\right)^{\deg Q}|Q|^{\deg\mathfrak{p}}(m+1)^{\deg\mathfrak{p}\,\deg Q} \le |F|^{\deg Q}|Q|^{\deg\mathfrak{p}}e^{m(r+1)\deg\mathfrak{p}\,\deg Q}.$$

Each point of the cube $|u_{ij}| \le 1$ is a limit of a certain sequence of points (u'_{ij}) satisfying conditions (25). We find by continuity that

$$\max_{|u_{ij}|\le 1}|G(u_{ij})| \le |F|^{\deg Q}|Q|^{\deg\mathfrak{p}}e^{m(r+1)\deg\mathfrak{p}\,\deg Q}.$$

In view of the inequality

$$|G| \le \max_{|u_{ij}|\le 1}|G(u_{ij})|,$$

this implies the assertion of Lemma 1.8 for $r \ge 2$.

Lemma 1.9. *The following inequality holds for the u-resultant of the associated form F of the prime ideal $\mathfrak{p} \subset \mathbb{K}[X]$, $x_0 \notin \mathfrak{p}$, and of the homogeneous polynomial $Q \subset \mathbb{K}[X]$:*

$$h(G) \le h(\mathfrak{p})\deg Q + h(Q)\deg\mathfrak{p} + \nu m(r+1)\deg\mathfrak{p}\,\deg Q,$$

where $r = 1 + \dim\mathfrak{p}$.

Proof. Assertion (2) of Lemma 1.5 and Lemma 1.8 yield

$$\begin{aligned} h(G) &= \sum_{v\in\mathcal{M}}\ln|G|_v \\ &\le \sum_{v\in\mathcal{M}}\big(\deg Q\,\ln|F|_v + \deg\mathfrak{p}\,\ln|Q|_v\big) + \sum_{v\in\mathcal{M}_\infty} m(r+1)\deg\mathfrak{p}\,\deg Q \\ &= h(\mathfrak{p})\deg Q + h(Q)\deg\mathfrak{p} + \nu m(r+1)\deg\mathfrak{p}\,\deg Q. \end{aligned}$$

Lemma 1.9 is proved.

1.7. The construction of zeros of an ideal which are sufficiently close to a given point. Let us agree to use the notation

$$\|\overline{\varphi}-\overline{\psi}\| = \Big(\max_{0\le i<j\le m}|\varphi_i\psi_j - \varphi_j\psi_i|\Big)|\overline{\varphi}|^{-1}|\overline{\psi}|^{-1}$$

for two nonzero points $\overline{\varphi} = (\varphi_0,\dots,\varphi_m) \in \mathcal{K}^{m+1}$ and $\overline{\psi} = (\psi_0,\dots,\psi_m) \in \mathcal{K}^{m+1}$ and call this quantity the *distance* between the points $\overline{\varphi}$ and $\overline{\psi}$. Obviously, the inequality

$$\|\overline{\varphi}-\overline{\psi}\| \le \begin{cases} 1 & \text{if } |\cdot| \text{ is non-Archimedean} \\ 2 & \text{if } |\cdot| \text{ is Archimedean} \end{cases}$$

always holds true.

Lemma 1.10. *Suppose that* $\mathfrak{p} \subset \mathbb{K}[X]$ *is a homogeneous prime ideal*, $r = 1 + \dim \mathfrak{p} \geq 1$, $x_0 \notin \mathfrak{p}$, $\overline{\omega} \in \mathcal{K}^{m+1}$, $|\overline{\omega}| = 1$, *and let* $\varkappa(a) \neq 0$. *Suppose, in addition, that* $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, *is an absolute value on the field* $\mathcal{K}$. *Then there exists a homomorphism*

$$\tau \colon B[a^{-1}] \to \mathcal{K}$$

which is identical on $\mathbb{K}$ *and such that for* $\beta_i^{(j)} = \tau(\alpha_i^{(j)}) \in \mathcal{K}$, *the vectors* $\overline{\beta}_j = (1, \beta_1^{(j)}, \dots, \beta_m^{(j)})$, $1 \leq j \leq \deg \mathfrak{p}$, *are zeros of the ideal* $\mathfrak{p}$, *and, moreover,*

(1) $|\tau(a)| \prod_{j=1}^{g} |\overline{\beta}_j| \leq |F| e^{2m^2 \deg \mathfrak{p}}$,

(2) $|\tau(a)| \prod_{j=1}^{g} \big(\|\overline{\omega} - \overline{\beta}_j\| \times |\overline{\beta}_j|\big) \leq |\mathfrak{p}(\overline{\omega})| \times |F| e^{2m^2 \deg \mathfrak{p}}$.

If $|\cdot|$ *is a non-Archimedean absolute value, then the factors* $e^{2m^2 \deg \mathfrak{p}}$ *can be omitted in these inequalities.*

In addition, for each polynomial $H \in \mathbb{K}[\overline{u}_1, \dots, \overline{u}_{r-1}]$, *the homomorphism* τ *can be chosen such that*

(3a) $|\varkappa(H)| = |\tau(H)|$ *if* $|\cdot|$ *is non-Archimedean, or*

(3b) $|\varkappa(H)| \leq (1 + \varepsilon)|\tau(H)|$ *for any a priori given* $\varepsilon > 0$ *if* $|\cdot|$ *is an Archimedean absolute value.*

Proof. For $r \geq 2$, we choose elements $\tau_{jk}^{(i)} \in \mathcal{K}$, $1 \leq i \leq r-1$, $0 \leq j \leq m$, $0 \leq k \leq m$, such that

$$\tau_{jk}^{(i)} + \tau_{kj}^{(i)} = 0, \qquad |\tau_{jk}^{(i)}| \leq 1,$$

and the polynomial $\varkappa(a)$ does not vanish when one replaces the variables $s_{jk}^{(i)}$ by the elements $\tau_{jk}^{(i)}$. Obviously, this can be done. Below we impose additional conditions on the choice of $\tau_{jk}^{(i)}$.

Now we define the homomorphism

$$\tau \colon \mathbb{K}[\overline{u}_1, \dots, \overline{u}_{r-1}] \to \mathcal{K}$$

assuming that this mapping is an embedding on $\mathbb{K}$ and is defined by the relations

$$\tau(u_{ij}) = u'_{ij} = \sum_{k=0}^{m} \tau_{jk}^{(i)} \omega_k, \qquad 1 \leq i \leq r-1, \quad 0 \leq j \leq m,$$

on the elements u_{ij}.

If the absolute value $|\cdot|$ is non-Archimedean, then we additionally choose elements $\tau_{jk}^{(i)}$ such that

$$|\varkappa(a)| = |\tau(a)|, \qquad |\varkappa(H)| = |\tau(H)|.$$

This can be done since the residue field of the absolute value $|\cdot|$ in the algebraically closed field $\mathcal{K}$ is itself algebraically closed, and therefore it is infinite.

Now if the absolute value $|\cdot|$ is Archimedean, we additionally choose elements $\tau_{jk}^{(i)}$ such that

$$|\varkappa(H)| \leq (1+\varepsilon)|\tau(H)|. \tag{27}$$

As was already indicated, in the case of an Archimedean absolute value one can assume that $\mathbb{K} \subset \mathcal{K} = \mathbb{C}$ and $|\cdot|$ is an ordinary absolute value on $\mathbb{C}$. In view of the multidimensional Cauchy formula, the value $|\varkappa(H)|$ does not exceed the maximum of the modulus of the polynomial $\varkappa(H)$ on the polydisk $|s_{jk}^{(i)}| \leq 1$ in this case. Therefore the choice of the numbers $\tau_{jk}^{(i)}$ which additionally satisfy (27) is possible.

We shall assume that the homomorphism τ is an embedding of the field $\mathbb{K}$ into $\mathcal{K}$ for $r = 1$.

Since $\tau(a) \neq 0$, by Lemma 1.7 the homomorphism τ is extended to the homomorphism of the ring $B[a^{-1}]$ into $\mathcal{K}$. The vectors $\beta_i^{(j)} = \tau(\alpha_i^{(j)})$ are zeros of the ideal $\mathfrak{p}$ by assertion (2) of Lemma 1.4. Relation (22) implies that

$$\tau(a) \prod_{j=1}^{g} (u_{r0} + \beta_1^{(j)} u_{r1} + \cdots + \beta_m^{(j)} u_{rm}) = \tau(F) \in \mathcal{K}[\overline{u}_r]. \tag{28}$$

We shall begin with the case where the absolute value $|\cdot|$ is Archimedean. As was already indicated, in this case we can assume that $\mathbb{K} \subset \mathcal{K} = \mathbb{C}$ and that $|\cdot|$ is the ordinary absolute value on $\mathbb{C}$. By inequality (7) in Lemma 1.1, we find from (28) that

$$|\tau(a)| \prod_{j=1}^{g} |\overline{\beta}_j| \leq e^{m \deg \mathfrak{p}} |\tau(F)|.$$

Since

$$|\tau(u_{ij})| \leq \sum_{k=0}^{m} |\tau_{jk}^{(i)}| \leq m+1,$$

we have

$$|\tau(F)| \leq |F|(m+1)^{(2r-2)\deg \mathfrak{p}} \leq |F| e^{(2r-2)m \deg \mathfrak{p}},$$

and therefore

$$|\tau(a)| \prod_{j=1}^{g} |\overline{\beta}_j| \leq |F| e^{2m^2 \deg \mathfrak{p}}.$$

The obtained inequality proves assertion (1).

Substituting the vector $S^{(r)}\overline{\omega}$ for $\overline{u}_r$ in relation (28), we obtain the relation

$$\tau(a) \prod_{j=1}^{g} \Bigg(\sum_{0 \leq i < k \leq m} s_{ik}^{(r)} (\beta_i^{(j)} \omega_k - \beta_k^{(j)} \omega_i) \Bigg) = \tau(F)\big|_{\overline{u}_r = S^{(r)}\overline{\omega}}.$$

We denote by C the polynomial from the ring $\mathcal{K}[S^{(r)}]$ which appears on the right-hand side of the last relation. With the aid of inequality (7), we find from Lemma 1.1 that

$$|\tau(a)| \prod_{j=1}^{g} \big(\|\overline{\omega} - \overline{\beta}_j\| \times |\overline{\beta}_j| \big) \leq e^{m^2 \deg \mathfrak{p}} |C|. \tag{29}$$

The polynomial C can be obtained from the polynomial $\varkappa(F)$ by replacing the variables $s_{jk}^{(i)}$ by the complex numbers $\tau_{jk}^{(i)}$, $1 \le i \le r-1$, $0 \le j < k \le m$. Therefore it follows from the definition of the value $|\mathfrak{p}(\overline{\omega})|$ that

$$|C| \le |\varkappa(F)| \Big(\frac{m^2+m}{2}\Big)^{r \deg \mathfrak{p}} = |\mathfrak{p}(\overline{\omega})| \times |F| \times \Big(\frac{m^2+m}{2}\Big)^{r \deg \mathfrak{p}} \le |\mathfrak{p}(\overline{\omega})| \times |F| \times e^{m^2 \deg \mathfrak{p}}.$$

The last inequality, together with (29), proves assertion (2). Thus, all assertions of Lemma 1.10 are proved for Archimedean absolute values.

The proof for a non-Archimedean absolute values can be carried out according to the same scheme as in the case of an Archimedean absolute value.

1.8. Construction of an ideal of a smaller dimension. We prove here the following

Proposition 1.4. *Let* $\mathfrak{p}$ *be a homogeneous prime ideal of the ring* $\mathbb{K}[X]$, $\dim \mathfrak{p} \ge 0$, *and let* Q *be a homogeneous polynomial from* $\mathbb{K}[X]$ *with* $Q \notin \mathfrak{p}$. *Assume, in addition, that* $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, *is an absolute value on* $\mathbb{K}$.

If $r = 1 + \dim \mathfrak{p} \ge 2$, *then there exists a homogeneous unmixed ideal* $J \subset \mathbb{K}[X]$, $\dim J = \dim \mathfrak{p} - 1$, *whose zeros coincide with those of the ideal* $(\mathfrak{p}, Q)$, *such that*

(1) $\deg J \le \deg \mathfrak{p} \deg Q$,

(2) $h(J) \le h(\mathfrak{p}) \deg Q + h(Q) \deg \mathfrak{p} + \nu m(r+1) \deg \mathfrak{p} \deg Q$,

(3) *for any point* $\overline{\omega} \in \mathcal{K}^{m+1}$ *and for* $\rho = \min \|\overline{\omega} - \overline{\beta}\|$, *where the minimum is taken over all nontrivial zeros* $\overline{\beta}$ *of the ideal* $\mathfrak{p}$ *in* $\mathcal{K}^{m+1}$, *the following inequality holds*:

$$\ln |J(\overline{\omega})| \le \ln \delta + h(Q) \deg \mathfrak{p} + h(\mathfrak{p}) \deg Q + 11 \nu m^2 \deg \mathfrak{p} \deg Q, \tag{30}$$

where

$$\delta = \begin{cases} \|Q\|_{\overline{\omega}} & \text{if } \rho < \|Q\|_{\overline{\omega}} \\ |\mathfrak{p}(\overline{\omega})|_{\overline{\omega}} & \text{if } \rho \ge \|Q\|_{\overline{\omega}}. \end{cases}$$

Inequality (30) *also holds for* $r = 1$ *if we formally assume that* $|J(\overline{\omega})| = 1$ *in this case.*

Proof. Since $\dim \mathfrak{p} \ge 0$, there exists a variable x_i which does not belong to the ideal $\mathfrak{p}$. We can assume without loss of generality that $x_0 \notin \mathfrak{p}$.

We shall first show how to construct the ideal J with the required properties in the case $r \ge 2$. To this end, we use the notation introduced in Lemma 1.6. For $j = 1, \dots, s$ we denote by I_j the primary ideal in the ring $\mathbb{K}[X]$ whose radical is equal to $\mathfrak{q}_j$ and whose exponent is r_j. For example, we can take the symbolic power $\mathfrak{q}_j^{(r_j)}$ (see [26, vol. 1, ch. 4, § 12]) as I_j. We set $J = I_1 \cap \dots \cap I_s$. Since all isolated prime ideals of the ideal $\mathfrak{a} = (\mathfrak{p}, Q)$ are of dimension $r-2$ (see [26, vol. 2, ch. 7, § 7, Theorem 21]), it follows from the second assertion of Lemma 1.6 that the varieties of zeros of the ideals J and $\mathfrak{a} = (\mathfrak{p}, Q)$ coincide. The ideal J is unmixed since $\dim \mathfrak{q}_i = r-2$, $i = 1, \dots, s$. By Proposition 1.1, we have

$$\overline{J}(r-1) = (E_1^{r_1} \cdots E_s^{r_s}),$$

and, hence, $G = E_1^{r_1} \cdots E_s^{r_s}$ is the associated form of the ideal J. Moreover, according to Theorem 1.6, the polynomial G is the u-resultant of the associated form F of the ideal $\mathfrak{p}$ and of the polynomial Q. Now, assertion (3) of Lemma 1.5 and Lemma 1.9 imply the first two assertions of Proposition 1.4.

To complete the proof of Proposition 1.4, we shall need one more lemma.

Lemma 1.11. *Let V and W be homogeneous polynomials from the ring $\mathbb{K}[X]$ of degree d. Then, for any nonzero vectors $\overline{\omega} = (\omega_0, \dots, \omega_m) \in \mathcal{K}^{m+1}$, $\overline{\xi} = (\xi_0, \dots, \xi_m) \in \mathcal{K}^{m+1}$, the following inequality holds*:

$$|V(\overline{\omega})W(\overline{\xi}) - V(\overline{\xi})W(\overline{\omega})| \le \|\overline{\omega} - \overline{\xi}\| \times |V| \times |W| \times |\overline{\omega}|^d |\overline{\xi}|^d (d+1)^{2m+1}.$$

If the absolute value $|\cdot|$ is non-Archimedean, then the factor $(d+1)^{2m+1}$ on the right-hand side of the last inequality can be omitted.

Proof. See [12, Lemma 7].

Corollary 1. *If V is a homogeneous polynomial of the ring $\mathbb{K}[X]$ and $\overline{\omega}, \overline{\xi} \in \mathcal{K}^{m+1}$ are nonzero points, and, moreover, if $V(\overline{\xi}) = 0$, then the following inequality holds*:

$$\|V\|_{\overline{\omega}} \le \|\overline{\omega} - \overline{\xi}\| \times e^{(2m+1)\deg V}.$$

If the absolute value $|\cdot|$ is non-Archimedean, then the factor $e^{(2m+1)\deg V}$ can be omitted.

Proof. Let $\overline{\xi} = (\xi_0, \dots, \xi_m)$. We apply Lemma 1.11 to the polynomials $V(X)$ and $W(X) = x_k^d$, where $d = \deg V$ and the subscript k is chosen such that $|\overline{\xi}| = |\xi_k|$. Using the fact that $V(\overline{\xi}) = 0$, we find from the inequality of Lemma 1.11 that

$$|V(\overline{\omega})\xi_k^d| \le \|\overline{\omega} - \overline{\xi}\| \times |V| \times |\overline{\omega}|^d \times |\overline{\xi}|^d \times e^{(2m+1)d};$$

this implies the required assertion by virtue of the definition of the value $\|V\|_{\overline{\omega}}$.

Corollary 2. *In the notation of Proposition* 1.4, *the following inequality holds*:

$$|\mathfrak{p}(\overline{\omega})| \le \rho e^{5m^2 \deg \mathfrak{p}}.$$

If the absolute value $|\cdot|$ is non-Archimedean, then the factor $e^{5m^2 \deg \mathfrak{p}}$ can be omitted.

Proof. We use the mapping ϑ which was introduced in the proof of Proposition 1.3. We represent the polynomial $\vartheta(F)$ as a polynomial in the variables $s_{jk}^{(i)}$ with coefficients from the ring $\mathbb{K}[x_0, \dots, x_m]$. Let $C(X) \in \mathbb{K}[x_0, \dots, x_m]$ be one of the coefficients of the polynomial $\vartheta(F)$. Then we have the inclusion $C(X) \in \mathfrak{p}$. Since the polynomial F in the variables u_{ij} consists of not more than

$$\binom{g+m}{m}^r \le (g+1)^{mr}$$

terms, $g = \deg \mathfrak{p}$, and $\vartheta(u_{ij})$ consists of not more than $m+1$ terms, we have the inequality

$$|C| \le |F|(m+1)^{rg}(g+1)^{mr} \le |F| e^{2m^2 \deg \mathfrak{p}}.$$

In addition,

$$\deg C(X) = rg \le m \deg \mathfrak{p}.$$

Applying Corollary 1 to the polynomial $C(X)$, we obtain

$$|C(\overline{\omega})| \le \rho |C| \times |\overline{\omega}|^{rg} \times e^{3mrg} \le \rho |F| \times |\overline{\omega}|^{rg} \times e^{5m^2 \deg \mathfrak{p}}.$$

The obtained inequality means that

$$|\varkappa(F)| \leq \rho|F| \times |\overline{\omega}|^{rg} \times e^{5m^2 \deg \mathfrak{p}}$$

and

$$|\mathfrak{p}(\overline{\omega})| = |\varkappa(F)| \times |F|^{-1} \times |\overline{\omega}|^{-rg} \leq \rho e^{5m^2 \deg \mathfrak{p}}.$$

This completes the proof of Corollary 2.

Let us now prove the last assertion of Proposition 1.4. We can assume without loss of generality that $x_0 \notin \mathfrak{p}$, $|\overline{\omega}| = 1$. By the definition of the value $|I(\overline{\omega})|$, we can easily obtain the inequality

$$|I(\overline{\omega})| \leq e^{\nu m r \deg I}$$

for any ideal I. By virtue of (1) and (2) of Proposition 1.4, this signifies the validity of (30) for $\delta \geq 2$. Therefore, for $\|Q\|_{\overline{\omega}} > 2$, inequality (30) holds for trivially reasons. We assume furthermore that $\|Q\|_{\overline{\omega}} \leq 2$.

As before, we shall use the following notation: F is the associated form of the ideal $\mathfrak{p}$ and $a \in \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_{r-1}]$ is the polynomial defined in the formulation of Lemma 1.4.

If the relation $\vartheta(a) = 0$ holds in the case $r \geq 2$, then we have the inclusion $a \in \overline{\mathfrak{p}}(r-1)$. However, this is impossible. Thus, $\vartheta(a) \neq 0$, and therefore there exist points $\overline{\omega}$ such that

$$\varkappa(a) \neq 0 \tag{31}$$

(the mapping $\varkappa$ is defined in Subsection 1.3). Obviously, each point $\overline{\omega} \in \mathcal{K}^{m+1}$ is a limit of the sequence of points $\overline{\omega}_n \in \mathcal{K}^{m+1}$ satisfying condition (31). Both sides of the inequality in assertion (3) of Proposition 1.4 are continuous functions of $\overline{\omega}$, and therefore it suffices to prove this assertion under the assumption that the point $\overline{\omega}$ satisfies condition (31).

We fix the positive number ε and define the homomorphism τ by means of Lemma 1.10 so that the following inequality holds:

$$|\varkappa(G)| \leq (1+\varepsilon)|\tau(G)|,$$

where the polynomial $G \in \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_{r-1}]$ is defined by relation (23). As in the proof of Lemma 1.10, we shall assume for $r = 1$ that the homomorphism τ is an embedding of the field $\mathbb{K}$ into $\mathcal{K}$.

Applying Lemma 1.11 to the points $\overline{\omega}$, $\overline{\xi} = \overline{\beta}_j$ and to the polynomials $V = Q$, $W = x_k^d$, $d = \deg Q$, where the subscript k is chosen such that $|\omega_k| = |\overline{\omega}| = 1$, we find that

$$|Q(\overline{\beta}_j)\omega_k^d - Q(\overline{\omega})(\beta_k^{(j)})^d| \leq \|\overline{\omega} - \overline{\beta}_j\| \times |Q| \times |\overline{\beta}_j|^d (d+1)^{2m+1},$$

and therefore,

$$\begin{aligned} |Q(\overline{\beta}_j)| &\leq |Q(\overline{\omega})| \times |\overline{\beta}_j|^d + \|\overline{\omega} - \overline{\beta}_j\| \times |Q| \times |\overline{\beta}_j|^d (d+1)^{2m+1} \\ &\leq |Q| \times |\overline{\beta}_j|^d \times e^{(2m+2)d} \max\bigl\{\|\overline{\omega} - \overline{\beta}_j\|, \|Q\|_{\overline{\omega}}\bigr\}. \end{aligned} \tag{32}$$

Note that if the absolute value $|\cdot|$ is non-Archimedean, then the last inequality can be improved:

$$|Q(\overline{\beta}_j)| \leq |Q| \times |\overline{\beta}_j|^d \times \max\bigl\{\|\overline{\omega} - \overline{\beta}_j\|, \|Q\|_{\overline{\omega}}\bigr\}.$$

By virtue of the relation

$$\tau(G)=\tau(a)^d\prod_{j=1}^{g}Q(1,\beta_1^{(j)},\dots,\beta_m^{(j)}),$$

inequalities (32), and also by assertions (1) and (2) of Lemma 1.10, we obtain

$$|\tau(G)|\le|Q|^{\deg\mathfrak{p}}\Big(|\tau(a)|\prod_{j=1}^{g}|\overline{\beta}_j|\Big)^d\times e^{(2m+2)d\deg\mathfrak{p}}\prod_{j=1}^{g}\max\{\|\overline{\varpi}-\overline{\beta}_j\|,\|Q\|_{\overline{\varpi}}\}. \tag{33}$$

This inequality also can be improved for the non-Archimedean absolute value $|\cdot|$:

$$|\tau(G)|\le|Q|^{\deg\mathfrak{p}}\Big(|\tau(a)|\prod_{j=1}^{g}|\overline{\beta}_j|\Big)^d\times\prod_{j=1}^{g}\max\{\|\overline{\varpi}-\overline{\beta}_j\|,\|Q\|_{\overline{\varpi}}\}. \tag{34}$$

We shall prove assertion (3) of Proposition 1.4 under the assumption that $|\cdot|$ is an Archimedean absolute value. If this absolute value is non-Archimedean, then the proof is carried out precisely in the same way but with the replacement of inequality (33) by (34). So, in what follows, we assume that $|\cdot|$ is an Archimedean absolute value.

We can assume without loss of generality that

$$\|\overline{\varpi}-\overline{\beta}_1\|\le\|\overline{\varpi}-\overline{\beta}_2\|\le\dots\le\|\overline{\varpi}-\overline{\beta}_g\|.$$

If $\|\overline{\varpi}-\overline{\beta}_1\|\ge\|Q\|_{\overline{\varpi}}$, then, by the inequalities of Lemma 1.10, we find from inequality (33) that

$$\begin{aligned}|\tau(G)|&\le|Q|^{\deg\mathfrak{p}}\Big(|\tau(a)|\prod_{j=1}^{g}|\overline{\beta}_j|\Big)^{d-1}\Big(|\tau(a)|\prod_{j=1}^{g}\big(\|\overline{\varpi}-\overline{\beta}_j\|\times|\overline{\beta}_j|\big)\Big)e^{(2m+2)d\deg\mathfrak{p}}\\&\le|\mathfrak{p}(\overline{\varpi})|\times|Q|^{\deg\mathfrak{p}}\times|F|^{\deg Q}\times e^{6m^2\deg Q\,\deg\mathfrak{p}}.\end{aligned}\tag{35}$$

Now if $\|\overline{\varpi}-\overline{\beta}_1\|<\|Q\|_{\overline{\varpi}}$, then, precisely in the same way, taking into account the inequalities $\|\overline{\varpi}-\overline{\beta}_j\|\le2$, $\|Q\|_{\overline{\varpi}}\le2$, we obtain

$$|\tau(G)|\le\|Q\|_{\overline{\varpi}}\times|Q|^{\deg\mathfrak{p}}\times|F|^{\deg Q}\times e^{7m^2\deg Q\,\deg\mathfrak{p}}. \tag{36}$$

We also use the inequality

$$-\ln|G|\le\sum_{\substack{v\in\mathcal{M}\\v\ne w}}\ln|G|_v, \tag{37}$$

which holds due to the fact that $h(G)\ge0$.

In the case $\|\overline{\varpi}-\overline{\beta}_1\|\ge\|Q\|_{\overline{\varpi}}$, with the aid of Lemmas 1.5 and 1.8, we find from (35) that

$$\begin{aligned}\ln|J(\overline{\varpi})|&=\ln|\varkappa(G)|-\ln|G|\le\ln(1+\varepsilon)+\ln|\tau(G)|+\sum_{\substack{v\in\mathcal{M}\\v\ne w}}\ln|G|_v\\&\le\ln(1+\varepsilon)+\ln|\mathfrak{p}(\overline{\varpi})|+\deg\mathfrak{p}\,\ln|Q|+\deg Q\,\ln|F|+6m^2\deg\mathfrak{p}\,\deg Q\\&\quad+\sum_{\substack{v\in\mathcal{M}\\v\ne w}}\big(\deg Q\,\ln|F|_v+\deg\mathfrak{p}\,\ln|Q|_v\big)+2(\nu-1)m^2\deg\mathfrak{p}\,\deg Q\\&\le\ln(1+\varepsilon)+\ln|\mathfrak{p}(\overline{\varpi})|+\deg\mathfrak{p}\,h(Q)+\deg Q\,h(F)+6\nu m^2\deg\mathfrak{p}\,\deg Q.\end{aligned}\tag{38}$$

Turning ε to zero in the obtained inequality, we get, in particular, assertion (3) of Proposition 1.4 for $\rho \ge \|Q\|_{\overline{\omega}}$.

Let us now consider the case $\rho < \|Q\|_{\overline{\omega}} \le \|\overline{\omega} - \overline{\beta}_1\|$. According to the corollary of Lemma 1.11, we have

$$\ln|\mathfrak{p}(\overline{\omega})| \le \ln\|Q\|_{\overline{\omega}} + 5m^2 \deg\mathfrak{p}$$

so that for ε tending to zero inequality (38) yields

$$\ln|J(\overline{\omega})| \le \ln\|Q\|_{\overline{\omega}} + \deg\mathfrak{p}\, h(Q) + \deg Q\, h(F) + 11\nu m^2 \deg\mathfrak{p}\ \deg Q;$$

this proves assertion (3) in the case under consideration.

In the case $\rho < \|\overline{\omega} - \overline{\beta}_1\| < \|Q\|_{\overline{\omega}}$, with the aid of inequality (37) and Lemmas 1.5 and 1.8, we find from (36) that

$$\begin{aligned}
\ln|J(\overline{\omega})| = \ln|\varkappa(G)| - \ln|G| &\le \ln(1+\varepsilon) + \ln|\tau(G)| + \sum_{\substack{v\in\mathcal{M}\\ v\ne w}} \ln|G|_v \\
&\le \ln(1+\varepsilon) + \ln\|Q\|_{\overline{\omega}} + \deg\mathfrak{p}\ \ln|Q| + \deg Q\ \ln|F| + 7m^2\deg\mathfrak{p}\ \deg Q \\
&\quad + \sum_{\substack{v\in\mathcal{M}\\ v\ne w}} \big(\deg Q\ \ln|F|_v + \deg\mathfrak{p}\ \ln|Q|_v\big) + 2(\nu-1)m^2\deg\mathfrak{p}\ \deg Q \\
&\le \ln(1+\varepsilon) + \ln\|Q\|_{\overline{\omega}} + \deg\mathfrak{p}\, h(Q) + \deg Q\, h(F) + 7\nu m^2 \deg\mathfrak{p}\ \deg Q.
\end{aligned}$$

Now, turning ε to zero, we find that inequality (30) also holds in this case.

Proposition 1.4 is completely proved.

In applications, we shall use the following assertion, which can easily be deduced from Proposition 1.4.

Corollary 3. *Suppose that $P \in \mathbb{K}[X]$, $P \not\equiv 0$, is a homogeneous polynomial, $\mathfrak{p}$ is a homogeneous prime ideal of the ring $\mathbb{K}[X]$, $\dim\mathfrak{p} \ge 0$, $P \notin \mathfrak{p}$, and let the point $\overline{\omega} \in \mathcal{K}^{m+1}$ and the absolute value $|\cdot|$ on the field $\mathcal{K}$ be such that*

$$|\mathfrak{p}(\overline{\omega})| \le e^{-S}, \quad S > 0, \qquad \|P\|_{\overline{\omega}} \le e^{-2m\nu\deg P}.$$

If the integer $\eta > 0$ satisfies the inequality

$$-\eta\ln\|P\|_{\overline{\omega}} \ge 2\min\Big\{S,\ \ln\frac{1}{\rho}\Big\}, \tag{39}$$

where $\rho = \min\|\overline{\omega} - \overline{\beta}\| < 1$ and the last minimum is taken over all nontrivial zeros $\overline{\beta}$ of the ideal $\mathfrak{p}$ in $\mathcal{K}^{m+1}$, then, for $r = 1 + \dim\mathfrak{p} \ge 2$, there exists a homogeneous unmixed ideal $J \subset \mathbb{K}[X]$, $\dim J = \dim\mathfrak{p} - 1$, whose zeros coincide with those of the ideal $(\mathfrak{p}, P)$, the ideal being such that

(1) $\deg J \le \eta\deg\mathfrak{p}\ \deg P$,

(2) $h(J) \le \eta\big(h(\mathfrak{p})\deg P + h(P)\deg\mathfrak{p} + \nu m(r+2)\deg\mathfrak{p}\ \deg P\big)$,

(3) $\ln|J(\overline{\omega})| \le -S + \eta\big(h(P)\deg\mathfrak{p} + h(\mathfrak{p})\deg P + 12\nu m^2\deg\mathfrak{p}\ \deg P\big)$.

Inequality (3) *is also valid for* $r = 1$ *if we formally assume that* $|J(\overline{\omega})| = 1$ *in this case.*

Indeed, let us apply Proposition 1.4 to the ideal $\mathfrak{p}$ and to the polynomial $Q = P^\eta$. Since $\deg Q = \eta \deg P$, and, according to Lemma 1.2,

$$h(Q) \leq \eta h(P) + \nu m \deg Q = \eta\big(h(P) + \nu m \deg P\big),$$

inequalities (1) and (2) of Proposition 1.4 yield inequalities (1) and (2) of Corollary 3.

In the case $\rho \geq \|Q\|_{\overline{\omega}}$, inequality (3) follows immediately from (30). Now if $\rho < \|Q\|_{\overline{\omega}}$, then we have

$$\ln\frac{1}{\rho} > -\ln\|Q\|_{\overline{\omega}} \geq -\eta\ln\|P\|_{\overline{\omega}} - \nu\eta m \deg P \geq -\frac{\eta}{2}\ln\|P\|_{\overline{\omega}} \geq \min\Big\{S,\ \ln\frac{1}{\rho}\Big\};$$

this implies that the minimum on the right-hand side of the last inequality is equal to S, and therefore $-\ln\|Q\|_{\overline{\omega}} \geq S$ so that $\ln\|Q\|_{\overline{\omega}} \leq -S$, and inequality (30) also yields assertion (3) in this case.

The same reasoning proves (3) for $r = 1$.

Note that in order to obtain better estimates for the quantities $\deg J$ and $h(J)$, it is more efficient to choose the minimal integer η for which (39) is valid.

1.9. The value of a prime ideal and the distance from a point to its variety of zeros. In this subsection, we shall study the relationships between $|I(\overline{\omega})|$ and the distance from the point $\overline{\omega}$ to the variety of zeros of the homogeneous unmixed ideal I. Intuitively, it is clear that if the value of an ideal at some point is sufficiently small, then this point must be sufficiently close to the variety of zeros of this ideal. The following assertion yields the exact characteristic of this closeness. In a certain sense, this assertion is converse to Corollary 2 of Lemma 1.11.

Proposition 1.5. *Let* $I \subset \mathbb{K}[X]$ *be a homogeneous unmixed ideal,* $r = 1 + \dim I \geq 1$, *and let* $|\cdot|$ *be an absolute value. For each nonzero point* $\overline{\omega} \in \mathcal{K}^{m+1}$, *there exists a zero* $\overline{\beta} \in \mathcal{K}^{m+1}$ *of the ideal* I *such that*

$$\deg I \times \ln\|\overline{\omega} - \overline{\beta}\| \leq \frac{1}{r}\ln|I(\overline{\omega})| + \frac{1}{r}h(I) + (\nu + 3)m^3 \deg I.$$

If the absolute value $|\cdot| = |\cdot|_w$, $w \in \mathcal{M}$, *is non-Archimedean, then the factor* $\nu + 3$ *on the right-hand side of this inequality can be replaced by* ν.

Proof. We can assume without loss of generality that $|\overline{\omega}| = 1$. Assume that there exist ideals which satisfy the hypothesis of Proposition 1.5 such that the assertion of this proposition is not longer true. Let I be one of these ideals, namely, the ideal whose dimension $\dim I$ is minimal. If $\overline{\omega}$ is a zero of the ideal I, then the assertion is obviously valid with $\overline{\beta} = \overline{\omega}$. Otherwise, there is a zero $\overline{\beta}$ among those of I for which $\overline{\omega} - \overline{\beta}$ assumes the minimal value, i.e., $\rho = \|\overline{\omega} - \overline{\beta}\| > 0$ in the notation of Proposition 1.4.

For the ideal I, we define homogeneous prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ and natural numbers $k_1, \ldots, k_s$ just as in the formulation of Proposition 1.2. Let us prove the inequalities

$$\deg\mathfrak{p}_j \times \ln\|\overline{\omega} - \overline{\beta}\| \leq \frac{1}{r}\ln|\mathfrak{p}_j(\overline{\omega})| + \frac{1}{r}h(\mathfrak{p}_j) + \Big(\nu + 3 - \frac{\nu+1}{r}\Big)m^3 \deg\mathfrak{p}_j, \qquad j = 1, \ldots, s. \tag{40}$$

Let $\mathfrak{p}$ be one of the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Since $\dim\mathfrak{p} = r - 1 \geq 0$, there exists a subscript i such that $x_i \notin \mathfrak{p}$. We can assume without loss of generality that $x_0 \notin \mathfrak{p}$. We shall use the notation introduced at the beginning of Subsection 1.3 before the statement of Proposition 1.2.

In the case $r = 1$ we have $a \in \mathbb{K}$. Since $\beta_0^{(j)} = 1$, and therefore $|\overline{\beta}_j| \geq 1$, it follows from inequality (2) of Lemma 1.10, that the following inequality holds:

$$\ln|a| + \sum_{j=1}^{\deg \mathfrak{p}} \ln\|\overline{\omega} - \overline{\beta}_j\| \leq \ln|\mathfrak{p}(\overline{\omega})| + \ln|F| + 2m^2 \deg \mathfrak{p}.$$

Using the fact that for any absolute value $|\cdot|_v$ the inequality $|a|_v \leq |F|_v$ holds and also the fact that

$$\sum_{v\in\mathcal{M}} \ln|a|_v = 0,$$

we find from the last relation that

$$\begin{aligned}\deg \mathfrak{p} \times \ln\|\overline{\omega} - \overline{\beta}\| &\leq \ln|a| + \sum_{\substack{v\in\mathcal{M}\\ v\neq w}} \ln|a|_v + \sum_{j=1}^{\deg \mathfrak{p}} \ln\|\overline{\omega} - \overline{\beta}_j\| \\ &\leq \ln|\mathfrak{p}(\overline{\omega})| + \ln|F| + 2m^2 \deg \mathfrak{p} + \sum_{\substack{v\in\mathcal{M}\\ v\neq w}} \ln|F|_v = \ln|\mathfrak{p}(\overline{\omega})| + h(\mathfrak{p}) + 2m^2 \deg \mathfrak{p}.\end{aligned}$$

Note that if the absolute value $|\cdot|$ is non-Archimedean, then the term $2m^2 \deg \mathfrak{p}$ is absent in the last inequality. This proves inequalities (40) in the case $r = 1$.

Next we shall assume that $r \geq 2$. Two cases are possible here.

(a) Assume that

$$\ln|\varkappa(a)| \geq \Big(1 - \frac{1}{r}\Big) \ln|\mathfrak{p}(\overline{\omega})| - (\nu+1)\Big(1 - \frac{1}{r}\Big)m^3 \deg \mathfrak{p} - \frac{1}{r}h(\mathfrak{p}) + \ln|F|.$$

It follows from Lemma 1.10 that there exist a homomorphism

$$\tau\colon \mathbb{K}[\overline{u}_1, \ldots, \overline{u}_{r-1}, a^{-1}, \alpha_1^{(1)}, \ldots, \alpha_m^{(g)}] \to \mathcal{K}$$

and zeros $\overline{\beta}_j, 1 \leq j \leq \deg \mathfrak{p}$, of the ideal $\mathfrak{p}$ such that $|\overline{\beta}_j| \geq 1$ and

$$\Big(1 - \frac{1}{r}\Big) \ln|\mathfrak{p}(\overline{\omega})| - (\nu+1)\Big(1 - \frac{1}{r}\Big)m^3 \deg \mathfrak{p} - \frac{1}{r}h(\mathfrak{p}) + \ln|F| \leq \ln|\varkappa(a)| \leq \ln|\tau(a)| + \ln(1+\varepsilon), \tag{41}$$

$$\ln|\tau(a)| + \sum_{j=1}^{\deg \mathfrak{p}} \ln\big(\|\overline{\omega} - \overline{\beta}_j\| \times |\overline{\beta}_j|\big) \leq \ln|\mathfrak{p}(\overline{\omega})| + \ln|F| + 2m^2 \deg \mathfrak{p}. \tag{42}$$

Since $\overline{\beta}_j$ are zeros of the ideal I, we have $\|\overline{\omega} - \overline{\beta}\| \leq \|\overline{\omega} - \overline{\beta}_j\|$, and, by the inequalities $|\overline{\beta}_j| \geq 1$ and the arbitrariness of $\varepsilon > 0$, we find from (41) and (42) that

$$\begin{gathered}\Big(1 - \frac{1}{r}\Big) \ln|\mathfrak{p}(\overline{\omega})| - (\nu+1)\Big(1 - \frac{1}{r}\Big)m^3 \deg \mathfrak{p} - \frac{1}{r}h(\mathfrak{p}) + \ln|F| + \deg \mathfrak{p}\, \ln\|\overline{\omega} - \overline{\beta}\| \\ \leq \ln|\mathfrak{p}(\overline{\omega})| + \ln|F| + 2m^2 \deg \mathfrak{p},\end{gathered}$$

i.e., we obtain inequality (40) for the ideal $\mathfrak{p}$.

(b) Assume that

$$\ln|\varkappa(a)| < \Big(1-\frac{1}{r}\Big)\ln|\mathfrak{p}(\overline{\omega})| - (\nu+1)\Big(1-\frac{1}{r}\Big)m^3\deg\mathfrak{p} - \frac{1}{r}h(\mathfrak{p}) + \ln|F|.$$

We denote the homogeneous unmixed ideal which was constructed in Proposition 1.4 for the ideal $\mathfrak{p}$ and the polynomial $Q = x_0 \notin \mathfrak{p}$ by J. It follows from this proposition that $\deg J \le \deg\mathfrak{p}$. Since the associated form G of the ideal J is defined by relation (23), i.e., $G = a$, and since for any $v \in \mathcal{M}$, the inequality $|a|_v \le |F|_v$ holds, for $|\cdot| = |\cdot|_w$ we have

$$\begin{aligned}
\ln|J(\overline{\omega})| &= \ln|\varkappa(a)| - \ln|a| \\
&< \Big(1-\frac{1}{r}\Big)\ln|\mathfrak{p}(\overline{\omega})| - (\nu+1)\Big(1-\frac{1}{r}\Big)m^3\deg\mathfrak{p} - \frac{1}{r}h(\mathfrak{p}) + \ln|F| + \sum_{\substack{v\in\mathcal{M}\\ v\ne w}}\ln|F|_v - h(J) \\
&= \Big(1-\frac{1}{r}\Big)\ln|\mathfrak{p}(\overline{\omega})| - (\nu+1)\Big(1-\frac{1}{r}\Big)m^3\deg\mathfrak{p} + \Big(1-\frac{1}{r}\Big)h(\mathfrak{p}) - h(J). \qquad (43)
\end{aligned}$$

Since $\dim J < \dim\mathfrak{p}$, the assertion of Proposition 1.5 holds for the ideal J, and therefore there exists a zero $\overline{\gamma} \in \mathcal{K}^{m+1}$ of the ideal J such that

$$\deg J \times \ln\|\overline{\omega} - \overline{\gamma}\| \le \frac{1}{r-1}\ln|J(\overline{\omega})| + \frac{1}{r-1}h(J) + (\nu+3)m^3\deg J. \qquad (44)$$

According to Proposition 1.4, the point $\overline{\gamma}$ is also a zero of the ideal $\mathfrak{p}$; therefore $\|\overline{\omega} - \overline{\beta}\| \le \|\overline{\omega} - \overline{\gamma}\|$, and, taking into account that $\|\overline{\omega} - \overline{\beta}\| \le 2$ and $\deg J \le \deg\mathfrak{p}$, $h(J) \le h(\mathfrak{p})$, we again obtain from (43) and (44) inequality (40) for the ideal $\mathfrak{p}$.

Thus, inequalities (40) hold for all j. With the aid of (40) and Proposition 1.2, we find that

$$\begin{aligned}
\deg I \times \ln\|\overline{\omega} - \overline{\beta}\| &= \sum_{j=1}^{\deg\mathfrak{p}} k_j\deg\mathfrak{p}_j\ln\|\overline{\omega} - \overline{\beta}\| \\
&\le \frac{1}{r}\sum_{j=1}^{\deg\mathfrak{p}} k_j\ln|\mathfrak{p}_j(\overline{\omega})| + \frac{1}{r}\sum_{j=1}^{\deg\mathfrak{p}} k_j h(\mathfrak{p}_j) + \Big(\nu+3-\frac{\nu+1}{r}\Big)m^3\sum_{j=1}^{\deg\mathfrak{p}} k_j\deg\mathfrak{p}_j \\
&\le \frac{1}{r}\ln|I(\overline{\omega})| + \frac{1}{r}h(I) + (\nu+3)m^3\deg I.
\end{aligned}$$

Thus the assertion of Proposition 1.5 holds for the ideal I and the point $\overline{\omega} \in \mathcal{K}^{m+1}$. The obtained contradiction with the choice of the ideal I means the validity of Proposition 1.5 for Archimedean absolute values. If $|\cdot|$ is a non-Archimedean absolute value, then the proof is the same, but instead of the coefficient $\nu + 3 - (\nu+1)/r$ in inequalities (40), we must have the coefficient $\nu - \nu/r$.

2. CONSTRUCTION OF AUXILIARY POLYNOMIALS

The content of this section is close to that of Section 2 in [3]. The difference is that instead of a sequence of auxiliary polynomials, as in [3], we construct here a two-parameter family of these polynomials. Here and in the next sections, the letters c with subscripts denote positive absolute constants.

Lemma 2.1. *For all numbers $L_1 \geq 1$, $L_2 \geq 1$ such that the sum $L_1 + L_2$ is sufficiently large, there exists a polynomial $A \in \mathbb{Z}[z, x_1, x_2, x_3]$, $A \not\equiv 0$, such that*

$$\deg_z A \leq L_1, \qquad \deg_{x_i} A \leq L_2, \quad i = 1, 2, 3, \qquad \ln H(A) \leq 85 L_2 \ln(L_1 + L_2),$$

where $H(A)$ is the maximum of the moduli of the coefficients of A, and the function

$$F(z) = A\bigl(z, P(z), Q(z), R(z)\bigr)$$

satisfies the relations

$$F^{(k)}(0) = 0, \qquad k = 0, 1, \dots, \Bigl[\frac{1}{2} L_1 L_2^3\Bigr] - 1.$$

Here and in the sequel (except for references) we denote by $[\,\cdot\,]$ the integral part of a number.

Proof. It is proved in [3] that the functions $P(z), Q(z)$, and $R(z)$ can be majorized as follows:

$$P(z) \ll \frac{24 \times 2!}{(1-z)^3}, \qquad Q(z) \ll \frac{240 \times 4!}{(1-z)^5}, \qquad R(z) \ll \frac{504 \times 6!}{(1-z)^7}.$$

In addition, for any natural k we have

$$z^k \ll \frac{1}{1-z}.$$

We introduce the set

$$\mathfrak{M} = \bigl\{\overline{k} = (k_0, k_1, k_2, k_3) \in \mathbb{Z}^4,\ 0 \leq k_0 \leq L_1, 0 \leq k_i \leq L_2,\ i = 1, 2, 3\bigr\}.$$

It follows from the majorants written above that for any vector $\overline{k} \in \mathfrak{M}$ we have

$$z^{k_0} P(z)^{k_1} Q(z)^{k_2} R(z)^{k_3} = \sum_{n=0}^{\infty} d(\overline{k}, n) z^n \ll c_1^{3L_2} (1-z)^{-16L_2},$$

where $c_1 = 504 \times 6!$ and $d(\overline{k}, n) \in \mathbb{Z}$. From this we find the estimate

$$|d(\overline{k}, n)| \leq c_1^{3L_2} (n + 16L_2)^{16L_2} \leq (c_2 n L_2)^{16L_2}, \quad n \geq 1, \qquad |d(\overline{k}, 0)| \leq 1. \tag{45}$$

Let now

$$A = \sum_{\overline{k} \in \mathfrak{M}} a(\overline{k}) z^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3},$$

where the set of integers $a(\overline{k})$ is chosen as a nontrivial solution to the system of linear homogeneous equations

$$\sum_{\overline{k} \in \mathfrak{M}} d(\overline{k}, n) a(\overline{k}) = 0, \qquad n = 0, 1, \dots, \Bigl[\frac{1}{2} L_1 L_2^3\Bigr] - 1.$$

The number of variables $u = [L_1 + 1] \times [L_2 + 1]^3$ and the number of equations $v = \bigl[\frac{1}{2} L_1 L_2^3\bigr]$ of this system are connected by the inequality $2v \leq u$. Therefore, by the Siegel lemma on the solution to a system of homogeneous linear equations (see, e.g., [25]) and by (45), this system has a nontrivial integer solution satisfying the inequality

$$\max_{\overline{k}\in\mathfrak{M}} |a(\overline{k})| \le L_1L_2^3(c_2L_1L_2^4)^{16L_2} \le (L_1+L_2)^{85L_2}$$

whenever the sum L_1+L_2 is sufficiently large. Lemma 2.1 is proved.

Now we introduce the notation

$$M = \operatorname{ord} F(z).$$

Lemma 2.1 and Theorem 3 imply the inequalities

$$\left[\frac{1}{2}L_1L_2^3\right] \le M \le cL_1L_2^3, \tag{46}$$

where $c = 10^{47}$.

We also set

$$r = \min\left\{\frac{1+|q|}{2}, 2|q|\right\};$$

and then $|q| < r < 1$.

Lemma 2.2. *If the number L_1+L_2 is sufficiently large, then, for all $z \in \mathbb{C}$, $|z| \le r$, the following estimate holds*:

$$|F(z)| \le |z|^M(L_1+L_2)^{189L_2}.$$

Proof. Assume that the function $F(z)$ has the following Taylor series expansion at the origin:

$$F(z) = \sum_{n=M}^{\infty} b_nz^n.$$

Then

$$b_n = \sum_{\overline{k}\in\mathfrak{M}} d(\overline{k},n)a(\overline{k}) \in \mathbb{Z},$$

and by (45) we have

$$|b_n| \le \sum_{\overline{k}\in\mathfrak{M}} (c_2nL_2)^{16L_2}(L_1+L_2)^{85L_2} \le n^{16L_2}(L_1+L_2)^{103L_2}, \qquad n \ge M.$$

For $|z| \le r$ we have

$$\begin{aligned}|F(z)| &\le \sum_{n=M}^{\infty}|b_n| \times |z|^n = \sum_{n=0}^{\infty}|b_{n+M}| \times |z|^{n+M} \le |z|^M(L_1+L_2)^{103L_2}\sum_{n=0}^{\infty}(n+M)^{16L_2}|z|^n\\ &\le |z|^M(L_1+L_2)^{103L_2}(M+1)^{16L_2}\left(1+\sum_{n=1}^{\infty}n^{16L_2}|z|^n\right)\\ &\le |z|^M(L_1+L_2)^{103L_2}(cL_1L_2^3+1)^{16L_2}(16L_2)!(1-r)^{-16L_2-1} \le |z|^M(L_1+L_2)^{189L_2}\end{aligned}$$

if L_1+L_2 is sufficiently large. Lemma 2.2 is proved.

Lemma 2.3. *There exists an integer T, $0 \le T \le \gamma L_2 \ln(L_1 + L_2)$, for which the following inequality holds*:

$$|F^{(T)}(q)| > \Big(\frac{1}{2}|q|\Big)^{2M},$$

where $\gamma = 190\big(\ln|r/q|\big)^{-1}$.

The proof of this lemma repeats literally the arguments used for proving Lemma 2.3 in [3]. It is only necessary to replace the expressions N^N and $N\ln N$ by $(L_1+L_2)^{L_2}$ and $L_2\ln(L_1+L_2)$, respectively, in several places.

Proposition 2.1. *Let $q \in \mathbb{C}$, $0 < |q| < 1$. For all numbers $L_1 \ge 1$, $L_2 \ge 1$ such that the sum $L_1 + L_2$ is sufficiently large there exists a polynomial $B \in \mathbb{Z}[z, x_1, x_2, x_3]$ such that*

$$\deg_z B \le L_1, \qquad \deg_{x_i} B \le 2\gamma L_2 \ln(L_1+L_2), \quad i = 1,2,3, \tag{47}$$

$$\ln H(B) \le 2\gamma L_2 \ln^2(L_1+L_2), \tag{48}$$

$$\exp(-\varkappa_2 L_1 L_2^3) \le \big|B\big(q, P(q), Q(q), R(q)\big)\big| \le \exp(-\varkappa_1 L_1 L_2^3), \tag{49}$$

where

$$\varkappa_1 = \frac{1}{4}\,\ln\frac{1}{r}, \qquad \varkappa_2 = 3c\ln\frac{2}{|q|},$$

and c is the constant from Theorem 3.

Proof. For any polynomial $E \in \mathbb{C}[z, x_1, x_2, x_3]$, the following identity holds:

$$\frac{d}{dz}E\big(z, P(z), Q(z), R(z)\big) = z^{-1}DE\big(z, P(z), Q(z), R(z)\big), \tag{50}$$

where

$$D = z\frac{\partial}{\partial z} + \frac{1}{12}(x_1^2 - x_2)\frac{\partial}{\partial x_1} + \frac{1}{3}(x_1x_2 - x_3)\frac{\partial}{\partial x_2} + \frac{1}{2}(x_1x_3 - x_2^2)\frac{\partial}{\partial x_3}$$

is an operator corresponding to the system of differential equations (1). Let us define the polynomial B by setting

$$B(z, x_1, x_2, x_3) = (12z)^T (z^{-1}D)^T A(z, x_1, x_2, x_3).$$

Here A is the polynomial which was constructed in Lemma 2.1 and T is the integer which was defined in Lemma 2.3. It is easy to prove by induction the identity

$$(z^{-1}D)^n = z^{-n}\prod_{k=0}^{n-1}(D-k), \qquad n \ge 1, \tag{51}$$

so that $B \in \mathbb{Z}[z, x_1, x_2, x_3]$. It follows from (50) that

$$B\big(z, P(z), Q(z), R(z)\big) = (12z)^T F^{(T)}(z),$$

where the function $F(z)$ is defined in the formulation of Lemma 2.1.

Lemma 2.3 and inequalities (46) yield the lower bound

$$\bigl|B\bigl(q, P(q), Q(q), R(q)\bigr)\bigr| \geq \Bigl(\frac{1}{2}|q|\Bigr)^{3M} \geq \exp(-\varkappa_2 L_1 L_2^3).$$

To obtain the upper bound, we use the formula

$$F^{(T)}(q) = \frac{T!}{2\pi i}\int\limits_{C_2} \frac{F(z)}{(z-q)^{T+1}}\,dz,$$

where C_2 is the circle $|z-q| = r - |q|$. It follows from the inequalities $|z| \leq |z-q| + |q| = r$ and also from (46), Lemma 2.2, and Lemma 2.3 that

$$\bigl|B\bigl(q, P(q), Q(q), R(q)\bigr)\bigr| \leq 12^T \times T! \times \bigl(r - |q|\bigr)^{-T} r^M \times (L_1+L_2)^{189L_2} \leq \exp(-\varkappa_1 L_1 L_2^3).$$

This proves inequalities (49).

In order to prove conditions (47) and (48), we use identity (51). If $E \in \mathbb{C}[z, x_1, x_2, x_3]$ and

$$E \ll H(1+x_1+x_2+x_3)^S(1+z+\cdots+z^L),$$

then, for any integer k, $|k| \leq T$, we have

$$\begin{aligned}(D+k)E \ll{} & |k|H(1+x_1+x_2+x_3)^S(1+z+\cdots+z^L)\\ & + HS(1+x_1+x_2+x_3)^{S-1}\bigl((x_1^2+x_2)+(x_1x_2+x_3)+(x_1x_3+x_2^2)\bigr)(1+z+\cdots+z^L)\\ & + HL(1+x_1+x_2+x_3)^S(1+z+\cdots+z^L)\\ \ll{} & H(S+L+T)(1+x_1+x_2+x_3)^{S+1}(1+z+\cdots+z^L).\end{aligned}$$

Therefore

$$12^T \prod_{k=0}^{T-1}(D-k)E(z, x_1, x_2, x_3) \ll 12^T H(S+L+2T)^T(1+x_1+x_2+x_3)^{S+T}(1+z+\cdots+z^L),$$

and since

$$A \ll (L_1+L_2)^{85L_2}(1+x_1+x_2+x_3)^{3[L_2]}\bigl(1+z+\cdots+z^{[L_1]}\bigr),$$

we obtain

$$B(z, x_1, x_2, x_3) \ll (L_1+L_2)^{85L_2}(36L_2+12L_1+24T)^T(1+x_1+x_2+x_3)^{3[L_2]+T}\bigl(1+z+\cdots+z^{[L_1]}\bigr).$$

This means that

$$\deg_z B \leq L_1, \qquad \deg_{x_i} B \leq 3L_2 + T \leq 2\gamma L_2 \ln(L_1+L_2),$$
$$H(B) \leq (L_1+L_2)^{85L_2}(36L_2+12L_1+24T)^T \times 4^{3[L_2]+T} \times (L_1+1) \leq \exp\bigl(2\gamma L_2 \ln^2(L_1+L_2)\bigr),$$

and proves estimates (47) and (48). Proposition 2.1 is proved.

3. PROOF OF THEOREM 1

In this section, we shall apply the auxiliary assertions proved above in the case $\mathbb{K} = \mathbb{Q}$ to the ordinary absolute value $|\cdot|$ on $\mathbb{Q}$. Theorem 1 will be deduced from the following assertion whose proof occupies the major part of this section.

Theorem 4. *Let $q \in \mathbb{C}$, $0 < |q| < 1$. Then, for each integer r, $1 \le r \le 3$, there exists a constant $\mu_r > 0$ such that for any homogeneous unmixed ideal $I \subset \mathbb{Q}[x_0, \dots, x_4]$, $\dim I = r - 1$, the following inequality holds*:

$$\ln |I(\overline{\xi})| \ge -\mu_r D^{\frac{4}{4-r}} (\ln D)^{\frac{3r}{4-r}},$$

where D is an arbitrary number satisfying the inequality

$$D \ge \max\bigl\{h(I) + \deg I \times \ln\bigl(h(I) + \deg I\bigr), e\bigr\}$$

and $\overline{\xi} = \bigl(1, q, P(q), Q(q), R(q)\bigr)$.

Corollary (see [3], Theorem 1). *For each $q \in \mathbb{C}$, $0 < |q| < 1$, there exist at least three numbers among $q, P(q), Q(q)$, and $R(q)$ that are algebraically independent over $\mathbb{Q}$.*

Proof. Let $\mathfrak{p}$ be a homogeneous prime ideal generated by all homogeneous polynomials in the ring $\mathbb{Q}[x_0, \dots, x_4]$ that vanish at $\overline{\xi}$. Then $\dim \mathfrak{p} = \operatorname{tr}\deg_{\mathbb{Q}}\bigl(q, P(q), Q(q), R(q)\bigr)$. If $\dim \mathfrak{p} \le 2$, then, by Theorem 4, we obtain inequality $|\mathfrak{p}(\overline{\xi})| > 0$, which contradicts Corollary 2 of Lemma 1.1.

We shall show how Theorem 1 can be derived from Theorem 4. We denote by $\mathfrak{p}$ a homogeneous prime ideal generated by all homogeneous polynomials of the ring $\mathbb{Q}[x_0, \dots, x_4]$ which vanish at the point $\overline{\xi}$. The values of $h(\mathfrak{p})$ and $\deg \mathfrak{p}$ can be uniquely determined from $\overline{\xi}$. According to the Corollary of Theorem 4 and to the conditions of Theorem 1, we have $\dim \mathfrak{p} = 3$.

For any polynomial $B \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ with $B(\overline{\xi}) \ne 0$, we denote by

$$C(x_0, x_1, x_2, x_3, x_4) = x_0^{\deg B} B\Bigl(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0}\Bigr)$$

a homogeneous polynomial which is not contained in $\mathfrak{p}$. Then $C(\overline{\xi}) = B(\overline{\xi})$. We denote by J the homogeneous unmixed ideal in the ring $\mathbb{Q}[X] = \mathbb{Q}[x_0, \dots, x_4]$ which is constructed for $\mathfrak{p}$ and C in accordance with Proposition 1.4. Then $\dim J = 2$,

$$\deg J \le c_3 \deg C, \qquad h(J) \le c_3\bigl(h(C) + \deg C\bigr).$$

In addition,

$$\ln |J(\overline{\xi})| \le \ln \|C\|_{\overline{\xi}} + c_3\bigl(h(C) + \deg C\bigr).$$

Applying Theorem 4 with $D = c_4\bigl(t(B) + \deg B \times \ln t(B)\bigr)$ to the ideal J for a sufficiently large constant c_4 and taking into account that $\deg C = \deg B$, $h(C) = h(B) \le \ln H(B)$, $r = 3$, we obtain the inequality

$$\ln\bigl|B\bigl(q, P(q), Q(q), R(q)\bigr)\bigr| \ge -c_5\bigl(t(B) + \deg B\ \ln t(B)\bigr)^4 \ln^9 t(B). \tag{52}$$

It follows from the hypothesis of Theorem 1 that all numbers ω_1, ω_2, and ω_3 are algebraic over the field $\mathbb{Q}\bigl(q, P(q), Q(q), R(q)\bigr)$. If the polynomial $E \in \mathbb{Q}[x_1, \dots, x_4]$ is such that with the number

$d = E\big(q, P(q), Q(q), R(q)\big)$ all numbers $d\omega_i$ are algebraic integers over the ring $\mathbb{Q}[q, P(q), Q(q), R(q)]$, then

$$\operatorname{Norm}\big(d^{\deg A} A(\omega_1, \omega_2, \omega_3)\big) = B\big(q, P(q), Q(q), R(q)\big),$$

where $B(x_1, x_2, x_3, x_4)$ is a polynomial with integer coefficients. Taking into account that

$$\deg B \le c_6 \deg A, \qquad \ln H(B) \le c_6 \ln H(A),$$

we easily obtain the estimate from Theorem 1 with the aid of (52).

We shall prove Theorem 4 by induction in r. Let r, $1 \le r \le 3$, be the least number for which the assertion of Theorem 4 is no longer true. We choose and fix a sufficiently large number λ.

The set of real numbers D for which there exists a prime ideal $\mathfrak{p}$ with $\dim \mathfrak{p} = r - 1$,

$$h(\mathfrak{p}) + \deg \mathfrak{p} \times \ln\big(h(\mathfrak{p}) + \deg \mathfrak{p}\big) \le D, \tag{53}$$

$$\ln|\mathfrak{p}(\overline{\xi})| < -2\lambda^{12} D^{\frac{4}{4-r}} (\ln D)^{\frac{3r}{4-r}}, \tag{54}$$

is unbounded. Indeed, otherwise, the inequality

$$\ln|\mathfrak{p}(\overline{\xi})| \ge -\gamma_1 \varphi(\mathfrak{p})^{\frac{4}{4-r}} \big(\ln \varphi(\mathfrak{p})\big)^{\frac{3r}{4-r}},$$

where $\varphi(\mathfrak{p}) = h(\mathfrak{p}) + \deg \mathfrak{p} \times \ln\big(h(\mathfrak{p}) + \deg \mathfrak{p}\big)$, would hold with a certain positive constant γ_1 for all homogeneous prime ideals $\mathfrak{p} \subset \mathbb{Q}[X]$, $\dim \mathfrak{p} = r - 1$. Applying now Proposition 1.2 to the arbitrary homogeneous unmixed ideal $I \subset \mathbb{Q}[X]$ of dimension $r - 1$, we successively obtain the inequalities

$$h(\mathfrak{p}_j) + \deg \mathfrak{p}_j \le 17\big(h(I) + \deg I\big), \qquad j = 1, \dots, s,$$

$$\sum_{j=1}^{s} k_j \varphi(\mathfrak{p}_j) \le \sum_{j=1}^{s} k_j \big(h(\mathfrak{p}_j) + \deg \mathfrak{p}_j \times \ln\big(17(h(I) + \deg I)\big)\big) \le c_7 \varphi(I).$$

Using now the fact that the function

$$x^{\frac{r}{4-r}} (\ln x)^{\frac{3r}{4-r}}$$

monotonically increases on the set $x \ge 1$, we infer that

$$\sum_{j=1}^{s} k_j \varphi(\mathfrak{p}_j)^{\frac{4}{4-r}} \big(\ln \varphi(\mathfrak{p}_j)\big)^{\frac{3r}{4-r}} \le \big(c_7 \varphi(I)\big)^{\frac{4}{4-r}} \big(\ln\big(c_7 \varphi(I)\big)\big)^{\frac{3r}{4-r}}.$$

Now the last inequality of Proposition 1.2 yields

$$\begin{aligned}
\ln|I(\overline{\xi})| &\ge \sum_{j=1}^{s} k_j \ln|\mathfrak{p}_j(\overline{\xi})| - 64 \deg I \\
&\ge -\gamma_1 \sum_{j=1}^{s} k_j \varphi(\mathfrak{p}_j)^{\frac{4}{4-r}} \big(\ln \varphi(\mathfrak{p}_j)\big)^{\frac{3r}{4-r}} - 64 \deg I \ge -\gamma_1 c_8 \big(\varphi(I)\big)^{\frac{4}{4-r}} \big(\ln\big(\varphi(I)\big)\big)^{\frac{3r}{4-r}}.
\end{aligned}$$

But this contradicts the assumption that the assertion of Theorem 4 does not hold for ideals of dimension $r - 1$.

Thus, the set of real numbers D for which there exists a prime homogeneous ideal $\mathfrak{p}$ of dimension $r-1$ satisfying conditions (53) and (54) is unbounded. Let now D be sufficiently large, and let $\mathfrak{p}$ be a prime homogeneous ideal of dimension $r-1$ satisfying conditions (53) and (54).

Let us define the number L by the relation

$$2\varkappa_2 L^4 \ln L = \min\Big\{2\lambda^{12} D^{\frac{4}{4-r}} (\ln D)^{\frac{3r}{4-r}},\ \ln\frac{1}{\rho}\Big\}, \tag{55}$$

where ρ is the distance from the point $\overline{\xi}$ to the variety of zeros of the ideal $\mathfrak{p}$ which was defined in Section 1, and $\varkappa_2$ is the constant from Proposition 2.1. It follows from Proposition 1.5 that when D increases, the quantity L also increases up to infinity.

Now we set $L_1 = L \ln L$, $L_2 = L$ and define the polynomial $B \in \mathbb{Z}[z, x_1, x_2, x_3]$ with the aid of Proposition 2.1. The degree of the polynomial B in the totality of variables does not exceed $n = [7\gamma L \ln L]$. Let us now define the homogeneous polynomial

$$E = x_0^n B\Big(\frac{x_1}{x_0}, \dots, \frac{x_4}{x_0}\Big) \in \mathbb{Z}[x_0, \dots, x_4].$$

We shall prove that E is not contained in the ideal $\mathfrak{p}$. If the inclusion $E \in \mathfrak{p}$ held, then, by Corollary 1 of Lemma 1.11, the inequality

$$|E(\overline{\xi})| \le \rho H(E) |\overline{\xi}|^n e^{9n}$$

would hold. Since $H(E) = H(B)$ and since $\ln H(B) \le 3\gamma L \ln^2 L$ by (48), it would follow from the obtained inequality in view of (55) that

$$\begin{aligned} \ln|B(\overline{\xi})| &= \ln|E(\overline{\xi})| \le \ln\rho + \ln H(B) + n\big(9 + \ln|\overline{\xi}|\big) \\ &\le -2\varkappa_2 L^4 \ln L + 3\gamma L \ln^2 L + 7\gamma\big(9 + \ln|\overline{\xi}|\big) L \ln L < -\varkappa_2 L^4 \ln L = -\varkappa_2 L_1 L_2^3. \end{aligned}$$

However, this contradicts the left inequality in (49). Thus, $E \notin \mathfrak{p}$.

Let us now apply Corollary 3 of Proposition 1.4 to the ideal $\mathfrak{p}$ and to the polynomial E. To this end, we set $\eta = 1 + [4\varkappa_2/\varkappa_1]$, where $\varkappa_i$ are the constants from Proposition 2.1. Since

$$-\eta \ln\|E\|_{\overline{\xi}} \ge -\eta \ln|B(\overline{\xi})| \ge \eta\varkappa_1 L_1 L_2^3 > 4\varkappa_2 L^4 \ln L = 2\min\Big\{2\lambda^{12} D^{\frac{4}{4-r}} (\ln D)^{\frac{3r}{4-r}},\ \ln\frac{1}{\rho}\Big\}$$

according to (49), inequality (39) holds. By Corollary 3 of Proposition 1.4, we can assert that for $r \ge 2$, there exists a homogeneous unmixed ideal J of the ring $\mathbb{Q}[x_0, \dots, x_4]$ for which

$$\deg J \le \eta n \deg\mathfrak{p}, \tag{56}$$

$$h(J) \le \eta\big(n h(\mathfrak{p}) + h(E)\deg\mathfrak{p} + 24n\deg\mathfrak{p}\big), \tag{57}$$

$$\ln|J(\overline{\xi})| \le -2\lambda^{12} D^{\frac{4}{4-r}} (\ln D)^{\frac{3r}{4-r}} + \eta\big(h(E)\deg\mathfrak{p} + n h(\mathfrak{p}) + 192n\deg\mathfrak{p}\big). \tag{58}$$

Inequality (58) in which we formally set $|J(\overline{\xi})| = 1$ also holds for $r = 1$.

It follows from (55) that

$$L \le \lambda^4 D^{\frac{1}{4-r}} (\ln D)^{\frac{r-1}{4-r}}, \tag{59}$$

and (53) implies

$$h(\mathfrak{p}) \le D, \qquad \deg \mathfrak{p} \le 2D(\ln D)^{-1}. \tag{60}$$

Using these inequalities and also inequalities (48) and $h(E) \le \ln H(B)$, we find that

$$\begin{aligned}
&\eta\big(h(E)\deg\mathfrak{p} + nh(\mathfrak{p}) + 192n\deg\mathfrak{p}\big) \\
&\quad\le \eta\big(3\gamma L\ln^2 L \times \deg\mathfrak{p} + 7\gamma L\ln L \times h(\mathfrak{p}) + 192 \times 7\gamma L\ln L \times \deg\mathfrak{p}\big) \\
&\quad\le 14\gamma\eta DL\ln L \le \lambda^5 D^{\frac{5-r}{4-r}}(\ln D)^{\frac{3}{4-r}}.
\end{aligned} \tag{61}$$

The obtained inequality, together with (58), yields

$$\ln|J(\overline{\xi})| \le -\lambda^{12} D^{\frac{4}{4-r}}(\ln D)^{\frac{3r}{4-r}}. \tag{62}$$

In the case $r = 1$, inequality (62) does not hold; therefore, in what follows, we assume that $r \ge 2$.

It follows from inequalities (56), (59), and (60) that

$$\deg J \le 14\gamma\eta D\ln^{-1} D\, L\ln L \le 28\gamma\eta DL \le \lambda^5 D^{\frac{5-r}{4-r}}(\ln D)^{\frac{r-1}{4-r}},$$

and we find from (57) and (61) that

$$h(J) \le \lambda^5 D^{\frac{5-r}{4-r}}(\ln D)^{\frac{3}{4-r}}.$$

The obtained inequalities mean that

$$\varphi(J) = h(J) + \deg J \times \ln\big(h(J) + \deg J\big) \le 4\lambda^5 D^{\frac{5-r}{4-r}}(\ln D)^{\frac{3}{4-r}},$$

$$\varphi(J)^{\frac{4}{5-r}}\big(\ln\varphi(J)\big)^{\frac{3r-3}{5-r}} \le \lambda^{11} D^{\frac{4}{4-r}}(\ln D)^{\frac{3r}{4-r}}.$$

Since $\dim J = r - 2$, according to the definition of the number r, the inequality

$$\ln|J(\overline{\xi})| \ge -\mu_{r-1}\varphi(J)^{\frac{4}{5-r}}\big(\ln\varphi(J)\big)^{\frac{3r-3}{5-r}} \ge -\mu_{r-1}\lambda^{11} D^{\frac{4}{4-r}}(\ln D)^{\frac{3r}{4-r}}$$

must hold for the ideal J. If we choose $\lambda > \mu_{r-1}$, then this inequality contradicts inequality (62). The obtained contradiction completes the proof of Theorem 4.

4. PROOF OF THEOREM 2

The method for proving Theorem 2 is similar to that used in Section 3. However, in this case we shall deal with the polynomial ring $\mathbb{K}[X] = \mathbb{K}[x_0, x_1, x_2, x_3]$, where the field $\mathbb{K} = \mathbb{Q}(q)$ is a finite extension of the field of rational numbers which is obtained by the addition of the algebraic number q. Then $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{C}$, $\nu = [\mathbb{K} : \mathbb{Q}]$; we assume that $|\cdot|$ is the ordinary absolute value on the field $\mathbb{C}$. The following assertion holds.

Theorem 5. *Let q, $0 < |q| < 1$, be an algebraic number. Then, for each integer r, $1 \le r \le 3$, there exists a constant $\mu'_r > 0$ such that for any homogeneous unmixed ideal $I \subset \mathbb{K}[x_0, x_1, x_2, x_3]$, $\dim I = r - 1$, the following inequality holds*:

$$\ln|I(\overline{\xi})| \ge -\mu'_r\big(h + d\ln(h+d)\big)d^{\frac{r}{4-r}}\big(\ln(h+d)\big)^{\frac{3r}{4-r}},$$

where d and h are arbitrary numbers satisfying the inequalities

$$\deg I \le d, \qquad h(I) \le h,$$

and $\overline{\xi} = \bigl(1, P(q), Q(q), R(q)\bigr)$.

Theorem 2 can be derived from this theorem in the following way. For the arbitrary polynomial $B \in \mathbb{Z}[x_1, x_2, x_3]$, we define the homogeneous polynomial

$$C(x_0, x_1, x_2, x_3) = x_0^{\deg B} B\Bigl(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\Bigr).$$

Then $C(\overline{\xi}) = B(\overline{\xi})$. We denote by $I = (C)$ the principal ideal of the ring $\mathbb{K}[X]$. Then $\dim I = 2$, and, according to Proposition 1.3, we have

$$\deg I = \deg C, \qquad h(I) \le 9\nu\bigl(h(C) + \deg C\bigr), \tag{63}$$

$$\ln|I(\overline{\xi})| \le \ln\|C\|_{\overline{\xi}} + 18\deg C. \tag{64}$$

Taking into account that $h(C) \le \nu \ln H(B)$ and $\deg C = \deg B$, we determine

$$d = \deg B, \qquad h = 9\nu^2\bigl(\ln H(B) + \deg B\bigr).$$

Then (63) implies the inequalities $\deg I \le d$ and $h(I) \le h$. Now, with the aid of Theorem 5 and inequality (64), we find that

$$\ln\|C\|_{\overline{\xi}} \ge \ln|I(\overline{\xi})| - 18\deg C \ge -\mu_3' c_9\bigl(\ln H(B) + d(B)\ln t(B)\bigr) d(B)^3 \ln^9 t(B).$$

In order to obtain an estimate for the polynomials at the point $\overline{\omega} \in \mathbb{C}^3$, we must use the same reasoning with the norm as in Section 3.

Proof of Theorem 5. The remaining part of this section is devoted to the proof of Theorem 5. Just as Theorem 4, we shall prove it by induction in r. Let r, $1 \le r \le 3$, be the least number for which the assertion of Theorem 5 is no longer true. We choose and fix a sufficiently large number λ.

The set of real numbers S for which there exist a pair of numbers d, h and a prime homogeneous ideal $\mathfrak{p} \subset \mathbb{K}[X]$ satisfying the conditions $\dim \mathfrak{p} = r - 1$,

$$\deg \mathfrak{p} \le d, \quad h(\mathfrak{p}) \le h, \qquad h + d\ln(h+d) = S, \qquad \ln|\mathfrak{p}(\overline{\xi})| < -2\lambda^{12} S d^{\frac{r}{4-r}} (\ln S)^{\frac{3r}{4-r}}, \tag{65}$$

is unbounded. Indeed, otherwise, the inequality

$$\ln|\mathfrak{p}(\overline{\xi})| \ge -\gamma_2 \varphi(\mathfrak{p}) (\deg \mathfrak{p})^{\frac{r}{4-r}} \bigl(\ln \varphi(\mathfrak{p})\bigr)^{\frac{3r}{4-r}},$$

where $\varphi(\mathfrak{p}) = h(\mathfrak{p}) + \deg \mathfrak{p} \times \ln\bigl(h(\mathfrak{p}) + \deg \mathfrak{p}\bigr)$, would hold with a certain positive constant γ_2 for all homogeneous prime ideals $\mathfrak{p} \subset \mathbb{Q}[X]$, $\dim \mathfrak{p} = r - 1$. Applying now Proposition 1.2 to the arbitrary unmixed ideal $I \subset \mathbb{K}[X]$ of dimension $r - 1$, we obtain, as in Section 3, the inequality

$$\sum_{j=1}^{s} k_j \varphi(\mathfrak{p}_j) \le c_{10} \varphi(I).$$

Using this inequality and also the fact that $\deg \mathfrak{p}_j \le \deg I$, we infer that

$$\sum_{j=1}^{s} k_j \varphi(\mathfrak{p}_j)(\deg \mathfrak{p}_j)^{\frac{r}{4-r}}\bigl(\ln \varphi(\mathfrak{p}_j)\bigr)^{\frac{3r}{4-r}} \le c_{10}\varphi(I)(\deg I)^{\frac{r}{4-r}}\bigl(\ln\bigl(c_{10}\varphi(I)\bigr)\bigr)^{\frac{3r}{4-r}}.$$

Now, the last inequality of Proposition 1.2 yields

$$\ln|I(\overline{\xi})| \ge \sum_{j=1}^{s} k_j \ln|\mathfrak{p}_j(\overline{\xi})| - 27\deg I \ge -\gamma_2 \sum_{j=1}^{s} k_j\varphi(\mathfrak{p}_j)(\deg \mathfrak{p}_j)^{\frac{r}{4-r}}\bigl(\ln\varphi(\mathfrak{p}_j)\bigr)^{\frac{3r}{4-r}} - 27\deg I$$
$$\ge -\gamma_2 c_{11}\varphi(I)(\deg I)^{\frac{r}{4-r}}\bigl(\ln(\varphi(I)\bigr)^{\frac{3r}{4-r}}.$$

However, this contradicts the assumption that the assertion of Theorem 4 does not hold for ideals of dimension $r-1$.

Thus, the set of real numbers S for which there exist a pair of numbers d, h and a prime homogeneous ideal $\mathfrak{p} \subset \mathbb{K}[X]$ of dimension $r-1$ with conditions (65) is unbounded. Let now S be sufficiently large, and let the numbers d, h, and the prime homogeneous ideal $\mathfrak{p}$ of dimension $r-1$ satisfy conditions (65).

We define the numbers L_1 and L_2 by the relations

$$L_1 = \lambda^3 S d^{\frac{r-3}{4-r}}(\ln S)^{\frac{3}{4-r}}, \tag{66}$$

$$2\varkappa_2 L_1 L_2^3 = \min\Bigl\{2\lambda^{12} S d^{\frac{r}{4-r}}(\ln S)^{\frac{3r}{4-r}},\ \ln\frac{1}{\rho}\Bigr\}, \tag{67}$$

where ρ is the distance from the point $\overline{\xi}$ to the variety of zeros of the ideal $\mathfrak{p}$ which was defined in Section 1 and $\varkappa_2$ is the constant from Proposition 2.1. It follows from (65) and Proposition 1.5 that

$$\varkappa_2 L_1 L_2^3 \ge \lambda^{11} S d^{\frac{2r-4}{4-r}}(\ln S)^{\frac{3r}{4-r}},$$

and therefore, for a sufficiently large λ, we have $L_1 \ge 1$ and $L_2 \ge 1$. Relations (66) and (67) give the estimate

$$L_2 \le \lambda^3 d^{\frac{1}{4-r}}(\ln S)^{\frac{r-1}{4-r}}.$$

Let us define the polynomial $B \in \mathbb{Z}[z, x_1, x_2, x_3]$ with the aid of Proposition 2.1. The degree n of the polynomial B in the totality of variables x_1, x_2, x_3 satisfies the inequality

$$n \le 12\gamma L_2 \ln S \le 12\gamma\lambda^3 d^{\frac{1}{4-r}}(\ln S)^{\frac{3}{4-r}}. \tag{68}$$

Let us now define the homogeneous polynomial $E \in \mathbb{K}[X]$ by the relation

$$E = x_0^n B\Bigl(q, \frac{x_1}{x_0}, \frac{x_1}{x_0}, \frac{x_3}{x_0}\Bigr).$$

If the polynomial B has the form

$$B = \sum_{\overline{k}} a_{\overline{k}}(z) x_1^{k_1} x_2^{k_2} x_3^{k_3}, \qquad a_{\overline{k}}(z) \in \mathbb{Z}[z],$$

then, for any absolute value $|\cdot|_v$, $v \in \mathcal{M}$, the following inequalities hold:

$$\max_{\overline{k}} |a_{\overline{k}}(q)|_v \le |B|_v \big(\max\{1, |q|_v\}\big)^{L_1}(1+L_1); \tag{69}$$

the factor $(1+L_1)$ can be omitted if $v \notin \mathcal{M}_\infty$, i.e., if the absolute value is non-Archimedean. Hence we find that

$$\begin{aligned} h(E) &= \sum_{v\in\mathcal{M}} \ln\Big(\max_{\overline{k}} |a_{\overline{k}}(q)|_v\Big) \le \sum_{v\in\mathcal{M}} \ln|B|_v + L_1 \sum_{v\in\mathcal{M}} \ln\big(\max\{1,|q|_v\}\big) + \nu\ln(1+L_1) \\ &\le h(B) + c_{12}L_1 \le \nu \ln H(B) + c_{12}L_1 \le c_{13}\lambda^3\Big(d^{\frac{1}{4-r}}(\ln S)^{\frac{7-r}{4-r}} + Sd^{\frac{r-3}{4-r}}(\ln S)^{\frac{3}{4-r}}\Big) \\ &\le \lambda^4 Sd^{\frac{r-3}{4-r}}(\ln S)^{\frac{3}{4-r}}. \end{aligned}$$

Let us prove that E is not contained in the ideal $\mathfrak{p}$. If the inclusion $E \in \mathfrak{p}$ held, then, by Corollary 1 of Lemma 1.11 being applied to E, the inequality

$$|E(\overline{\xi})| \le \rho |E| \times |\overline{\xi}|^n e^{7n} \tag{70}$$

would hold. With the aid of (69), we find that

$$\ln|E| \le \ln H(B) + L_1\big(\ln(1+|q|)+1\big) \le \lambda^4 Sd^{\frac{r-3}{4-r}}(\ln S)^{\frac{3}{4-r}}.$$

Now, by virtue of (67) and (68), inequality (70) implies

$$\begin{aligned} \ln|B(\overline{\xi})| &= \ln|E(\overline{\xi})| \le \ln\rho + \ln|E| + n\big(7+\ln|\overline{\xi}|\big) \\ &\le -2\varkappa_2 L_1 L_2^3 + \lambda^4 Sd^{\frac{r-3}{4-r}}(\ln S)^{\frac{3}{4-r}} + \big(7+\ln|\overline{\xi}|\big)12\gamma L_2 \ln S < -\varkappa_2 L_1 L_2^3. \end{aligned}$$

However, this contradicts the left inequality of (49). Thus, $E \notin \mathfrak{p}$.

Let us now apply Corollary 3 of Proposition 1.4 to the ideal $\mathfrak{p}$ and the polynomial E. By the Liouville theorem, the following inequality holds:

$$\ln|E| \ge -c\big(\ln H(B) + \deg_z B\big) \ge -\lambda^4 Sd^{\frac{r-3}{4-r}}(\ln S)^{\frac{3}{4-r}}.$$

We set $\eta = 1 + [5\varkappa_2/\varkappa_1]$, where $\varkappa_i$ is the constant from Proposition 2.1. Since

$$\begin{aligned} -\eta \ln\|E\|_{\overline{\xi}} &= -\eta\ln\big(|B(\overline{\xi})| \times |E|^{-1} \times |\overline{\xi}|^{-n}\big) \ge \eta\Big(\varkappa_1 L_1 L_2^3 + n\ln|\overline{\xi}| + \lambda^4 Sd^{\frac{r-3}{4-r}}(\ln S)^{\frac{3}{4-r}}\Big) \\ &> 4\varkappa_2 L_1 L_2^3 = 2\min\Big\{2\lambda^{12} Sd^{\frac{r}{4-r}}(\ln S)^{\frac{3r}{4-r}},\ \ln\frac{1}{\rho}\Big\} \end{aligned}$$

according to (49), inequality (39) holds. By Corollary 3 of Proposition 1.4, we can assert that for $r \ge 2$, there exists a homogeneous unmixed ideal J of the ring $\mathbb{K}[x_0, x_1, x_2, x_3]$ for which

$$\deg J \le \eta n \deg \mathfrak{p}, \tag{71}$$

$$h(J) \le \eta\big(h(E)\deg\mathfrak{p} + nh(\mathfrak{p}) + 15\nu n \deg\mathfrak{p}\big), \tag{72}$$

$$\ln|J(\overline{\xi})| \le -2\lambda^{12} Sd^{\frac{r}{4-r}}(\ln S)^{\frac{3r}{4-r}} + \eta\big(h(E)\deg\mathfrak{p} + nh(\mathfrak{p}) + 108\nu n\deg\mathfrak{p}\big). \tag{73}$$

For $r = 1$, we have inequality (73) in which we formally set $|J(\overline{\xi})| = 1$.

Using the inequalities proved above, we find that

$$\eta\big(h(E)\deg\mathfrak{p} + nh(\mathfrak{p}) + 108n\deg\mathfrak{p}\big) \le \eta\big(\lambda^4 S d^{\frac{r-3}{4-r}}(\ln S)^{\frac{3}{4-r}} d + 12\gamma\lambda^3 d^{\frac{1}{4-r}}(\ln S)^{\frac{3}{4-r}}(h + 108d)\big)$$
$$\le \lambda^5 S d^{\frac{1}{4-r}}(\ln S)^{\frac{3}{4-r}}. \tag{74}$$

This inequality, together with (73), yields

$$\ln|J(\overline{\xi})| \le -\lambda^{12} S d^{\frac{r}{4-r}}(\ln S)^{\frac{3r}{4-r}}. \tag{75}$$

In the case $r = 1$, inequality (75) does not hold; therefore, in what follows, we assume that $r \ge 2$.

Inequalities (71) and (68) imply

$$\deg J \le 12\gamma\eta\lambda^3 d^{\frac{5-r}{4-r}}(\ln S)^{\frac{3}{4-r}},$$

and from (72) and (74) we find that

$$h(J) \le \lambda^5 S d^{\frac{1}{4-r}}(\ln S)^{\frac{3}{4-r}}.$$

The obtained inequalities mean that

$$\varphi(J) = h(J) + \deg J \times \ln\big(h(J) + \deg J\big) \le 4\lambda^5 S d^{\frac{1}{4-r}}(\ln S)^{\frac{3}{4-r}},$$
$$\varphi(J)(\deg J)^{\frac{r-1}{5-r}}\big(\ln\varphi(J)\big)^{\frac{3r-3}{5-r}} \le \lambda^{10} S d^{\frac{r}{4-r}}(\ln S)^{\frac{3r}{4-r}}.$$

Since $\dim J = r - 2$, the inequality

$$\ln|J(\overline{\xi})| \ge -\mu'_{r-1}\varphi(J)(\deg J)^{\frac{r-1}{5-r}}\big(\ln\varphi(J)\big)^{\frac{3r-3}{5-r}} \ge -\mu'_{r-1}\lambda^{10} S d^{\frac{r}{4-r}}(\ln S)^{\frac{3r}{4-r}}$$

holds for the ideal J in accordance with the definition of the number r. If we choose $\lambda^2 > \mu'_{r-1}$, this inequality contradicts inequality (75). And this contradiction completes the proof of Theorem 5.

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