Proceedings of the Steklov Institute of Mathematics, Vol. 218, 1997, pp. 340–351.
Translated from Trudy Matematicheskogo Instituta imeni V. A. Steklova, Vol. 218, 1997, pp. 343–353.

Meromorphic Continuation of Dirichlet Series Connected with the Distribution of Numbers Modulo 1

A. I. Pavlov

Received February, 1997

In 1921 Hecke proved the following statement in his work [1]. Let $\alpha = \sqrt{D}$, where the integer $D \geq 1$ is a square-free number under the conditions $D \not\equiv 1 \pmod 4$. Then the series

$$F(s) = \sum_{n=1}^{\infty} \frac{\{\alpha n\} - 1/2}{n^s},$$

where $\{\cdot\}$ is the fractional part of the number, which converges for $\operatorname{Re} s > 1$, can be analytically continued to the whole plane of the complex variable s, except for the points $s_{q,r} = -2q + 2\pi i r/\ln\eta$, where $q = 0, 1, 2, \dots$, $r = 0, \pm 1, \pm 2, \dots$, η is the principal unit of the field $\mathbb{K}(\sqrt{D})$, where $\mathbb{K}$ is the field of rational numbers. At the points $s_{q,r}$ the function $F(s)$ has simple poles.

In 1923 Hardy and Littlewood, in their work [2], generalized Hecke's result to the case of any quadratic irrationality α.

Let M be the set of all functions which are analytic in the domain $\operatorname{Re} s > 1$ and which admit a meromorphic continuation to the whole plane of the variables s. We shall prove here the following statements.

Theorem 1. *If the integer $q \geq 2$ and $P(x)$ is an arbitrary polynomial, then*

$$\sum_{n=1}^{\infty} \frac{P(\{n^{1/q}\})}{n^s} \in M.$$

Theorem 2. *If the integer $q \geq 2$ and $P(x)$ is an arbitrary polynomial, then*

$$\sum_{n=1}^{\infty} \frac{P(\{\log_q n\})}{n^s} \in M.$$

Remark 1. In Theorems 1 and 2, in the case of $P(x) \equiv x$, the arrangement of poles of the corresponding function is indicated.

Theorem 3. *If the integer $q \geq 2$, then there exists an infinite set of irrational α such that*

$$\sum_{n=1}^{\infty} \frac{\{\alpha q^n\}}{n^s} \in M.$$

Remark 2. For each α from Theorem 3 we can indicate the arrangement of poles of the corresponding functions.

Proof of Theorem 1. We shall begin with a special case which is interesting in itself since much (if not all) can be calculated here to the end. According to the definition of the integral part for $\operatorname{Re} s > 3/2$ we have

$$\sum_{n=1}^{\infty} \frac{\{\sqrt{n}\}}{n^s} = \sum_{k=1}^{\infty} \sum_{k^2 \le n < (k+1)^2} \frac{\sqrt{n} - k}{n^s} = \zeta\Big(s - \frac{1}{2}\Big) - \sum_{k=1}^{\infty} k\Big(\sum_{n=k^2}^{\infty} n^{-s} - \sum_{n=(k+1)^2} n^{-s}\Big)$$
$$= \zeta\Big(s - \frac{1}{2}\Big) - f_1(s),$$

where

$$f_1(s) = \sum_{k=1}^{\infty} \sum_{n=k^2}^{\infty} n^{-s}. \tag{1}$$

Using for $\operatorname{Re} s > 0$ the relation

$$n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-nx}\, dx \tag{2}$$

(here and in the sequel $\Gamma(s)$ is the gamma-function), we get from (1) the relation

$$f_1(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{\psi(x/\pi)}{1 - e^{-x}}\, dx,$$

where

$$\psi(x) = \sum_{k=1}^{\infty} e^{-\pi x k^2}.$$

Let

$$f_2(s) = \frac{1}{\Gamma(s)} \int_0^{1} x^{s-1} \frac{\psi(x/\pi)}{1 - e^{-x}}\, dx, \qquad f_3(s) = \frac{1}{\Gamma(s)} \int_1^{\infty} x^{s-1} \frac{\psi(x/\pi)}{1 - e^{-x}}\, dx. \tag{3}$$

Then $f_1(s) = f_2(s) + f_3(s)$, and, as it is easy to see, the function $f_3(s)$ is entire. We know that

$$\sum_{k=-\infty}^{\infty} e^{-\pi x k^2} = \frac{1}{\sqrt{x}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/x}$$

for $x > 0$. Consequently, for $x > 0$ we have

$$\psi(x) = \frac{1}{\sqrt{x}} \psi\Big(\frac{1}{x}\Big) + \frac{1}{2}\Big(\frac{1}{\sqrt{x}} - 1\Big). \tag{4}$$

From (3) and (4) we obtain

$$f_2(s) = \frac{1}{2\Gamma(s)} \int_0^{1} \frac{x^{s-1}}{1 - e^{-x}} \Big(\sqrt{\frac{\pi}{x}} - 1\Big)\, dx + f_4(s), \tag{5}$$

where

$$f_4(s)=\frac{\sqrt{\pi}}{\Gamma(s)}\int\limits_0^1 x^{s-3/2}\frac{\psi(\pi/x)}{1-e^{-x}}\,dx$$

and the function $f_4(s)$ is entire. It is easy to see that

$$\begin{aligned}\int\limits_0^1 \frac{x^{s-1}}{1-e^{-x}}\left(\sqrt{\frac{\pi}{x}}-1\right)dx&=\int\limits_0^1\left(\sqrt{\pi}x^{s-3/2}-x^{s-1}\right)dx+\int\limits_0^1\frac{\sqrt{\pi}x^{s-3/2}-x^{s-1}}{e^x-1}\,dx\\&=\frac{\sqrt{\pi}}{s-1/2}-\frac{1}{s}+\int\limits_0^\infty\frac{\sqrt{\pi}x^{s-3/2}-x^{s-1}}{e^x-1}\,dx-f_5(s),\end{aligned}\tag{6}$$

where

$$f_5(s)=\int\limits_1^\infty\frac{\sqrt{\pi}x^{s-3/2}-x^{s-1}}{e^x-1}\,dx,$$

and the function $f_5(s)$ is entire. Hence, using (5), (6), and the relation

$$\zeta(s)\Gamma(s)=\int\limits_0^\infty\frac{x^{s-1}}{e^x-1}\,dx,\qquad \operatorname{Re}s>1,\tag{7}$$

we obtain

$$f_2(s)=\frac{1}{\Gamma(s)}\left(\frac{\sqrt{\pi}}{2s-1}-\frac{1}{2s}\right)+\frac{1}{2\Gamma(s)}\left(\sqrt{\pi}\Gamma\Big(s-\frac12\Big)\zeta\Big(s-\frac12\Big)-\Gamma(s)\zeta(s)\right)-\frac{f_5(s)}{2\Gamma(s)}.\tag{8}$$

We know that the function $1/\Gamma(s)$ is entire. Therefore, using relations (3) and (8), we obtain for $\operatorname{Re}s>3/2$ the relation

$$\sum_{n=1}^\infty\frac{\{\sqrt{n}\}}{n^s}=\left(1-\frac{\sqrt{\pi}\Gamma(s-1/2)}{2\Gamma(s)}\right)\zeta\Big(s-\frac12\Big)+\frac{\zeta(s)}{2}-\frac{\sqrt{\pi}}{(2s-1)\Gamma(s)}+\Phi(s),\tag{9}$$

where the funtion $\Phi(s)$ is entire. It is easy to see that the right-hand side of relation (9) is analytic function throughout the plane of the complex variable s, except for the simple poles at the points 1, $1/2$, and $1/2-k$, where k is an odd positive integer.

Let us now consider the general case. Obviously, it suffices to prove Theorem 1 for the series

$$\sum_{n=1}^\infty\frac{\{n^{1/q}\}^m}{n^s},$$

where the integer $m\ge1$. Since

$$\{n^{1/q}\}=(n^{1/q}-[n^{1/q}])^m=\sum_{\nu=0}^m(-1)^\nu\binom{m}{\nu}[n^{1/q}]^\nu n^{(m-\nu)/q},$$

it follows that

$$\sum_{n=1}^{\infty}\frac{\{n^{1/q}\}^m}{n^s}=\sum_{\nu=0}^{m}(-1)^{\nu}\binom{m}{\nu}\sum_{n=1}^{\infty}[n^{1/q}]^{\nu}n^{-(s-(m-\nu)/q)}.$$

This means that everything reduces to the analytic continuation of the series

$$f_{q,m}(s)=\sum_{n=1}^{\infty}[n^{1/q}]^m n^{-s},$$

where the integers $q\geq 2$ and $m\geq 1$. However, for $\operatorname{Re}s>1+m/q$ we have

$$\begin{aligned}f_{q,m}(s)&=\sum_{k=1}^{\infty}\sum_{k^q\leq n<(k+1)^q}[n^{1/q}]n^{-s}=\sum_{k=1}^{\infty}k^m\sum_{k^q\leq n<(k+1)^q}n^{-s}\\&=\sum_{k=1}^{\infty}\bigl(k^m-(k-1)^m\bigr)\sum_{n=k^q}^{\infty}n^{-s}=\sum_{\nu+1}^{m}(-1)^{\nu+1}\binom{m}{\nu}\sum_{k=1}^{\infty}k^{m-\nu}\sum_{n=k^q}^{\infty}n^{-s}.\end{aligned}$$

Therefore, in order to prove Theorem 1, it suffices, for any integer $\nu\geq 0$, to continue meromorphically the series

$$\varphi_{q,\nu}(s)=\sum_{k=1}^{\infty}k^{\nu}\sum_{n=k^q}n^{-s}$$

to the whole plane of the variable s. Using relation (2), for any integer $N\geq 1$ we obtain

$$\sum_{n=N}^{\infty}n^{-s}=\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}\frac{e^{-Nx}}{1-e^{-x}}\,dx.$$

Therefore

$$\varphi_{q,\nu}(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}\frac{\psi_{q,\nu}(x)}{1-e^{-x}}\,dx,$$

where

$$\psi_{q,\nu}(x)=\sum_{k=1}^{\infty}k^{\nu}e^{-xk^q},\qquad x>0.$$

Let

$$\varphi_{q,\nu}^{(1)}=\frac{1}{\Gamma(s)}\int_0^{1}x^{s-1}\frac{\psi_{q,\nu}(x)}{1-e^{-x}}\,dx,\qquad \varphi_{q,\nu}^{(2)}=\frac{1}{\Gamma(s)}\int_1^{\infty}x^{s-1}\frac{\psi_{q,\nu}(x)}{1-e^{-x}}\,dx.$$

Then $\varphi_{q,\nu}(s)=\varphi_{q,\nu}^{(1)}(s)+\varphi_{q,\nu}^{(2)}(s)$, and $\varphi_{q,\nu}^{(2)}$ is an entire function of s. It is known [3, p. 83] that for $x\to 0$, $x>0$, and for any fixed integer $N\geq 1$ the relation

$$\psi_{q,\nu}(x)=\frac{1}{q}\Gamma\Bigl(\frac{\nu+1}{q}\Bigr)x^{-(\nu+1)/q}+\sum_{k=0}^{N}c_kx^k+O(x^{N+1/2})$$

is satisfied, where the constants c_k do not depend on x under the sign O either. Consequently, for any integer $N \geq 1$ we have

$$\varphi_{q,\nu}^{(1)} = \frac{1}{q\Gamma(s)}\Gamma\left(\frac{\nu+1}{q}\right)\int_0^1 \frac{x^{s-(\nu+1)/q}}{1-e^{-x}}\,dx + \frac{1}{\Gamma(s)}\sum_{k=0}^{N} c_k \int_0^1 \frac{x^{s+k-1}}{1-e^{-x}}\,dx + R_N^{(1)}(s), \tag{10}$$

where the function $R_N^{(1)}(s)$ is analytic for $\operatorname{Re} s > -(N+1/2)$. It is easy to see that for any real α the relation

$$\int_0^1 \frac{x^{s+\alpha-1}}{1-e^{-x}}\,dx = \frac{1}{s+\alpha} + \int_0^\infty \frac{x^{s+\alpha-1}}{e^x-1}\,dx - g_\alpha(s),$$

where

$$g_\alpha(s) = \int_1^\infty \frac{x^{s+\alpha-1}}{e^x-1}\,dx$$

is entire function of s for any α, holds true for any $\operatorname{Re} s > -\alpha$. Consequently, using relation (7), we obtain

$$\int_0^1 \frac{x^{s+\alpha-1}}{1-e^{-x}}\,dx = \frac{1}{s+\alpha} + \Gamma(s+\alpha)\zeta(s+\alpha) - g_\alpha(s). \tag{11}$$

Returning to the function $\varphi_{q,\nu}(s)$, with the aid of relations (10) and (11), for $\operatorname{Re} s > \max\bigl(-(1+\nu)/q, -(N+1/2)\bigr)$ we arrive at the relation

$$\begin{aligned}\varphi_{q,\nu}(s) = \frac{1}{q}\Gamma\left(\frac{1+\nu}{q}\right)\left(s-\frac{1+\nu}{q}\right)^{-1} + \frac{1}{q}\Gamma\left(\frac{1+\nu}{q}\right)\frac{\Gamma(s-(\nu+1)/q)\zeta(s-(\nu+1)/q)}{\Gamma(s)}\\ + \frac{1}{\Gamma(s)}\sum_{k=0}^{N} c_k\left(\frac{1}{s+k} + \Gamma(s+k)\zeta(s+k)\right) + G_N(s),\end{aligned} \tag{12}$$

where the function $G_N(s)$ is analytic for $\operatorname{Re} s > -(N+1/2)$. Since N is an arbitrary positive integer, it follows from (12) that the function $\varphi_{q,\nu}(s)$ is analytic for any complex s, except for the simple poles at the points $(\nu+1)/q$ and $(\nu+1)/q-k$, where k is an odd positive integer, and $-k$, where k is a nonnegative integer.

Proof of Theorem 2. We shall begin with a special case where $P(x) \equiv x$. Then, for $\operatorname{Re} s > 1$, we have

$$\sum_{n=1}^\infty \frac{\{\log_q n\}}{n^s} = \sum_{n=1}^\infty \frac{\log_q n - [\log_q n]}{n^s} = -\log_q e \times \zeta'(s) - f_6(s),$$

where

$$f_6(s) = \sum_{n=1}^\infty \frac{[\log_q n]}{n^s} = \sum_{k=0}^\infty \sum_{q^k \leq n < q^{k+1}} k n^{-s} = \sum_{k=1}^\infty \sum_{n=q^k}^\infty n^{-s}.$$

Using relation (2), we get, for $\operatorname{Re} s > 1$, the relation

$$f_6(s) = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{\psi_q(x)}{1-e^{-x}}\,dx,$$

where

$$\psi_q(x) = \sum_{k=1}^{\infty} e^{-xq^k}.$$

We can show (see the lemma below) that the relation

$$\psi_q(x) = -\frac{\ln x}{\ln q} + c_0 + \sum_{k=1}^{N} \frac{(-1)^{k+1} q^k}{(q^k - 1)k!} x^k + \frac{1}{\ln q} \sum_{k=-\infty}^{\infty} x^{\gamma_k} \Gamma(\gamma_k) + O(x^{N+1/2}), \tag{13}$$

where $\gamma_k = 2\pi i k / \ln q$ and c_0 is a constant dependent only on q, is satisfied for any integer $N \geq 1$ when $x > 0$ and $x \to 0$. We shall show that the infinite series from the right-hand side of relation (13) converges. Hence, if

$$f_7(s) = \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \frac{\psi_q(x)}{1 - e^{-x}}\, dx, \qquad f_8(s) = \frac{1}{\Gamma(s)} \int_1^{\infty} x^{s-1} \frac{\psi_q(x)}{1 - e^{-x}}\, dx,$$

then $f_6(s) = f_7(s) + f_8(s)$, the function $f_8(s)$ is entire, and according to (13), the relation

$$\begin{aligned} f_7(s) = &-\frac{1}{\Gamma(s) \ln q} \int_0^1 \frac{x^{s-1} \ln x}{1 - e^{-x}}\, dx + \frac{c_0}{\Gamma(s)} \int_0^1 \frac{x^{s-1}}{1 - e^{-x}}\, dx + \frac{1}{\Gamma(s)} \sum_{k=1}^{N} c_k \int_0^1 \frac{x^{s+k-1}}{1 - e^{-x}}\, dx \\ &+ \frac{1}{\Gamma(s)} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\Gamma(\gamma_k)}{\ln q} \int_0^1 \frac{x^{s-\gamma_k-1}}{1 - e^{-x}} + R_N^{(2)}(s), \end{aligned}$$

where $c_k = (-1)^{k+1} q^k / \big((q^k - 1)k!\big)$ and the function $R_N^{(2)}(s)$ is analytic for $\operatorname{Re} s > -(N + 1/2)$, holds true. Since the relations

$$\int_0^1 x^{s-1} \ln x\, dx = -\frac{1}{s^2}, \qquad \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} \ln x}{e^x - 1}\, dx = \zeta'(s) + \frac{\Gamma'(s)}{\Gamma(s)} \zeta(s)$$

are satisfied for $\operatorname{Re} s > 1$, we find by analogy that

$$\begin{aligned} f_7(s) = &\frac{1}{s^2 \Gamma(s) \ln q} - \frac{1}{\ln q} \left(\zeta'(s) + \frac{\Gamma'(s)}{\Gamma(s)} \zeta(s) \right) + \frac{c_0}{s \Gamma(s)} + c_0 \zeta(s) + \frac{1}{\Gamma(s)} \sum_{k=1}^{N} \frac{c_k}{s+k} \\ &+ \frac{1}{\Gamma(s)} \sum_{k=1}^{N} c_k \Gamma(s+k) \zeta(s+k) + \frac{1}{\Gamma(s)} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\Gamma(\gamma_k)}{(s + \gamma_k) \ln q} \\ &+ \frac{1}{\Gamma(s) \ln q} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \Gamma(\gamma_k) \Gamma(s - \gamma_k) \zeta(s - \gamma_k) + R_N^{(3)}(s), \end{aligned}$$

where the function $R_N^{(3)}(s)$ is analytic for $\operatorname{Re} s > -(N + 1/2)$. It follows from the last relation that the function $f_7(s)$ can be analytically continued to the whole plane of the variable s, except for the simple poles at the points $1, 0, -1, -2, \dots$ and at the points $-m + \gamma_k$, where $k = \pm 1, \pm 2, \dots$ and $m = 0, -1, -3, -5, \dots$.

Before passing to the general case, we shall prove the following statement.

Lemma. *If the integer $\nu \geq 0$ and the real $q > 1$, then the relation*

$$\psi_{\nu,q}(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^{\nu} e^{-xq^k} = \sum_{k=0}^{\nu+1} c_k (\ln x)^k + \sum_{k=1}^{N} \frac{(-1)^{k+1} q^k}{(q^k-1)k!} x^k$$
$$+ \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} x^{-\gamma_k} \sum_{\mu=0}^{\nu} (-\ln x)^{\mu} \sum_{m=1+\mu}^{1+\nu} a_{\nu,\mu,m} \Gamma^{(m-1-\mu)}(\gamma_k) + O(x^{N+1/2}),$$

where $\gamma_k = 2\pi i k/\ln q$ and the constants c_k and $a_{\nu,\mu,m}$ do not depend on x, is satisfied for any positive integer $N \geq 1$ when $x > 0$ and $x \to 0$.

Proof. It is known [4, p. 50] that if $|z| < 1$ and the integer $\nu \geq 0$, then

$$\sum_{k=1}^{\infty} k^{\nu} z^k = \sum_{m=1}^{\nu+1} c_{\nu,m} \left(\frac{z}{z-1}\right)^m.$$

Consequently, for $\operatorname{Re} w > 0$ we have

$$\sum_{k=1}^{\infty} k^{\nu} q^{-kw} = \sum_{m=1}^{\nu+1} c_{\nu,m} (q^w - 1)^{-m}.$$

Therefore, using the relation

$$e^{-x} = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} x^{-w} \Gamma(w)\, dw, \qquad \sigma > 0,$$

we get, for $x > 0$, the relation

$$\psi_{\nu,q}(x) = \sum_{m=1}^{\nu+1} c_{\nu,m} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(w)}{x^w (q^w-1)^m}\, dw. \tag{14}$$

The function $\Gamma(w)/\big(x^w(q^w-1)^m\big)$ of w has the following singularities:

a pole of order $m+1$ at the point $w = 0$ with the residue $(\ln q)^{-m} \dfrac{(-\ln x)^m}{m!}$;

poles of order m at the points $w = \gamma_k$, where the integer $k \neq 0$, with the residues $\dfrac{(\ln q)^{-m}}{(m-1)!} \left(\dfrac{\Gamma(w)}{x^w}\right)^{(m-1)} \Bigg|_{w=\gamma_k}$;

simple poles at the points $w = -k$, where the integer $k \geq 1$, with the residue $\dfrac{(-1)^{k+1}(qx)^k}{(q^k-1)k!}$.

Therefore, replacing the straight line $\operatorname{Re} w = \sigma$ in relation (14) by the straight line $\operatorname{Re} w = -(N+1/2)$, what we can do if we use the relation

$$\ln|\Gamma(x+iy)| = -\frac{\pi}{2}|y| + \left(x - \frac{1}{2}\right) \ln|y| + O(1), \tag{15}$$

for the fixed x and $|y| \to \infty$, we obtain from (14), for $x > 0$ and $x \to 0$, the relation

$$\psi_{\nu,q}(x) = \sum_{k=0}^{\nu+1} c_k (\ln x)^k + \sum_{k=1}^{N} \frac{(-1)^{k+1} q^k}{(q^k - 1)k!} x^k + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} x^{-\gamma_k} \sum_{m=1}^{\nu+1} \frac{c_{\nu,m}}{(m-1)!(\ln q)^m} \sum_{\mu=0}^{m-1} \binom{m-1}{\mu} \Gamma^{(\mu)}(\gamma_k)(-\ln x)^{m-1-\mu} + O(x^{N+1/2}). \tag{16}$$

Let us show that for $0 \le \mu \le \nu$ the series

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} x^{-\gamma_k} \Gamma^{(\mu)}(\gamma_k) \tag{17}$$

converges. Indeed, relation (15) gives the estimate $|\Gamma(iy)| = O\big(\exp\{-\frac{\pi}{3}|y|\}\big)$, where y is real and $|y| \to \infty$. It is known that

$$\frac{\Gamma'(z)}{\Gamma(z)} = \gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)},$$

where γ is Euler's constant. Consequently, by induction on $\mu \ge 0$, we obtain

$$|\Gamma^{(\mu)}(iy)| = O\left(\exp\left\{-\frac{\pi}{4}|y|\right\}\right), \qquad |y| \to \infty.$$

Therefore the convergence of series (17) follows from the definition of γ_k. Let

$$C_{\nu,m} = \frac{c_{\nu,m}}{(m-1)!(\ln q)^m}.$$

Then

$$\sum_{m=1}^{\nu+1} C_{\nu,m} \sum_{\mu=0}^{m-1} \binom{m-1}{\mu} \Gamma^{(\mu)}(\gamma_k)(-\ln x)^{m-1-\mu} = \sum_{\mu=0}^{\nu} (-1)^{\mu} (\ln x)^{\mu} \sum_{m=1+\mu}^{\nu+1} C_{\nu,m} \binom{m-1}{\mu} \Gamma^{(m-1-\mu)}(\gamma_k).$$

Setting

$$C_{\nu,m} \binom{m-1}{\mu} = a_{\nu,\mu,m},$$

we get the statement of the lemma.

Passing to the general case of Theorem 2, we note that it suffices to prove the theorem for the case $P(x) \equiv x^m$, where the integer $m \ge 1$. For $\operatorname{Re} s > 1$ the relation

$$f_8(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\{\log_q n\}^m}{n^s} = \sum_{n=1}^{\infty} n^{-s} (\log_q n - [\log_q n])^m = \sum_{n=1}^{\infty} n^{-s} \sum_{\mu=0}^{m} \binom{m}{\mu} (-1)^{\mu} [\log_q n]^{\mu} (\log_q n)^{m-\mu}.$$

is satisfied.

It is easy to see that if

$$f_{\mu,q}(s)=\sum_{n=1}^{\infty}[\log_q n]^{\mu}n^{-s},$$

then

$$f_8(s)=\sum_{\mu=0}^{m}(-1)^m\binom{m}{\mu}(\ln q)^{\mu-m}\big(f_{\mu,q}(s)\big)^{(m-\mu)}$$

Consequently, it is sufficient to continue analytically the series $f_{\mu,q}(s)$ for every integer $\mu\ge 1$. We have

$$f_{\mu,q}(s)=\sum_{k=0}^{\infty}\sum_{q^k\le n<q^{k+1}}k^{\mu}n^{-s}=\sum_{k=1}^{\infty}\big(k^{\mu}-(k+1)^{\mu}\big)\sum_{n=q^k}^{\infty}n^{-s}=\sum_{\nu=0}^{\mu-1}(-1)^{\mu-\nu+1}\binom{\mu}{\nu}\sum_{k=1}^{\infty}k^{\nu}\sum_{n=q^k}n^{-s}.$$

Therefore, in order to prove Theorem 2, it suffices to continue meromorphically the series

$$\varphi_{\nu,q}(s)=\sum_{k=1}^{\infty}k^{\nu}\sum_{n=q^k}n^{-s}$$

for any integer $\nu\ge 0$.

Using relation (2), we obtain, for $\operatorname{Re}s>1$, the relation

$$\varphi_{\nu,q}(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}\frac{\psi_{\nu,q}(x)}{1-e^{-x}}\,dx,$$

where

$$\psi_{\nu,q}(x)=\sum_{k=1}^{\infty}k^{\nu}e^{-xq^k}.$$

Let

$$\varphi^{(1)}_{\nu,q}(s)=\frac{1}{\Gamma(s)}\int_0^{1}x^{s-1}\frac{\psi_{\nu,q}(x)}{1-e^{-x}}\,dx,\qquad \varphi^{(2)}_{\nu,q}(s)=\frac{1}{\Gamma(s)}\int_1^{\infty}x^{s-1}\frac{\psi_{\nu,q}(x)}{1-e^{-x}}\,dx.$$

Then $\varphi_{\nu,q}(s)=\varphi^{(1)}_{\nu,q}(s)+\varphi^{(2)}_{\nu,q}(s)$, and the function $\varphi^{(2)}_{\nu,q}(s)$ is entire. Using the lemma, we obtain

$$\varphi^{(1)}_{\nu,q}(s)=\sum_{k=0}^{\infty}c_k\frac{1}{\Gamma(s)}\int_0^1\frac{x^{s-1}(\ln x)^k}{1-e^{-x}}\,dx+\sum_{k=1}^{N}\frac{(-1)^{k+1}q^k}{(q^k-1)k!\Gamma(s)}\int_0^1\frac{x^{s+k-1}}{1-e^{-x}}\,dx$$

$$+\sum_{\substack{k=-\infty\\k\ne 0}}^{\infty}\sum_{\mu=0}^{\nu}(-1)^{\mu}\sum_{m=1+\mu}^{1+\nu}a_{\nu,\mu,m}\Gamma^{(m-1-\mu)}(\gamma_k)\frac{1}{\Gamma(s)}\int_0^1\frac{x^{s-\gamma_k-1}(\ln x)^{\mu}}{1-e^{-x}}\,dx+R_N^{(2)}(s),$$

where the function $R_N^{(2)}(s)$ is analytic for $\operatorname{Re} s > -(N+1/2)$. If $\operatorname{Re} s > 1$, then the relations

$$\frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s+k-1}}{1-e^{-x}}\,dx = \frac{1}{(s+k)\Gamma(s)} + \frac{\Gamma(s+k)\zeta(s+k)}{\Gamma(s)} - \frac{1}{\Gamma(s)}\int_1^\infty \frac{x^{s+k-1}}{e^x-1}\,dx,$$

$$\frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-1}(\ln x)^k}{1-e^{-x}}\,dx = \frac{(-1)^k k!}{s^{k+1}\Gamma(s)} + \frac{(\Gamma(s)\zeta(s))^{(k)}}{\Gamma(s)} - \frac{1}{\Gamma(s)}\int_1^\infty \frac{x^{s-1}(\ln x)^k}{e^x-1}\,dx, \tag{18}$$

$$\frac{1}{\Gamma(s)}\int_0^1 \frac{x^{s-\gamma_k-1}(\ln x)^\mu}{1-e^{-x}}\,dx = \frac{(-1)^\mu \mu!}{(s-\gamma_k)^{\mu+1}\Gamma(s)} + \frac{(\Gamma(s-\gamma_k)\zeta(s-\gamma_k))^{(\mu)}}{\Gamma(s)} - \frac{1}{\Gamma(s)}\int_1^\infty \frac{x^{s-\gamma_k-1}(\ln x)^\mu}{e^x-1}\,dx.$$

hold true. Note that the third term on the right-hand side of each of relations (18) is an entire function of s. Therefore, using relations (18), we get from (16) the relation

$$\begin{aligned}
\varphi_{\nu,q}^{(1)}(s) &= \frac{1}{\Gamma(s)}\sum_{k=0}^{N}\frac{(-1)^k k! c_k}{s^{k+1}} + \frac{1}{\Gamma(s)}\sum_{k=0}^{N} c_k^{(1)}\big(\Gamma(s)\zeta(s)\big)^{(k)} + \frac{1}{\Gamma(s)}\sum_{k=1}^{N}\frac{c_k^{(2)}}{s+k} \\
&+ \frac{1}{\Gamma(s)}\sum_{k=1}^{N} c_k^{(2)}\Gamma(s+k)\zeta(s+k) + \frac{1}{\Gamma(s)}\sum_{\mu=0}^{\nu}\mu!\sum_{m=1+\mu}^{1+\nu} a_{\nu,\mu,m}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{\Gamma^{(m-1-\mu)}(\gamma_k)}{(s-\gamma_k)^{\mu+1}} \\
&+ \frac{1}{\Gamma(s)}\sum_{\mu=0}^{\nu}(-1)^\mu\sum_{m=1+\mu}^{1+\nu} a_{\nu,\mu,m}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\Gamma^{(m-1-\mu)}(\gamma_k)\big(\Gamma(s-\gamma_k)\zeta(s-\gamma_k)\big)^{(\mu)} + R_N^{(2)}(s) + \Phi_1(s),
\end{aligned} \tag{19}$$

where $c_k^{(1)} = c_k$, $c_k^{(2)} = (-1)^{k+1}q^k/\big((q^k-1)k!\big)$ and $\Phi_1(s)$ is an entire function. The right-hand side of relation (19) is analytic for $\operatorname{Re} s > -(N+1/2)$, except for the points $0, -1, -2, \dots, -N$ and the points $\gamma_k - m$ and γ_k, where $k = \pm1, \pm2, \dots$ and m is an odd number. Since N is an arbitrary positive integer, this completes the proof of Theorem 2.

Proof of Theorem 3. Since

$$\{\alpha q^n\} = q^n \sum_{k=n+1}^{\infty}\frac{a_k}{q^k},$$

we have, for $\operatorname{Re} s > 1$, the relation

$$f_9(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty}\frac{\{\alpha q^n\}}{n^s} = \sum_{n=1}^{\infty} n^{-s}q^n \sum_{k=n+1}^{\infty}\frac{a_k}{q^k} = \sum_{k=2}^{\infty}\frac{a_k}{q^k}\sum_{n=1}^{k-1} q^n n^{-s}.$$

Using relation (2), we obtain

$$\sum_{n=1}^{k-1} q^n n^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\left(\sum_{n=1}^{k-1}(qe^{-x})^n\right)dx.$$

Consequently, for $\operatorname{Re} s > 1$ and $\alpha_1 = \{\alpha q\}$ we have

$$f_9(s) = \frac{1}{\Gamma(s)} \int\limits_0^\infty x^{s-1} \frac{\alpha_1 e^{-x} - \sum_{k=2}^\infty a_k e^{-kx}}{1 - qe^{-x}}\, dx. \tag{20}$$

We assume that for $x = \ln q$ the fraction on the right-hand side of relation (20) is equal to the number

$$\sum_{k=2}^\infty \frac{a_k(k-1)}{q^k}.$$

Let $a_{k_0+k} = 1$ for fixed integers $r \geq 2$ and $k_0 \geq 2$ if $k = m^r$ for a certain integer $m \geq 1$, and $a_{k_0+k} = 0$ if $k \neq m^r$ for any integer $m \geq 1$. Then

$$\sum_{k=2}^\infty a_k e^{-kx} = \sum_{k=2}^{k_0} a_k e^{-kx} + \Bigl(\sum_{m=1}^\infty e^{-xm^r}\Bigr) e^{-xk_0}.$$

This means that from (20) we have the relation

$$f_9(s) = -\frac{1}{\Gamma(s)} \int\limits_0^\infty x^{s-1} \frac{\psi_r(x) e^{-k_0 x}}{1 - qe^{-x}}\, dx - \frac{1}{\Gamma(s)} \sum_{k=2}^{k_0} a_k \int\limits_0^\infty \frac{x^{s-1} e^{-kx}}{1 - qe^{-x}}\, dx + \frac{\alpha_1}{\Gamma(s)} \int\limits_0^\infty \frac{x^{s-1} e^{-x}}{1 - qe^{-x}}\, dx,$$

where

$$\psi_r(x) = \sum_{m=1}^\infty e^{-xm^r}.$$

Let $x_q = \ln q/2$. Then

$$f_9(s) = \frac{\alpha_1}{\Gamma(s)} \int\limits_0^{x_q} \frac{x^{s-1} e^{-x}}{1 - qe^{-x}}\, dx - \frac{1}{\Gamma(s)} \int\limits_0^{x_q} x^{s-1} \frac{\psi_r(x) e^{-k_0 x}}{1 - qe^{-x}}\, dx - \frac{1}{\Gamma(s)} \sum_{k=2}^{k_0} a_k \int\limits_0^{x_q} \frac{x^{s-1} e^{-kx}}{1 - qe^{-x}}\, dx + \Phi_2(s), \tag{21}$$

where $\Phi_2(s)$ is an entire function. For any $k > 0$ the function

$$\frac{1}{\Gamma(s)} \int\limits_0^{x_q} \frac{x^{s-1} e^{-kx}}{1 - qe^{-x}}\, dx$$

is entire. This follows from the relation (the integer $m \geq 1$)

$$\frac{1}{\Gamma(s)} \int\limits_0^{x_q} \frac{x^{s-1} e^{-kx}}{1 - qe^{-x}}\, dx = \frac{(-1)^m}{s(s+1)\cdots(s+m-1)\Gamma(s)} \int\limits_0^{x_q} x^{s+m-1} \Bigl(\frac{e^{-kx}}{1 - qe^{-x}}\Bigr)^{(m)} dx + \Psi_k(s),$$

in which $\Psi_k(s)$ is an entire function and which can be easily proved by means of integration by parts. Using for $x > 0$ and $x \to 0$ the asymptotic relation

$$\psi_r(x) = \frac{1}{r}\Gamma\Bigl(\frac{1}{r}\Bigr) x^{-1/r} + \sum_{k=0}^N c_k x^k + O(x^{N+1/2}),$$

where the constants c_k do not depend on x and the integer $N \geq 1$ is fixed, we obtain

$$\frac{1}{\Gamma(s)} \int_0^{x_q} x^{s-1} \frac{\psi_r(x) e^{-k_0 x}}{1 - q e^{-x}}\, dx = \frac{1}{r} \frac{\Gamma(1/r)}{\Gamma(s)} \int_0^{x_q} \frac{x^{s-1/r-1} e^{-k_0 x}}{1 - q e^{-x}}\, dx + \frac{1}{\Gamma(s)} \sum_{k=0}^{N} c_k \int_0^{x_q} \frac{x^{s+k-1} e^{-k_0 x}}{1 - q e^{-x}}\, dx + R_N^{(3)}(s), \tag{22}$$

where the function $R_N^{(3)}(s)$ is analytic for $\operatorname{Re} s > -(N + 1/2)$. By means of integration by parts it is easy to prove that the first term on the right-hand side of relation (22) can be analytically continued to the whole plane of the variable s, except for the points $s = 1/r - m$, where m is a nonnegative integer, and the second term is an entire function. Since N is a positive integer, it suffices to use relations (21) and (22) in order to complete the proof of Theorem 3.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Fundamental Research (project no. 97-01-00721).

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Translated by I. Aleksandrova