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Spherical Convergence of the Fourier Series and Integral of the Indicator of a Two-Dimensional Domain

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1. INTRODUCTION

In [1, 2], attention was drawn to the problem of pointwise convergence of the method of summation over disks (balls) of Fourier series and integrals for indicators (characteristic functions) of domains in $\mathbb{R}^N$.

In [1] only spherically symmetric functions are considered, and it is shown that for dimension $N \geq 3$, the method of summation over disks [3–5] diverges for the characteristic function of a ball at the point which is the center of this ball. In [6], in the two-dimensional case, the pointwise convergence of the method of summation of the Fourier series over disks for the characteristic function of a domain with a convex boundary is proved. The author is not familiar with other results concerning the problem under consideration.

The main results of this paper refer to the two-dimensional case, but when we state the problem and describe the method, we consider the general case of an arbitrary dimension. This is due to the fact that the proposed approach is suitable for $N = 1, 2, 3$ and, possibly, for $N \geq 4$. In this paper, by examining the case $N = 2$, we demonstrate the difficulties that arise and the character of the results obtained. The case $N = 3$ is briefly discussed in Section 5.

We begin with the definition of the *general spherical method of summation of Fourier integrals and series* [5]. Everywhere in the sequel, we consider only bounded, absolutely integrable functions f on $\mathbb{R}^N$ with the support in the cube $[-\pi, \pi)^N$. We denote by $\langle x, y\rangle$ the inner product in $\mathbb{R}^N$, by $|x| = \sqrt{x_1^2 + \cdots + x_N^2}$ the norm of the element $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, and by

$$\widehat{f}(\omega) = \int\limits_{\mathbb{R}}^{N} e^{i\langle\omega,x\rangle} f(x)\,dx, \qquad \omega \in \mathbb{R}^N, \tag{1}$$

the Fourier transform of the function f. When considering the Fourier series, we always assume that $x \in [-\pi, \pi)^N$ and denote by f^T a periodic continuation of the function f from the cube $[-\pi, \pi)^N$ to $\mathbb{R}^N$. Thus, f^T is a function on the torus T^N which is obtained by the identification of the corresponding sides of the cube $[-\pi, \pi]^N$. In this case, the Fourier coefficients $a(n)$, $n \in \mathbb{Z}^N$, of the function f^T have the form

$$a(n) = (2\pi)^{-N} \widehat{f}(-n), \qquad n \in \mathbb{Z}^N. \tag{2}$$

By definition, the general spherically symmetric method A of summation of Fourier integrals and series is defined by the even function $A\colon \mathbb{R} \to \mathbb{R}$ and associates the function f_A with the function f, i.e.,

$$f_A(x) = (2\pi)^{-N} \int\limits_{\mathbb{R}}^{N} e^{-i\langle \omega,x\rangle} A(|\omega|)\widehat{f}(\omega)\,d\omega,$$

and the function f_A^T with the function f^T, i.e.,

$$f_A^T(x) = \sum_{n\in\mathbb{Z}^N} a(n)A(|n|)e^{i\langle n,x\rangle}. \tag{3}$$

In any case, both functions f_A and f_A^T exist if A is a bounded piecewise continuous function with compact support, $A(|\omega|) = 0$, $|\omega| > \Omega'$. Below we consider only the summation methods depending on one "large" parameter Ω for which

$$\begin{gathered} A(|\omega|) = G(\xi) = G(-\xi), \qquad \xi = \frac{|\omega|}{\Omega}, \\ G(\xi) \to 1, \quad \xi \to 0, \qquad G(\xi) = 0, \quad \xi > \xi_0. \end{gathered} \tag{4}$$

In particular,

$$G(\xi) = \begin{cases} 1, & |\xi| \le 1 \\ 0, & |\xi| > 1, \end{cases}$$

corresponds to the *summation over disks*, and

$$G(\xi) = \begin{cases} (1-\xi^2)^s, & |\xi| \le 1 \\ 0, & |\xi| > 1. \end{cases}$$

corresponds to the Riesz–Bochner summation [3, 4]. Below, for brevity, the method of summation over disks is called the Ω-*method* (Ω-*summation*), and, for this method, the functions f_A and f_A^T are denoted by f_Ω and f_Ω^T, respectively.

Let the function f be continuous in the neighborhood of the point x. The rate of convergence of the method A at the point x for the Fourier integral and series will be characterized by the quantities $\sigma_A(f|x)$ and $\sigma_A^T(f|x)$, respectively, which are defined as follows. Let

$$\Delta_A f(x) = f_A(x) - f(x), \qquad \Delta_A f^T(x) = f_A^T(x) - f^T(x). \tag{5}$$

Then $\sigma_A(f|x)$ and $\sigma_A^T(f|x)$ are *least upper bounds* of the quantities α and α' for which the following asymptotic estimates hold:

$$|\Delta_A f(x)| \le O(\Omega^{-\alpha}), \quad |\Delta_A f^T(x)| \le O(\Omega^{-\alpha'}), \qquad \Omega \to \infty.$$

By $\sigma_A(f)$ and $\sigma_A^T(f)$ we denote the *lower bounds of the quantities* $\sigma_A(f|x)$ *and* $\sigma_A^T(f|x)$ *on the set of points of continuity of the function* f. If the function f belongs to a certain class $\{f\}$, then $\sigma_A(\{f\})$ and $\sigma_A^T(\{f\})$ are *lower bounds of the quantities* $\sigma_A(f)$ *and* $\sigma_A^T(f)$ *on this class.* For the method of summation over disks, the corresponding quantities σ_A and σ_A^T are denoted by σ and σ^T.

In this paper, for $N = 2$, we consider the problem of computing the quantities $\sigma(f|x)$ and $\sigma^T(f|x)$ for $f \in \mathcal{F}_p$, where $\mathcal{F}_p$ is the class of characteristic functions of domains ($f = \chi_D$, $D \subset \mathbb{R}^2$) with a piecewise-smooth boundary ∂D. In what follows, we assume that the domains D are simply connected; obviously, this does not lead to the loss of generality. The exact definition of the class $\mathcal{F}_p$ will be given in Section 2. It should be pointed out that all the obtained results are directly extended to a wider class of functions $\{\mathcal{F}_p, I\}$, and $f \in \{\mathcal{F}_p, I\}$ if

$$f(x) = \sum_{i=1}^{I_0} g_i(x) f_i(x), \qquad f_i \in \mathcal{F}_p, \quad g_i \in C^\infty, \quad I_0 \le I.$$

Moreover,

$$\sigma_A(\{\mathcal{F}_p, I\}) = \sigma_A(\mathcal{F}_p), \qquad \sigma_A^T(\{\mathcal{F}_p, I\}) = \sigma_A^T(\mathcal{F}_p);$$

this follows immediately from the method of decomposition with respect to smoothness which is presented in [7].

It follows from the results of [6] that for a convex domain we have

$$\sigma^T(f|x) \ge \frac{1}{3}, \qquad N = 2.$$

A method is proposed in [8], which allows one to solve completely the problem of determination of the quantities $\sigma_A(f|x)$ for $N = 2$. In particular, it follows from the results of [8] that

$$\sigma(\mathcal{F}_p) = \frac{1}{2}, \qquad N = 2. \tag{6}$$

Let us formulate the results of [8] for the case $f = \chi_D$. Let ∂D be a general position smooth contour. This means that the *curvature* k of the curve ∂D as a function of the *natural parameter* s (s is the arc length) has only a finite number of isolated extrema (vertices) on ∂D, i.e., if $k^{(1)}(s_0) = 0$, then $k^{(2)}(s_0) \ne 0$. Any contour can be transferred to the general position by a small perturbation. For the general position contour, the *evolute* $\Gamma \equiv \Gamma(\partial D)$, which is defined as the envelope of normals or as the locus of curvature centers, has only simple cusps, as for an ellipse; we denote by Γ_0 the union of these cusps. Then, for $f = \chi_D$, $N = 2$, we have

$$\sigma(f|x) = \begin{cases} 1, & x \notin \Gamma \\ \dfrac{5}{6}, & x \in \Gamma, \quad x \notin \Gamma_0 \\ \dfrac{3}{4}, & x \in \Gamma_0. \end{cases}$$

For nongeneral position contours, $\sigma(f|x)$ can assume a certain sequence of rational numbers from the interval $[1/2, 1]$ as values, and the equality $\sigma(f|x) = 1/2$ is realized only for the characteristic function of a disk if x is the center of this disk.

Thus, the behavior of the quantity $\sigma(f|x)$ depends on the geometry of the boundary ∂D. It is interesting to clarify to what extent these results can be extended to Fourier series. One of the main results of this paper is the equality $\sigma^T(\mathcal{F}_p) = 1/2$, $N = 2$.

The approach to the problem of computing the quantities $\sigma(f|x)$ and $\sigma_A^T(f|x)$ presented below is based on the reduction of this problem to the problem of estimation of *oscillatory integrals* (OI) of the form

$$J[\Omega|u,\Phi] = \int_{\partial D} e^{i\Omega\Phi(s)} u(s)\,ds. \tag{7}$$

In the case of a Fourier series, the quantity $\Delta_A^T f(x)$ (4) can be represented (for $N=2$) in the form of a double series of these integrals. To estimate the sum of such a series, asymptotic estimates of OI which enter into this series are no longer sufficient, and uniform bounds of these integrals with the indication of constants in estimates, *estimates with constants*, are necessary. The methods of obtaining these estimates, which date back to van der Corput and Vinogradov, were developed in the works by Arkhipov, Karatsuba, and Chubarikov; the results obtained in these works can be found in [9, 10]. In turn, on the basis of [9, 10], estimates were obtained in [8] which are used below.

Everywhere in this paper, we accept the *convention concerning constants.*

Convention concerning constants. All constants whose values are not indicated are denoted by the letter C; one letter can be used to denote different constants. In all case, we indicate on what quantities the constant C depends; if these quantities are not indicated, then C is an absolute constant. All constants C can be given in explicit form. If constants C which are not indicated enter into the conditions of validity of a certain assertion or its formulation, then this means that there exist and can be explicitly indicated the values of the constants such that these conditions are fulfilled together with the corresponding assertions.

2. THE METHOD

We shall show how and in what sense the problem of convergence of the summation method for a Fourier series can be reduced to the corresponding problem for a Fourier integral. This is done with the aid of the Poisson integral. This reduction is widely used in the theory of Fourier series, for example, when solving the problem of equiconvergence of Fourier integral and series [4].

Lemma 1. *Assume that the following conditions hold*:

(1) *the method of summation is defined by a continuous even function A; moreover, $A(|\omega|) = 0$ $(|\omega| > \Omega')$,*

(2) *the function $f \in L^1$ has a support in the cube $[-\pi,\pi)^N$.*

If $x \in [-\pi,\pi)^N$, then

$$\Delta_A f^T(x) = \sum_{n\in\mathbb{Z}^N} \Delta_A f(x_n), \qquad x_n = x - 2\pi n, \tag{8}$$

under the condition of absolute convergence of the series on the right-hand side of this relation.

Proof. The Poisson formula [11]

$$\sum_{n\in\mathbb{Z}^N} g(n) = \sum_{n\in\mathbb{Z}^N} \widehat{g}(2\pi n) \tag{9}$$

holds if the following conditions are fulfilled:

(1) the function g is continuous at the points of the lattice, is integrable on each finite interval, and the series $\sum_n g(x+n)$ converges absolutely and uniformly in the cube $|x_i| \le 0.5$,

(2) the series on the right-hand side of the relation (9) converges.

We apply this formula to the series $f_A^T(x)$ (3). In this case, $g(\omega) = (2\pi)^{-N}\widehat{f}(\omega)A(|\omega|)e^{i\langle\omega,x\rangle}$, and the conditions of applicability of the Poisson formula hold by the virtue of assumptions that we have made. Thus,

$$f_A^T(x) = \sum_{n\in\mathbb{Z}^N} f_A(x_n)$$

and it remains to note that $f(x_n) = 0$ $(n \neq 0)$ for $x \in [-\pi,\pi)^N$.

For the method of summation over disks (Ω-summation), which is of interest to us, we have

$$f_\Omega^T(x) = \sum_{|n|\leq\Omega} a(n)e^{i\langle n,x\rangle}$$

and the conditions of Lemma 1 are not satisfied. In this case, we proceed as follows. Let A be a certain spherical summation method defined by a continuous even function A such that

$$A(|\omega|) \leq 1 \quad \forall\omega, \qquad A(|\omega|) = \begin{cases} 1, & |\omega| \leq \Omega \\ 0, & |\omega| > \Omega', \quad \Omega' = \Omega + \Delta\Omega, \quad \Delta\Omega > 0. \end{cases} \tag{10}$$

We introduce the quantity

$$\Delta_{A\Omega}f^T(x) = f_\Omega^T(x) - f_A^T(x) = -\sum_{\Omega<|n|\leq\Omega'} a(n)A(|n|)e^{i\langle n,x\rangle}, \tag{11}$$

for which, by (2) and (10), we have the estimate

$$|\Delta_{A\Omega}f^T(x)| \leq (2\pi)^{-N}\sum_{\Omega<|n|\leq\Omega'}|\widehat{f}(n)|. \tag{12}$$

By the definition of $\Delta_{A\Omega}f^T$,

$$\Delta_\Omega f^T(x) = \Delta_A f^T(x) + \Delta_{A\Omega}f^T(x).$$

Since relation (8) holds for $\Delta_A f^T(x)$, we have

$$|\Delta_A f^T(x)| \leq \sum_{n\in\mathbb{Z}^N}|\Delta_A f(x_n)|, \qquad x \in [-\pi,\pi)^N. \tag{13}$$

The estimate

$$|\Delta_\Omega f^T(x)| \leq |\Delta_A f^T(x)| + |\Delta_{A\Omega}f^T(x)|, \tag{14}$$

together with estimates (12) and (13), is a starting point for the estimation of the quantity $\Delta_\Omega f^T(x)$.

From this time on, we assume that $N = 2$ and use the vector notation $\mathbf{a}$, $\mathbf{r}$, $\mathbf{r}_0$, $\boldsymbol{\omega}$ for points from $\mathbb{R}^2$. Let us give the *definition of the class* $\mathcal{F}_p$ of functions being considered. We fix the integer $p \geq 0$. The contour $\partial D \subset \mathbb{R}^2$ is *p-piecewise-smooth* if it consists of a finite number of p-smooth arcs. An arc is *p-smooth* if it is a segment of a straight line or if it is defined by the function $\mathbf{r}(s)$ (s is a natural parameter) which has at least $p+3$ continuous derivatives, and if only a finite number of planar points s_i^0 $(k(s_i^0) = 0)$ can exist on this arc; moreover,

$$k^{(n)}(s_i^0) = 0, \quad n \leq p_i - 1, \qquad |k^{(p_i)}(s_i^0)| = K_i > 0, \quad p_i \leq p. \tag{15}$$

We say that $f \in \mathcal{F}_p$ if $f = \chi_D$ and ∂D is a p-piecewise-smooth contour.

The contour ∂D is said to be *strictly quasiconvex* if it is p-piecewise-smooth and its curvature k does not vanish anywhere, i.e.,

$$|k(s)| \geq k_0. \tag{16}$$

In this case, we can assume that $p = 0$. The contour ∂D is said to be *quasiconvex* if it consists of strictly quasiconvex arcs and segments of lines. Note that a smooth quasiconvex contour is convex.

The use of estimates (12) and (13) will be based on the fact that the quantities $\Delta_A f(x)$ and $\widehat{f}(\omega)$ can be represented (with the required accuracy) in the form of OI over the boundary (7), and, for $f \in \mathcal{F}_p$, these integrals admit an effective estimate [8].

Before considering these representations, note that everywhere below $\boldsymbol{\nu}$ is an *outer normal* to the contour ∂D; this contour is assumed to be positively oriented. In this case, the curvature k can assume both signs, and $k(s) > 0$ if the vector $\boldsymbol{\nu}$ is directed to the center of curvature. Moreover, the Serre–Frenet equations preserve their form

$$\frac{d\boldsymbol{\tau}}{ds} = k(s)\boldsymbol{\nu}(s), \quad \frac{d\boldsymbol{\nu}}{ds} = -k(s)\boldsymbol{\tau}(s), \qquad |\boldsymbol{\nu}| = 1, \quad |\boldsymbol{\tau}| = 1. \tag{17}$$

Here and in what follows, $\boldsymbol{\tau}(s)$ is a *tangent vector* to ∂D at the point $\mathbf{r}(s) \in \partial D$.

The representation of the quantity $\widehat{f}(\boldsymbol{\omega})$ (for $N = 2$) in the form of an OI results directly from definition (1). Passing to polar coordinates $\boldsymbol{\omega} = \lambda\boldsymbol{\eta}$, $\lambda = |\boldsymbol{\omega}|$, $\boldsymbol{\eta}(\cos\varphi, \sin\varphi)$ and applying the Green formula, we obtain

$$\widehat{f}(\lambda\boldsymbol{\eta}) = -\frac{i}{\lambda}\int\limits_{\partial D} e^{i\lambda\langle \mathbf{r}(s),\boldsymbol{\eta}\rangle}\langle \boldsymbol{\nu}(s), \boldsymbol{\eta}\rangle\, ds. \tag{18}$$

We are interested in the method of summation over disks; the consideration of other methods is of an auxiliary character and is connected only with the necessity of using estimate (14). Therefore, we shall assume below that the function G, (4), which defines the summation method, is piecewise-differentiable and can only have discontinuities of the first kind at the points $\xi_p \neq 0$. Moreover,

$$\sum_p |\Delta_p G| < \infty, \qquad \Delta_p G \equiv G(\xi_p + 0) - G(\xi_p - 0),$$

the derivative $G^{(1)}$ is piecewise-continuous, and $G^{(1)} \in L^1$. In this case, as is shown in [8], we have the representation

$$\begin{aligned}\Delta_A f(\mathbf{a}) = &\sum_{\xi_p \neq 0} \frac{\Delta_p G}{2\pi}\int\limits_{\partial D} ds\, \frac{\langle \mathbf{l}, \boldsymbol{\nu}\rangle}{|\mathbf{l}|^2} J_0(\xi_p\Omega|\mathbf{l}|) \\ &+ \frac{1}{2\pi}\int\limits_{\partial D} ds\, \frac{\langle \mathbf{l}, \boldsymbol{\nu}\rangle}{|\mathbf{l}|^2}\int\limits_0^\infty G^{(1)}(\xi) J_0(\Omega\xi|\mathbf{l}|)\, d\xi, \qquad \mathbf{l} \equiv \mathbf{r}(s) - \mathbf{a},\end{aligned} \tag{19}$$

where J_0 is a Bessel function. The possibility of this representation is based on the Gauss formula [12] according to which

$$\chi_D(\mathbf{a}) = \frac{1}{2\pi}\int\limits_{\partial D} \frac{\langle \mathbf{l}, \boldsymbol{\nu}\rangle}{|\mathbf{l}|^2}\, ds, \qquad N = 2, \quad \mathbf{a} \notin \partial D. \tag{20}$$

Relations (19) and (20) yield the required representation of the quantity $\Delta_\Omega f(\mathbf{a})$ in the form of an OI [8], i.e.,

$$\Delta_\Omega f(\mathbf{a}) = \operatorname{Re} \frac{1}{\sqrt{\Omega \times 2\pi^3}} \int\limits_{\partial D} \frac{\langle \mathbf{l}, \boldsymbol{\nu} \rangle}{|\mathbf{p}|^{5/2}} e^{i\Omega|\mathbf{l}|}\, ds + \Delta^1_\Omega f(\mathbf{a}), \qquad |\Delta^1_\Omega f(\mathbf{a})| \le \frac{CL}{\Omega^{3/2} d^{5/2}}. \tag{21}$$

Here and below L is the length of ∂D, and it is assumed that

$$|\mathbf{l}| \ge d, \qquad \Omega d \ge 1, \tag{22}$$

where d is the distance from the point $\mathbf{a}$ to ∂D.

For $N = 2, 3$, and, possibly, for all N, in order to consider the Ω-summation, it is sufficient to restrict ourselves by a special class of methods of summation (*q-summation*), which are given by the integer $q \ge 0$. For the q-summation,

$$G(\xi) = \begin{cases} 1, & 0 \le \xi \le 1 \\ H(\xi), & 1 \le \xi \le 1 + \delta, \quad \delta = \dfrac{\Delta\Omega}{\Omega} \\ 0, & \xi \ge 1 + \delta, \end{cases} \tag{23}$$

where H is a polynomial of degree $2q + 1$ satisfying the conditions

$$H(1) = 1, \quad H^{(1)}(1) = 0, \quad \dots, \quad H^{(q)}(1) = 0,$$
$$H(1+\delta) = 0, \quad H^{(1)}(1+\delta) = 0, \quad \dots, \quad H^{(q)}(1+\delta) = 0.$$

A polynomial of this kind does exist. It is unique, monotonic, and has the form

$$H(\xi) = h\left(1 - \frac{\xi - 1}{\delta}\right), \qquad h^{(1)}(y) = \frac{(2q+1)!}{(q!)^2} y^q (1-y)^q,$$
$$h(y) = \frac{(2q+1)!}{(q!)^2} \sum_{j=0}^{q} (-1)^j \binom{q}{j} \frac{y^{q+j+1}}{q+j+1}.$$

In particular,

$$h(y) = \begin{cases} y, & q = 0 \\ 3y^2 - 2y^3, & q = 1 \\ 10y^3 - 15y^3 + 6y^5, & q = 2. \end{cases}$$

When considering the q-summation for the quantities $\Delta_A f$ and $\Delta_A f^T$, we use the notation $\Delta_q f$, $\Delta_q f^T$.

It follows from the general results of [8] that

$$\Delta_q f(\mathbf{a}) = \operatorname{Im} \sum_{\xi_p} \sum_{j=q}^{2q} \frac{C_{jq}}{(\Delta\Omega)^{j+1}} \frac{1}{\sqrt{\xi_p \Omega}} J_{jp}(\mathbf{a}) + \Delta^1_q(\mathbf{a}), \qquad J_{jp}(\mathbf{a}) = \int\limits_{\partial D} ds\, \frac{\langle \mathbf{l}, \boldsymbol{\nu} \rangle}{|\mathbf{l}|^{j+7/2}} e^{i\xi_p \Omega |\mathbf{l}|}. \tag{24}$$

In this formula, ξ_p are points, where the smoothness of the function G is violated, i.e., $\xi_p = 1, 1+\delta$, and the estimate

$$|\Delta_q^1(\mathbf{a})| \le \frac{C_q L}{(\Omega d)^{3/2} d} \sum_{j=q}^{2q} \frac{1}{(\Delta\Omega \times d)^{j+1}}$$

holds for the quantity $\Delta_q^1(\mathbf{a})$. In particular, for $q = 0$ we have

$$\Delta_0 f(\mathbf{a}) = \operatorname{Im} \frac{A_0}{\Delta\Omega\Omega^{1/2}} \int\limits_{\partial D} ds \, \frac{\langle \mathbf{l}, \boldsymbol{\nu} \rangle}{|\mathbf{l}|^{7/2}} \left[\frac{e^{i\Omega(1+\delta)|\mathbf{l}|}}{\sqrt{1+\delta}} - e^{i\Omega|\mathbf{l}|} \right] + \Delta_0^1(\mathbf{a}),$$
$$|\Delta_0^1(\mathbf{a})| \le \frac{CL}{(\Omega d)^{3/2} d^2 \Delta\Omega}, \qquad A_0 = \frac{1}{\pi^{3/2}} e^{i3\pi/4}. \tag{25}$$

Thus, the estimation of the quantity $\Delta_q f(\mathbf{a})$ reduces to the estimation of one-dimensional OI of the form (7) with phase $|\mathbf{l}|$. It should be emphasized that the estimates with an explicit indication of constants in them (estimate with constants) are necessary since later on these estimates are substituted into series (13). These estimates will be considered in the next section. The required estimates for the quantities $\widehat{f}(\boldsymbol{\omega})$ (18) are already contained in [8] (see also [13]).

3. THE FOURIER INTEGRAL

Relations (21), (24), and (25) show that it is necessary to estimate one-dimensional OI of the form

$$J(\mathbf{a}) = \int\limits_{\partial D} ds \, e^{i\Omega\Phi(s)} u(s), \qquad u(s) = \frac{\langle \mathbf{l}, \boldsymbol{\nu} \rangle}{|\mathbf{l}|^t}, \tag{26}$$

with the phase $\Phi(s) = |\mathbf{l}(s)| = |\mathbf{a} - \mathbf{r}(s)|$, $\mathbf{r}(s) \in \partial D$, which does not depend on the summation method. When carrying out the estimation, we shall follow [8].

The estimation of OI begins with the analysis of critical points. In the case which is of interest to us, the critical points $s_c \equiv s_c(\mathbf{a})$ are determined from the conditions

$$\Phi^{(1)}(s_c) = 0 \iff \mathbf{l}(s_c) = \varepsilon_c |\mathbf{l}(s_c)| \boldsymbol{\nu}(s_c), \quad \varepsilon_c = \pm 1,$$

and at these points,

$$\Phi^{(2)}(s_c) = |\mathbf{l}(s_c)|^{-1} + \varepsilon_c k(s_c). \tag{27}$$

Let us consider the conditions of degeneration ($\Phi^{(2)}(s_c) = 0$). The evolute Γ of the curve ∂D is defined by the equation

$$\mathbf{a}_\Gamma(s) = \mathbf{r}(s) + \frac{\boldsymbol{\nu}(s)}{k(s)}, \qquad \mathbf{a}_\Gamma(s) \in \Gamma.$$

This implies that all critical points s_c are nondegenerate ($\Phi^{(2)}(s_c) \ne 0$) if $\mathbf{a} \notin \Gamma$, and there are degenerate critical points s_c^0 among them if $\mathbf{a} \in \Gamma$. Using the Serre–Frenet equations (17), we find that under the conditions

$$k^{(n)}(s_c^0) = 0, \quad 1 \le n \le j-1, \qquad |k^{(j)}(s_c^0)| = K_j^0, \quad j \ge 1, \tag{28}$$

we have

$$\Phi^{(n)}(s_c^0) = 0, \quad n \le j+1, \qquad \Phi^{(j+2)}(s_c^0) = \varepsilon_c k^{(j)}(s_c^0) \tag{29}$$

at this point. The case $j = 1$ corresponds to the fact that $\mathbf{a} \in \Gamma$, and the point $\mathbf{a}$ is not a singular point of the evolute ($\mathbf{a} \notin \Gamma_0$), where Γ_0 is a set of singular points. If $j \geq 2$, then $a \in \Gamma_0 \subset \Gamma$, $j = \infty$ if and only if ∂D is a circle and $\mathbf{a}$ is its center. The obtained results immediately imply

Theorem 1. *Let $f = \chi_D \in \mathcal{F}_p$, and suppose that we can draw a normal from the point $\mathbf{a}$ to ∂D. Then*

$$\sigma(f|\mathbf{a}) = \begin{cases} 1, & \mathbf{a} \notin \Gamma \\ \dfrac{5}{6}, & \mathbf{a} \in \Gamma, \quad \mathbf{a} \notin \Gamma_0 \\ \dfrac{1}{2} + \dfrac{1}{j+2}, & \mathbf{a} \in \Gamma_0, \end{cases} \tag{30}$$

where $j \equiv j(\mathbf{a})$ is the value of j, maximal for different points $s_c^0 \equiv s_c^0(\mathbf{a})$, such that conditions (28) *are fulfilled.*

If it is impossible to draw a normal from the point $\mathbf{a}$ to ∂D (∂D is a nonsmooth contour), then

$$\sigma(f|\mathbf{a}) = \frac{3}{2}. \tag{31}$$

Proof. To determine the quantity $\sigma(f|\mathbf{a})$, it suffices to know the asymptotics of the quantity $\Delta_\Omega f(\mathbf{a})$ as $\Omega \to \infty$. By (21), this asymptotic behavior is defined by the contribution of the critical point s_c of maximal degeneration (provided that critical points exist) [14]. Taking now into account (27) and (29), we immediately obtain (30). If it is impossible to draw a normal from the point $\mathbf{a}$ to ∂D, then the contour ∂D is nonsmooth, and the asymptotic behavior of the integral $J(\mathbf{a})$ is defined by the contribution of angle points. Integrating by parts, we obtain (31).

The knowledge of the asymptotics of the integral $J(\mathbf{a})$ is no longer sufficient for analyzing the quantity $\Delta_\Omega f^T(\mathbf{a})$, and the estimates with constants are necessary. Although the method of [8] allows us to obtain asymptotically exact estimates with constants for this integral at all $\mathbf{a}$ for an arbitrary p-piecewise-smooth contour ∂D, we shall prove only the simplest estimates.

Recall that we always assume that conditions (22) are fulfilled and $\partial D \subset [-\pi, \pi)^2$. By definition, for a strictly quasiconvex contour $|k(s)| \geq k_0$, (16), we shall assume that

$$k_0 \leq 1, \qquad |k(s)| \leq \overline{k}, \qquad |k^{(1)}(s)| \leq \overline{k^{(1)}}.$$

If $|\mathbf{l}(s)| \leq \overline{l}$, then it suffices to consider the estimates in the following two cases:

(1) $\overline{l} \leq C k_0^{-1}$,

(2) $d \geq C k_0^{-1}$,

since the condition $\overline{l} \geq C k_0^{-1}$ implies that $d \geq C k_0^{-1}$.

Lemma 2. *Let $f = \chi_D \in \mathcal{F}_p$, $p \geq 0$, and let ∂D be a strictly quasiconvex contour; suppose that we consider the domain of points $\mathbf{a}$ such that $|\mathbf{a} - \mathbf{a}_\Gamma| \geq d_\Gamma \quad \forall \mathbf{a}_\Gamma \in \Gamma$ (Γ is the evolute of the curve ∂D).*

Then for $\mathbf{l} \leq C k_0^{-1}$, we have the following estimate for the integral $J(\mathbf{a})$, (26):

$$|J(\mathbf{a})| \leq \frac{CN}{\Omega^{1/2}} \left(\frac{1}{d^{t-2}\sqrt{d_\Gamma}} + \frac{1}{d^{t-3/2}} \right) + \frac{B_1(\partial D)(N+1)}{\Omega d^{t+1} \widetilde{d}_\Gamma^4}, \qquad \widetilde{d}_\Gamma \equiv \min\{1, d, d_\Gamma\}; \tag{32}$$

and for $d \geq Ck_0^{-1}$ *we have*

$$|J(\mathbf{a})| \leq \frac{CN}{\Omega^{1/2} k_0^{1/2} d^{t-1}} + \frac{B_2(\partial D)(N+1)}{d^{t-1}\Omega}. \tag{33}$$

Here N *is the number of different normals from the point* $\mathbf{a}$ *to* ∂D *and the constants* $B_i(\partial D)$ *are indicated below* (see (36) and (38)).

Proof. Since $\mathbf{a} \notin \Gamma$ in the case being considered, all critical points are nondegenerate, and it follows from the results of [8] that

$$|J(\mathbf{a})| \leq C_1 \sum_{s_c} \frac{|u(s_c)| + \overline{u_c^{(1)}}\delta_c}{|\Omega\Phi^{(2)}(s_c)|^{1/2}} + \frac{C_2(\overline{u} + \|u^{(1)}\|_{L^1})}{\Omega\Phi_1}. \tag{34}$$

In this formula,

$$\Phi_1 = \min_{s_c, s_{ex}} \left\{ |\Phi^{(2)}(s_c)\delta_c|, \frac{(\psi^{(1)}(s_c)\Delta_c)^2}{\|\Phi^{(2)}\|_{L^1}}, \frac{|\psi(s_{ex})|^2}{\|\Phi^{(2)}\|_{L^1}} \right\}$$

and ψ is twice piecewise-differentiable function such that $|\Phi^{(1)}(s)| \geq |\psi^{(1)}(s)|$ and $s_{ex} \neq s_c$ are its extremal points ($\psi^{(1)}(s_{ex}) = 0$). We assume that

$$|u(s)| \leq \overline{u}, \quad |u^{(1)}(s)| \leq \overline{u_c^{(1)}}, \qquad |s - s_c| \leq \delta_c,$$
$$|\Phi^{(3)}(s)| \leq \overline{\Phi_c^{(3)}}, \quad |s - s_c| \leq \delta_c, \qquad \|\Phi^{(2)}\|_{L^1} \equiv \int |\Phi^{(2)}(s)|\, ds.$$

Estimate (34) holds for any $\delta_c > 0$ such that

$$\delta_c \leq \Delta_c \leq \min_{s_c} \left\{ \frac{|\Phi^{(2)}(s_c)|}{\overline{\Phi_c^{(3)}}}, \frac{\psi^{(1)}(s_c)}{\overline{\psi_c^{(2)}}} \right\}.$$

In the case which is of interest to us,

$$\begin{gathered} |u(s)| \leq Cd^{-t+1}, \qquad \|u^{(1)}\|_{L^1} \leq Cd^{-t+1}(\overline{k} + d^{-1})L, \\ |\Phi^{(2)}(s)| \leq C(|\mathbf{l}|^{-2} + \overline{k}|\mathbf{l}|^{-1} + \overline{k} + \overline{k}^{(1)}), \qquad \|\Phi^{(2)}\|_{L^1} \leq CL(\overline{k} + d^{-1}). \end{gathered} \tag{35}$$

The quantity $\Phi^{(2)}(s_c)$ in (27) can be written in the form

$$\Phi^{(2)}(s_c) = |\mathbf{l}(s_c)| \times |k(s_c)| \times |\mathbf{a}_\Gamma(s_c) - \mathbf{a}|$$

and therefore

$$|\Phi^{(2)}(s_c)| \geq \begin{cases} C|\mathbf{l}(s_c)|^{-2} d_\Gamma, & |k(s_c)\mathbf{l}(s_c)| \geq 0.5 \\ C|\mathbf{l}(s_c)|^{-1}, & |k(s_c)\mathbf{l}(s_c)| \leq 0.5. \end{cases}$$

In the first case (for $\overline{l} \leq Ck_0^{-1}$), the function ψ can be determined from the relation $\psi(s) = \overline{l}^{-1}\langle \mathbf{l}, \boldsymbol{\tau}\rangle$. Then $|\Phi^{(1)}(s)| \geq |\psi(s)|$, and it is easy to find the extremal points of the function ψ ($\langle a_\Gamma(s_{ex}) - \mathbf{a}, \boldsymbol{\nu}(s_c)\rangle = 0$), and therefore

$$|\psi(s_{ex})| \geq d_\Gamma \overline{l}^{-1}.$$

Since $\left|d|\mathbf{l}|/ds\right| \le 1$, it follows from the obtained estimates that we can take

$$\delta_{\mathrm{c}} = \Delta_{\mathrm{c}} = \frac{C\widetilde{d}_\Gamma}{g_0}, \qquad g_0 = 1 + \frac{\overline{k}}{k_0} + \frac{\overline{k}^{(1)}}{k_0^2} + \frac{\overline{k}^2}{k_0^2},$$

and then we obtain from (34) the required result (32) in which

$$B_{\mathrm{c}}(\partial D) = (L\overline{k}g_0)^2. \tag{36}$$

In the second case (for $d \ge Ck_0^{-1}$), we use the estimate

$$|\Phi^{(2)}(s_{\mathrm{c}})| \ge Ck_0, \tag{37}$$

which follows from (27). Now we can take $\psi = \Phi^{(1)}$ since the quantities $\Phi^{(1)}(s_{\mathrm{ex}})$ admit an effective lower bound. The equation $\Phi^{(2)}(s_{\mathrm{ex}}) = 0$ for extremal points can be written as

$$(1 + k\langle \mathbf{l}, \boldsymbol{\nu}\rangle)(s_{\mathrm{ex}}) = \frac{(\langle \mathbf{l}, \boldsymbol{\tau}\rangle(s_{\mathrm{ex}}))^2}{|\mathbf{l}(s_{\mathrm{ex}})|^2} = \left(\Phi^{(1)}(s_{\mathrm{ex}})\right)^2.$$

This implies that $|\Phi^{(1)}(s_{\mathrm{ex}})| \ge 0.5$ if $|k(s_{\mathrm{ex}})\langle \mathbf{l}, \boldsymbol{\nu}\rangle(s_{\mathrm{ex}})| \le 3/4$, or $|k(s_{\mathrm{ex}})\langle \mathbf{l}, \boldsymbol{\nu}\rangle(s_{\mathrm{ex}})| \ge 5/4$.

It remains to consider the case where $|k(s_{\mathrm{ex}})\langle \mathbf{l}, \boldsymbol{\nu}\rangle(s_{\mathrm{ex}})| \simeq 1$, and therefore $|\langle \mathbf{l}, \boldsymbol{\nu}\rangle(s_{\mathrm{ex}})| \le Ck_0^{-1}$. Since $\langle \mathbf{l}, \boldsymbol{\tau}\rangle^2 + \langle \mathbf{l}, \boldsymbol{\nu}\rangle^2 = |\mathbf{l}|^2$ and $|\mathbf{l}| \ge Ck_0^{-1}$, we have $|\langle \mathbf{l}, \boldsymbol{\tau}\rangle(s_{\mathrm{ex}})| \ge C|\mathbf{l}(s_{\mathrm{ex}})|$ in this case, and therefore $|\Phi^{(1)}(s_{\mathrm{ex}})| \ge C$. Using estimates (35) and (37), we find that one can take $\delta_{\mathrm{c}} = \Delta_{\mathrm{c}} = Ck_0^{-1}g^{-1}$, $g = 1 + \overline{k} + \overline{k}^2 + \overline{k}^{(1)}$. Now estimate (33) follows from (34); moreover,

$$B_2(\partial D) = \frac{(1 + L\overline{k})L\overline{k}g^2}{k_0^4}. \tag{38}$$

The obtained results can be used to estimate the quantity $\Delta_\Omega f(\mathbf{a})$ (21). In particular, these results imply that under the conditions of Lemma 2, there exist quantities $\alpha_i > 0$ such that if

$$\begin{gathered} d \ge \Omega^{-\alpha_1}, \qquad d_\Gamma \ge \Omega^{-\alpha_2}, \qquad k_0 \ge \Omega^{-\alpha_3}, \\ L \le \Omega^{\alpha_4}, \qquad \overline{k} \le \Omega^{\alpha_5}, \qquad \overline{k}^{(1)} \le \Omega^{\alpha_6}, \qquad N \le \Omega^{\alpha_7}, \end{gathered} \tag{39}$$

then the following estimate holds:

$$|\Delta_\Omega f(\mathbf{a})| \le \frac{CN}{\Omega d}\left(\frac{1}{d} + \frac{1}{\sqrt{dd_\Gamma}}\right). \tag{40}$$

It is shown in [8] that for a certain analog of conditions (39), estimate (40) is valid for an arbitrary $f \in \mathcal{F}_p$. With the aid of the results of [8], we can obtain similar estimates as $d_\Gamma \to 0$, but we shall not do this here.

Let us see how estimates (32) and (33) change for the case of a quasiconvex contour which has one or several plane segments of the boundary. Let a plane segment (a segment of a line) S correspond to the interval $s_1 \le s \le s_2$, and let the other part of the contour $\partial D \setminus S$ be strictly quasiconvex. Then, for the integral over $\partial D \setminus S$, estimates (32) and (33) hold true, since when deriving them, we never use the fact that ∂D is a closed curve. Thus, it remains to consider the contribution of the plane segment S, i.e., the integral

$$J_S = \int_{s_1}^{s_2} u(s) e^{i\Omega|\mathbf{l}|}\, ds.$$

When estimating the integral J_S, we can proceed precisely in the same way as in the proof of Lemma 2. Now the critical points s_c exist only if we can draw a normal from $\mathbf{a}$ to S. These critical points are nondegenerate, and there are no extremal points s_{ex} for $\psi = \Phi^{(1)}$. Setting $\delta_c = C\frac{d}{1+d}$, we find from (34) that

$$|J_S| \le \frac{N_S}{\Omega^{1/2} d^{t-3/2}} + \frac{N_S+1}{\Omega}\left(\frac{1}{d^{t-1}} + \frac{L_S}{d^t}\right)\left(1 + d + \frac{L_S}{d}\right), \tag{41}$$

where N_S is the number of normals from $\mathbf{a}$ to S and L_S is the length of the plane segment S. This estimate is also valid for an arbitrary finite number of plane segments if we understand S as the union of these segments. Thus, for the quasiconvex contour we have

$$|J(\mathbf{a})| \le |J_{\partial D \setminus S}| + |J_S|, \tag{42}$$

and estimates (33) and (34) are valid for $J_{\partial D\setminus S}$, and estimate (41) holds for J_S. If a contour consists only of plane segments, i.e., is a polygon, then we must set $J_{\partial D\setminus S} \equiv 0$ in (42).

4. A FOURIER SERIES

We begin the analysis of convergence of the method of summation over disks for a Fourier series with a rough estimation of the quantity. We obtain from (25) that

$$|\Delta_0 f(\mathbf{a})| \le \frac{C}{\Delta\Omega\Omega^{1/2}} \frac{L}{d^{5/2}}, \qquad |\mathbf{l}| \ge d. \tag{43}$$

Let $\mathbf{r}_0 \in [-\pi, \pi)^2$; we introduce the notation

$$\mathbf{a}_n = \mathbf{r}_0 + 2\pi n, \quad \mathbf{l}_n = \mathbf{r}(s) - \mathbf{a}_n, \qquad n \in \mathbb{Z}^2,$$
$$|\mathbf{l}_n| \ge d_n \equiv d(\mathbf{a}_n).$$

Then we can take

$$d_n = \begin{cases} d_0 = d, & n = 0, 1 \\ \pi|n|, & |n| \ge 2. \end{cases} \tag{44}$$

Recall that $d_0 \equiv d$ is the distance from $\mathbf{r}_0$ to ∂D, and we assume that $\Omega d > 1$.

It follows from (43) and (44) that the series

$$\sum_{n\in\mathbb{Z}^2} \Delta_0 f(\mathbf{a}_n)$$

absolutely converges. Thus, for $N = 2$, it suffices to consider the o-summation (23). We assume below that on each nonplane arc belonging to ∂D the following estimates hold:

$$|k^{(p)}(s)| \le C_p \overline{k} b^p, \qquad b \ge \overline{k}, \qquad |k(s)| \ge \widetilde{k} \qquad \left(|s - s_i^0| \ge C_i K_i (\overline{k} b^{p_i+1})^{-1}\right),$$

and in the notation $C(\partial D)$ we have $C(\partial D) \equiv C(p_i, K_i, K_j^0, \overline{k}, \widetilde{k}, k_0, L)$. In the case being considered, inequality (14) can be written as

$$|\Delta_\Omega f^T(\mathbf{r}_0)| \leq |\Delta_0 f(\mathbf{r}_0)| + |\Delta_{0\Omega} f^T(\mathbf{r}_0)|, \tag{45}$$

and it remains to estimate the quantity $\Delta_{A\Omega} f(\mathbf{r}_0)$, (11). To this end, we use estimate (12), which does not depend on the summation method, for $N = 2$ and $\Omega' = \Omega + \Delta\Omega$.

Lemma 3. *Let $f = \chi_D \in \mathcal{F}_p$. Then the following assertions hold for $\Omega \geq C(\partial D)$:*

(1) *if $C_1 \leq \Delta\Omega \leq C_2\Omega$ (say, $\frac{1}{2} \leq \Delta\Omega \leq \frac{1}{2}\Omega$), then*

$$\sum_{\Omega < |n| \leq \Omega + \Delta\Omega} |\widehat{f}(n)| \leq \frac{C(\partial D)\Delta\Omega}{\Omega^{1/2}} \tag{46}$$

and, in particular,

$$|\Delta_{0\Omega} f^T(\mathbf{r}_0)| \leq \frac{C(\partial D)\Delta\Omega}{\Omega^{1/2}}; \tag{47}$$

(2) *if $C_1\Omega^{-4/11} \leq \Delta\Omega \leq C_2\Omega$ and the constants γ_0 and γ_i are arbitrary but such that*

$$0 \leq \gamma_0 \leq 1, \qquad 0 \leq \gamma_i \leq \frac{p_i + 1}{p_i + 2},$$

then the following estimate holds:

$$\sum_{\Omega < |n| \leq \Omega + \Delta\Omega} |\widehat{f}(n)| \leq C(\partial D)\left\{\frac{\Delta\Omega}{\Omega^{1/2}} + \sum_i \left[\frac{\Delta\Omega}{\Omega^{1/2 - \gamma_i p_i/(2(p_i+1))}} + \frac{1}{\Omega^{\gamma_i + 1/(p_i+2)}}\right] + L_S(\Omega^{-1+\gamma_0}\Delta\Omega + \Omega^{-\gamma_0})\right\}, \tag{48}$$

where L_S is the total length of plane segments of the boundary ∂D and the summation over i is the summation over planar points s_i^0 of order p_i, (15).

Proof. We begin with the proof of assertion (1). We denote by $\square_n$ a closed square with side 1 and the center at the point $n \in \mathbb{Z}^2$. Let $Q \subset \mathbb{R}^2$ be a closed finite domain, and let

$$Q_\square = \bigcup_{n \in Q} \square_n.$$

We denote by $|\mathbf{r}, M|$ and $|M_1, M_2|$ the Euclidean distance from the point $\mathbf{r}$ to the set M and the distance between two closed subsets M_1 and M_2, respectively. Let the set $Q_1 \supset Q$ consist of points lying on the distance ≤ 1 from the set Q; thus, $|\mathbf{r}, Q| \leq 1$ for $\mathbf{r} \in Q_1 \setminus Q$. Let $g_f(\boldsymbol{\omega})$ be any function such that

$$g_f(\boldsymbol{\omega}) \geq |\widehat{f}(n)| \qquad \forall \boldsymbol{\omega} \in \square_n, \quad n \in Q_1. \tag{49}$$

Then

$$\sum_{n \in Q} |\widehat{f}(n)| \leq \int_{Q_1} g_f(\boldsymbol{\omega})|\, d\omega. \tag{50}$$

Indeed,

$$\sum_{n\in Q} |\widehat{f}(n)| = \sum_{n\in Q_{\square_n}} \int |\widehat{f}(n)|\,d\omega \le \sum_{n\in Q_{\square_n}} \int g_f(\boldsymbol{\omega})\,d\omega = \int_{Q_\square} g_f(\boldsymbol{\omega})\,d\omega \le \int_{Q_1} g_f(\boldsymbol{\omega})\,d\omega,$$

and the last inequality in this expression follows from the fact that $Q_\square \subset Q_1$ since $|\boldsymbol{\omega}, \partial Q| \le 2^{-1/2}$ if $\boldsymbol{\omega} \in Q_\square$, $\boldsymbol{\omega} \in Q$. Thus,

$$\sum_{\Omega<|n|\le\Omega+\Delta\Omega} |\widehat{f}(n)| \le \int_{\big||\boldsymbol{\omega}|-\Omega\big|\le C(1+\Delta\Omega)} g_f(\boldsymbol{\omega})\,d\omega, \tag{51}$$

and we have to find a function g_f satisfying (49). Actually, the question is a sufficiently good upper bound for the Fourier transform $\widehat{f}(\boldsymbol{\omega})$. All necessary estimates of the quantity $\widehat{f}(\boldsymbol{\omega})$ are contained in [8]; they are based on representation (18). In particular, it follows from the results of [8] that for a strictly quasiconvex contour we can take

$$g_f(\boldsymbol{\omega}) = \frac{C}{\Omega^{3/2} k_0^{1/2}} \qquad \text{if} \quad |\boldsymbol{\omega}| \ge 4\left(\overline{k}^2 + \frac{\overline{k}^{(1)}}{k_0^3}\right). \tag{52}$$

Assume that there are plane points s_i^0 on the contour ∂D and planar segments of the boundary. Below by the angle of a line we mean the angle φ in its equation $\langle \mathbf{r}, \boldsymbol{\eta}\rangle = q$, $\boldsymbol{\eta} = (\cos\varphi, \sin\varphi)$. Let $\varphi_{0\alpha}$ be an angle of the line constituting a plane segment of the boundary for which $s_{1\alpha} \le s \le s_{2\alpha}$, $s_{2\alpha} - s_{1\alpha} = L_\alpha$. The contribution of this segment to integral (18) can be easily estimated:

$$\left| -\frac{i}{\lambda} \int_{s_{1\alpha}}^{s_{2\alpha}} e^{i\lambda\langle \mathbf{r},\boldsymbol{\eta}\rangle} \langle \boldsymbol{\nu}, \boldsymbol{\eta}\rangle\,ds \right| \le \begin{cases} \dfrac{CL_\alpha}{\lambda}, & |\varphi - \varphi_{0\alpha}| \le \dfrac{C}{\lambda L_\alpha} \\[2ex] \dfrac{C}{\lambda^2 |\varphi - \varphi_{0\alpha}|}, & |\varphi - \varphi_{0\alpha}| \ge \dfrac{C}{\lambda L_\alpha}. \end{cases} \tag{53}$$

Let φ_i^0 be angles of tangents at planar points. It follows from (53) and the results of [8, 13] that for $\lambda \ge C(\partial D)$ the function g_f can be defined by the relation

$$g_f(\boldsymbol{\omega}) = \frac{C(\partial D)}{\lambda^{3/2}} \sum_i \frac{1}{|\varphi - \varphi_i^0|^{p_i/(2(p_i+1))}} + \begin{cases} \displaystyle\sum_\alpha \frac{C}{\lambda^2 |\varphi - \varphi_{0\alpha}|}, & |\varphi - \varphi_{0\alpha}| \ge \dfrac{C}{\lambda L_\alpha} \\[2ex] \displaystyle\sum_\alpha \frac{CL_\alpha}{\lambda}, & |\varphi - \varphi_{0\alpha}| \le \dfrac{C}{\lambda L_\alpha}. \end{cases} \tag{54}$$

The required result (46) now follows directly from (50).

We pass now to the proof of assertion (2). We introduce the notation

$$\Delta_i^0 = C_i(\partial D)\lambda^{-(p_i+1)/(p_i+2)}, \qquad \Delta_\alpha^0 = C(\lambda L_\alpha)^{-1}.$$

It follows from the results of [8] and estimate (53) that for $\lambda \ge C(\partial D)$ and $f = \chi_D \in \mathcal{F}_p$, the following inequalities hold for the Fourier transform $\widehat{f}(\boldsymbol{\omega})$, (1):

(1) if $|\varphi - \varphi_i^0| \ge \Delta_i^0$, $|\varphi - \varphi_{0\alpha}| \ge \Delta_\alpha^0$ $\forall \alpha, i$, then

$$|\widehat{f}(\lambda\boldsymbol{\eta})| \le \frac{C(\partial D)}{|\lambda|^{3/2}} \left[\sum_i \frac{1}{|\varphi - \varphi_i^0|^{p_i/(2(p_i+1))}} + \sum_\alpha \frac{1}{\lambda^{1/2} |\varphi - \varphi_{0\alpha}|} \right]; \tag{55}$$

(2) if $|\varphi - \varphi_j^0| \le \Delta_j^0$, then

$$|\widehat{f}(\lambda\boldsymbol{\eta})| \le \frac{C(\partial D)}{\lambda^{1+1/(p_j+2)}}; \tag{56}$$

(3) if $|\varphi - \varphi_{0\alpha}| \le \Delta_\alpha^0$, then

$$|\widehat{f}(\lambda\boldsymbol{\eta})| \le C L_\alpha \lambda^{-1}. \tag{57}$$

We denote by M the domain $\Omega \le |\boldsymbol{\omega}| \le \omega + \Delta\Omega$ ($\boldsymbol{\omega} \in \mathbb{R}^2$) which is of interest to us. We isolate the sectors $|\varphi - \varphi_i^0| \le \Delta_i$, $|\varphi - \varphi_{0\alpha}| \le \Delta_{0\alpha}$ and assume that $\Delta_i \ge \Delta_i^0$, $\Delta_{0\alpha} \ge \Delta_\alpha^0$. We denote the intersection of these sectors with M by M_i and $M_{0\alpha}$, respectively,

$$M = M' \cup \Big(\bigcup_i M_i\Big) \cup \Big(\bigcup_\alpha M_{0\alpha}\Big), \qquad M' = \big\{\boldsymbol{\omega} \in M \bigm| |\varphi - \varphi_i^0| > \Delta_i,\ |\varphi - \varphi_{0\alpha}| > \Delta_{0\alpha}\big\}.$$

The following estimate will be used as an assumption estimate:

$$\sum_{n\in M} |\widehat{f}(n)| \le \max_{n\in M} |\widehat{f}(n)| \times N(M') + \sum_i \max_{n\in M_i} |\widehat{f}(n)| \times N(M_i) + \sum_\alpha \max_{n\in M_\alpha} |\widehat{f}(n)| \times N(M_{0\alpha}), \tag{58}$$

where $N(Q)$ is the number of integer points in the domain Q. Let $S(Q)$ be the area of Q, and let $L(Q)$ be the length of the boundary ∂Q. We shall prove that

$$N(Q) \le S(Q) + C(L(Q) + 1). \tag{59}$$

We introduce the set

$$Q' = \bigcup_{n\in Q} \square_n \subset Q,$$

where the union is taken over all integer points $n \in Q$ such that $\square_n \subset Q$. Then $N(Q') = S(Q') \le S(Q)$, and it remains to consider the number of integral points in the domain $Q \setminus Q'$. If $\mathbf{r} \in Q \setminus Q'$, then $|\mathbf{r}, \partial Q| \le 2^{-1/2}$, since otherwise $\mathbf{r} \in Q'$. We choose a certain point $\mathbf{r}_0 \in \partial Q$; traversing the contour ∂Q in a certain direction, we put the points $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i, \dots$ on it in such a way that the distance from $\mathbf{r}_i$ to $\mathbf{r}_{i+1}$ along ∂Q be equal to 1. Then the number of these points $\le L(Q) + 1$. If Π_i is a disk of radius 1 centered at $\mathbf{r}_i$, then the part ∂Q_i of the boundary lying between the points $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ lies entirely in the intersection $\Pi_i \cap \Pi_{i+1}$. Hence, if we take the disks $\widetilde{\Pi}_i$ of radius $1 + 2^{-1/2}$ instead of Π_i, then $n \in Q \setminus Q'$, whence it follows that $n \in \bigcup_i \widetilde{\Pi}_i$. There are not more than nine integral points in each disk $\widetilde{\Pi}_i$, and therefore $N(Q \setminus Q') \le 9(L(Q) + 1)$. Thus, estimate (59) is proved with constant $C = 9$.

In (58), we use the estimates

$$\begin{gathered} N(M') \le N(M) = N(\Omega + \Delta\Omega) - N(\Omega), \\ N(M_i) \le C(S(M_i) + L(M_i) + 1), \qquad N(M_{0\alpha}) \le C(S(M_{0\alpha}) + L(M_{0\alpha}) + 1); \end{gathered} \tag{60}$$

in the first of which $N(\Omega)$ is the number of integral points in the disk of radius Ω. The following estimate is obtained in [15]:

$$N(\Omega) = \pi Q^2 + \Delta N(\Omega), \qquad |\Delta N(\Omega)| \le C\Omega^{-7/11},$$

from which it follows immediately that

$$N(\Omega+\Delta\Omega)-N(\Omega)\le C\Omega\Delta\Omega \tag{61}$$

for $C\Omega^{-4/11}\le\Delta\Omega\le C\Omega$, and thus, for $C\Omega^{-4/11}\le\Delta\Omega\le C\Omega$, we find from (58), (60), and (61) that

$$\begin{aligned}\sum_{\Omega<|n|\le\Omega+\Delta\Omega}|\widehat{f}(n)|\le C_1\max_{n\in M'}|\widehat{f}(n)|\Omega\Delta\Omega+C_2\sum_i\max_{n\in M_i}|\widehat{f}(n)|(S(M_i)+L(M_i)+1)\\ +C_3\sum_\alpha\max_{n\in M_{0\alpha}}|\widehat{f}(n)|(S(M_{0\alpha})+L(M_{0\alpha})+1).\end{aligned} \tag{62}$$

We take $\Delta_i=C_i\Omega^{-\gamma_i}$, $\Delta_{0\alpha}=C\Omega^{-\gamma_0}$ and assume that $|\varphi_i^0-\varphi_j^0|\ge\Delta_i$, $|\varphi_i^0-\varphi_{0\alpha}|\ge\Delta_i$, $|\varphi_{0\alpha}-\varphi_{0\beta}|\ge\Delta_{0\alpha}$. The estimates

$$\begin{gathered}\max_{n\in M'}|\widehat{f}(n)|\le C(\partial D)\left[\Omega^{-3/2}\sum_i\Omega^{\gamma_i p_i/(2(p_i+1))}+\Omega^{-2+\gamma_0}\right],\\ \max_{n\in M_i}|\widehat{f}(n)|\le\frac{C(\partial D)}{\Omega^{1+1/(p_i+2)}},\qquad\max_{n\in M_{0\alpha}}|\widehat{f}(n)|\le\frac{CL_\alpha}{\Omega}\end{gathered} \tag{63}$$

follow from (55)–(57). Note that

$$S(M_i)\le C\Delta_i\Omega\Delta\Omega\le C\Omega^{1-\gamma_i}\Delta\Omega,\qquad L(M_i)\le C(\Delta_i\Omega+\Delta\Omega)\le C(\Omega^{1-\gamma_i}+\Delta\Omega) \tag{64}$$

and similar estimates for $S(M_{0\alpha})$ and $L(M_{0\alpha})$ are obtained by the substitution of γ_0 for γ_i. Substituting (63) and (64) into (62), we obtain (48).

Let us formulate and prove the fundamental theorem.

Theorem 2. *We have* $\sigma^T(\mathcal{F}_p)=1/2$.

Proof. It follows from (45), with due account of (43) and (47), that for $f=\chi_D\in\mathcal{F}_p$ and $\Delta\Omega\sim1$,

$$|\Delta_\Omega f^T(\mathbf{r}_0)|\le\frac{C(\partial D)}{\Omega^{1/2}d^{5/2}},\qquad\Omega\ge C(\partial D) \tag{65}$$

and therefore,

$$\sigma^T(f|\mathbf{r}_0)\ge\frac{1}{2},\qquad f\in\mathcal{F}_p. \tag{66}$$

It remains to prove that there exist a contour ∂D and a point $\mathbf{r}_0$ such that the sign of equality is realized in (66). We can take a circle of radius $R<\pi$ (say, $R=1$) centered at the origin as such a contour and set $\mathbf{r}_0=0$. To prove this, we note that in the case of a circle of radius R, it follows from (52) that

$$|\Delta_{0\Omega}f^T(\mathbf{r}_0)|\le\frac{C(R)\Delta\Omega}{\Omega^{1/2}},\qquad C\Omega^{-4/11}\le\Delta\Omega\le C\Omega. \tag{67}$$

Let $f=\chi_D$, D be a disk centered at the origin, and let $\mathbf{r}_0=0$. Relation (8) implies the equality

$$\Delta_0 f(0)=\sum_{n\in\mathbb{Z}^2}\Delta_0 f(\mathbf{a}_n),\qquad\mathbf{a}_n=2\pi n.$$

Since we can assume that $d_n \geq Ck_0^{-1}$ ($k_0 = R^{-1}$) for $|n| \geq 1$, we obtain from (43) and (44) the estimate

$$|\Delta_0 f(\mathbf{a}_n)| \leq \frac{C}{|n|^{5/2}\Omega\Delta\Omega}, \qquad |n| \neq 0.$$

On the other hand (see (25)),

$$\Delta_0 f^T(0) = \operatorname{Im} \frac{A_0}{\Delta\Omega\Omega^{1/2}} \times \frac{e^{i\Omega R}}{R^{5/2}}\left(\frac{e^{iR\Delta\Omega}}{\sqrt{1+\Delta\Omega/\Omega}} - 1\right) + \Delta_0^1(0),$$

$$A_0 = \frac{1}{\sqrt{2}\pi^{3/2}}(-1+i), \qquad |\Delta_0^1(0)| \leq \frac{C}{\Omega^{3/2}R^{7/2}\Delta\Omega}.$$

This implies that for any fixed R, there exists a sequence of values Ω_j, $\Omega_j \to \infty$, $j \to \infty$, such that

$$|\Delta_0 f^T(0)| \geq \frac{A}{\Omega_j^{1/2}\Delta\Omega}$$

for $\Omega = \Omega_j$. Taking into account estimate (67) and choosing $\Delta\Omega$, for example, from the condition $(\Delta\Omega)^2 \leq 10^{-1}AC(R)$, we find that $|\Delta_\Omega f^T(0)| \geq C\Omega^{-1/2}$ ($\Omega = \Omega_j$).

Estimate (65) can be improved with the use of various assumptions concerning the shape of the contour ∂D and the position of the point $\mathbf{r}_0$. The roughness of estimates of the quantity $\Delta_{A\Omega} f$ given by Lemma 3 is the main obstacle for obtaining of exact results.

Let us formulate one of the possible results. To this end, we define the set $\widetilde{\Gamma} \in [-\pi,\pi]^2$. Let $\Gamma = \Gamma(\partial D)$ be the evolute of the curve ∂D. The evolute Γ defines the curve Γ' on the torus T^2, which is regarded as the square $[-\pi,\pi]^2$ with identified sides. Unrolling the torus T^2 into a square, we obtain the set $\widetilde{\Gamma}$ from Γ'. For the quasiconvex curve ∂D this set consists of a finite number of arcs. This is due to the fact that the diameter of the evolute $\Gamma \ \leq Ck_0^{-1}$. If there exist planar points, the set $\widetilde{\Gamma}$ can everywhere densely cover the square $[-\pi,\pi]^2$.

Lemma 4. *Let $f = \chi_D \in \mathcal{F}_p$, $p \geq 0$, ∂D be a strictly quasiconvex contour, and let $|\mathbf{r}_0 - \mathbf{a}_{\widetilde{\Gamma}}| \geq d_{\widetilde{\Gamma}}$ $\forall \mathbf{a}_{\widetilde{\Gamma}} \in \widetilde{\Gamma}$. Then there exist numbers $\alpha_1 > 0$ and $\alpha_2 > 0$ such that for $d \geq \Omega^{-\alpha_1}$, $d_{\widetilde{\Gamma}} \geq \Omega^{-\alpha_2}$, the following estimate holds*:

$$|\Delta_\Omega f^T(\mathbf{r}_0)| \leq \frac{C(\partial D)}{\Omega^{3/4} d}\left(\frac{1}{\sqrt{d d_{\widetilde{\Gamma}}}} + \frac{1}{d}\right), \qquad d \equiv |\mathbf{r}_0, \partial D|.$$

Proof. We shall again proceed from estimate (45). Applying Lemma 2 and taking into account (25) and (44), we find from (13) that under our assumptions

$$|\Delta_0 f^T(\mathbf{r}_0)| \leq \frac{C(\partial D)}{\Delta\Omega\Omega d}\left(\frac{1}{\sqrt{d d_{\widetilde{\Gamma}}}} + \frac{1}{d}\right). \tag{68}$$

To estimate the quantity $\Delta_{0\Omega} f^T(\mathbf{r}_0)$, it remains to apply the analog of estimate (67) and set $\Delta\Omega = \Omega^{-1/4}$.

For the quasiconvex contour ∂D with plane segments, the estimate of J_S (41) implies that it is necessary to consider the q-summation with $q \geq 1$ in order to use Lemma 1, and it is sufficient to consider the case $q = 1$. Then, instead of (68), we obtain

$$|\Delta_1 f^T(\mathbf{r}_0)| \leq C(\partial D, d, d_{\widetilde{\Gamma}})(\Delta\Omega)^{-3}\Omega^{-1}, \qquad \mathbf{r}_0 \notin \partial D \cup \widetilde{\Gamma}.$$

In this case, estimate (48) yields

$$|\Delta_{1\Omega} f^T(\mathbf{r}_0)| \leq C(\partial D)\left[\frac{\Delta\Omega}{\Omega^{1/2}} + L_S(\Omega^{-1+\gamma_0}\Delta\Omega + \Omega^{-\gamma_0})\right].$$

Choosing $\Delta\Omega = \Omega^{-1/8}$, $\gamma_0 = 9/16$, we find

$$|\Delta_\Omega f^T(\mathbf{r}_0)| \leq \frac{C(\partial D, d, d_{\widetilde{\Gamma}})}{\Omega^{9/16}}, \qquad \mathbf{r}_0 \notin \partial D \cup \widetilde{\Gamma},$$

for a quasiconvex contour. Of course, in this estimate, $C(\partial D, d, d_{\widetilde{\Gamma}}) \to \infty$ as $d \to 0$ or $d_{\widetilde{\Gamma}} \to 0$.

As one more example, we consider the case where ∂D is a polygon (not necessarily convex). In this case, we can reject the first term on the right-hand side of expression (54) for $g_f(\boldsymbol{\omega})$, and then, for $\Delta\Omega \geq C$, (51) implies that

$$\sum_{\Omega<|n|\leq\Omega+\Delta\Omega} |\widehat{f}(n)| \leq C\Omega^{-1}.$$

Using estimate (41) for J_S and choosing $q = 1$, we obtain

$$|\Delta_1 f^T(\mathbf{r}_0)| \leq \frac{C(\partial D)}{\Omega d^5}, \qquad \Omega \geq C(\partial D),$$

and, thus, if f is a characteristic function of the polygon, then

$$|\Delta_\Omega f^T(\mathbf{r}_0)| \leq \frac{C}{\Omega d^5}, \qquad \Omega \geq C(\partial D).$$

We can also slightly improve estimate (65) for general position contours. In this case, $p_i = 1$, (15), and $j \leq 2$, (28). Let $f = \chi_D$, and let the boundary ∂D consist of plane segments and arcs of the general position. In this case, the estimation of the quantity $\Delta_\Omega f^T$ is performed in the same way as above. The required estimate of integral (26), when the point $\mathbf{a}$ is near the evolute, is obtained with the aid of the results of [8], and for $d \geq \Omega^{-\alpha}$ we obtain

$$|\Delta_\Omega f^T(\mathbf{r}_0)| \leq C(\partial D)\Omega^{-29/54};$$

if $d \geq \Omega^{-\alpha_1}$, $|\mathbf{r}_0, \widetilde{\Gamma}_0| \geq \Omega^{-\alpha_2}$, then

$$|\Delta_\Omega f^T(\mathbf{r}_0)| \leq C(\partial D)\Omega^{-31/57} d^{-9/2},$$

where $\widetilde{\Gamma}_0$ is the image, under the mapping $\Gamma \to \widetilde{\Gamma}$, of the set Γ_0 of the singular points of the evolute Γ.

5. CONCLUDING REMARKS

The results obtained above refer to the two-dimensional case. Note that in a one-dimensional case, the proposed approach is also applicable and yields the estimates

$$|\Delta_\Omega f(x)| \le \frac{C}{\Omega d}, \qquad |\Delta_\Omega f^T(x)| \le \frac{C}{\Omega d^2}.$$

Naturally, this approach is also applicable for analyzing the general spherically symmetric summation methods. In particular, it follows immediately from the results of [8] concerning the estimation of the quantity $\Delta_A f(\mathbf{r}_0)$ ($f = \chi_D \in \mathcal{F}_p$) that for any M there exists a summation method such that $|\Delta_A f^T(\mathbf{r}_0)| \le C(d, \partial D)\Omega^{-M}$. The greatest difficulty for summation is precisely the summation over disks (Ω-summation). As was already mentioned, the main obstacle to the obtaining of exact results concerning the rate of convergence of the Ω-summation for the Fourier series is the absence of good estimates for the quantity $\Delta_{A\Omega} f^T$ (11). If $|\Delta_{q\Omega} f| \le C\Omega^{-1-\varepsilon}$ for $\Delta\Omega \sim 1$, then (for $N = 2$) we can obtain an analog of Theorem 1 for the Fourier series.

The obtained results can be regarded in the framework of the following general problem: with what accuracy can one reconstruct the characteristic function χ_D ($D \in \mathbb{R}^2$) for $d > \Omega^{-\alpha}$ if $\sim \Omega^2$ functionals of this function are known? It was shown above that with the use of $\sim \Omega^2$ Fourier coefficients even using the Ω-summation, one can reconstruct χ_D (D is a general position domain) with an accuracy of $\Omega^{-29/54}$. On the other hand, the maximal accuracy of reconstruction of this domain with the use of $\sim \Omega^2$ values of the Radon transform does not exceed $\Omega^{-1/3}$ or $\Omega^{-1/2}$ at the point of the general position [7].

Consider the following question: to what extent the obtained results can be extended to the three-dimensional case ($N = 3$)? For $N = 3$, we can also consider the class $\mathcal{F}_p$ of characteristic functions of domains with a piecewise-smooth boundary. By the results of [1], $\sigma(\mathcal{F}_p) = \sigma^T(\mathcal{F}_p) \le 0$ (the Fourier series diverges at the center of the ball), and it is necessary to consider the quantities of the type $\sigma(f|\mathbf{r}_0)$, $\sigma^T(f|\mathbf{r}_0)$, $\mathbf{r}_0 \in \mathbb{R}^3$, $f = \chi_D \in \mathcal{F}_p$. For $N = 3$, we have again the Gauss formula

$$\chi_D(\mathbf{a}) = \frac{1}{4\pi} \int\limits_{\partial D} dS\, \frac{\langle \mathbf{l}, \boldsymbol{\nu} \rangle}{|\mathbf{l}|^3}$$

and analogs of formulas (18) and (19),

$$\widehat{f}(\lambda \boldsymbol{\eta}) = -\frac{i}{\lambda} \int\limits_{\partial D} dS\, \langle \mathbf{l}, \boldsymbol{\nu} \rangle e^{i\lambda \langle \mathbf{r}, \boldsymbol{\eta} \rangle}, \qquad \mathbf{r} \in \partial D, \quad \boldsymbol{\omega} = \lambda \boldsymbol{\eta}, \quad \lambda = |\boldsymbol{\omega}|,$$

$$\Delta_A f(\mathbf{a}) = \frac{1}{2\pi^2} \int\limits_{\partial D} dS\, \frac{\langle \mathbf{l}, \boldsymbol{\nu} \rangle}{|\mathbf{l}|^3} \sum_p \Delta_p G\, \sin \Omega \xi_p |\mathbf{l}| + \frac{1}{2\pi^2} \int\limits_{\partial D} dS\, \frac{\langle \mathbf{l}, \boldsymbol{\nu} \rangle}{|\mathbf{l}|^3} \int\limits_0^\infty \frac{G(\xi) + \xi G^{(1)} - 1}{\xi} \sin \Omega \xi |\mathbf{l}|\, d\xi, \tag{69}$$

($\mathbf{l} = \mathbf{r} - \mathbf{a}$, $\mathbf{r} \in \partial D$, dS is an element of the surface area of ∂D).

With the aid of these formulas, the problem of estimation of the quantities $\sigma(f|\mathbf{r}_0)$ and $\sigma^T(f|\mathbf{r}_0)$ reduces to the problem of estimation of two-dimensional OI over the boundary ∂D with phases $\langle \mathbf{r}, \boldsymbol{\eta} \rangle$ and $|\mathbf{l}|$. Using the results from the singularity theory, [16], we can prove the analog of Theorem 1 (at least for general position surfaces). The enveloping surface of the family of normals to ∂D — $\Gamma(\partial D)$ plays the role of the evolute in this case.

As to the Fourier series, in order to apply formula (8) for $N = 3$, it suffices to consider the q-summation (23) for $q = 1$. A preliminary analysis on the basis of formulas (69) shows us that at least for convex domains $D \subset \mathbb{R}^3$, the method of summation over disks converges for $\mathbf{r}_0 \notin \widetilde{\Gamma}$, and the set $\widetilde{\Gamma} \subset [-\pi, \pi]^3$ can be constructed from Γ precisely in the same way as in the two-dimensional case.

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