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# On the Mean Value of the Ternary Function of Divisors on a Sparse Sequence

**H. M. Saliba**

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In the present paper, we obtain the asymptotic formula for the mean value of the ternary function of divisors on the sequence of integral parts of nonintegral powers of the series of natural numbers. As usual, one aims to obtain formulas of such a type for the maximal possible value of the power (see, e.g., [1, 2, 7, 10–13, 17, 18, 23–26]).

Here we prove the following main result.

**Theorem.** *For $1 < c < 9/8$ the following asymptotic formula holds:*

$$\sum_{n\le T} \tau_3([n^c]) = TP_2(\ln T) + O\left(\frac{T}{\ln T}\right),$$

*where $\tau_3(n)$ is the number of solutions to the equation $xyz = n$* in integers $x, y, z$, *$P_2(x)$ is a certain polynomial of the second degree, and the symbol $[\,\cdot\,]$ denotes the integral part of a number.*

In literature, the method for the study of problems of such a type is called the van der Corput method (see [19–21]). The recent results obtained by the van der Corput method are presented in the monographs by S. W. Graham and G. Kolesnik [10], A. Ivič [14], E. Krätzel [22], and also in the paper by M. N. Huxley [27].

After a principally new approach to these problems, which appeared in 80th in the paper by A. A. Karatsuba (see [15] and also [16]) and then in papers by E. Bombiery and H. Iwaniec [5, 6], there appears a series of papers in which early results are improved.

We also note that the behavior of trigonometrical sums of the van der Corput type was studied by J.-M. Deshouillers [12] with the help of a computer.

The problem solved in the dissertation by A. Zakzak [13] is the closest object of study to the present paper. This author found the asymptotic formula for the mean value of the Dirichlet divisor function. In other words, for $1 < c < 100/87$, he obtained the asymptotic formula

$$\sum_{n\le T} \tau([n^c]) = cT\ln T + 2\gamma T - cT + O\left(\frac{T}{\ln T}\right),$$

where $\gamma$ is the Euler constant.

One more problem connected with the asymptotic formula for the number of primes in the sequence $[n^c]$ was studied in [11, 17, 18, 23, 24, 26]. The best result in this problem belongs to G. Kolesnik. He obtained the asymptotic for $1 < c < 39/34$.

We also mention a close problem on the distribution of a sequence of the form $[n^c]$ in the arithmetic progression with the increasing difference. Here, the best results were obtained by

Deshouillers [11] and Kh. M. Abud [1]. Regretfully, the approach of these authors to the solution of problems, which are similar to the problem considered here, gives nothing, since there is a sufficiently large residue in the asymptotic. On the other hand, in our problem, it is necessary to know the distribution of these residues in the mean with respect to arithmetic progressions. We use this fact in our considerations.

We pass now to the *proof of the theorem.*

The problem consists in searching for the asymptotical formula of the value

$$A(T) = \sum_{n\le T} \tau_3([n^c])$$

as $T \to \infty$. This value is equal to the number of couples $(x, y, z)$ of natural numbers $x, y, z$ with the conditions $xyz \le [n^c]$, $n \le T$, or, what is the same, to the number of solutions $(x, y, z)$ of the inequalities $xyz \le n^c < xyz + 1$, $n \le T$.

Set $\gamma = 1/c$. Then the latter inequalities can be rewritten in the form

$$(xyz)^\gamma \le n < (xyz + 1)^\gamma.$$

For $c > 1$, the closed interval $[(xyz)^\gamma, (xyz + 1)^\gamma)$ can contain not more than one natural number since the length of this interval is less than one. Therefore, we have the following expression for the value $A(T)$:

$$A(T) = \sum_{xyz\le T^c} \big([-(xyz)^\gamma] - [-(xyz + 1)^\gamma]\big).$$

Let $\rho(t) = \frac{1}{2} - \{t\}$, where the value $\{\cdot\}$ denotes the fractional part of a number; moreover (see, e.g., [3, 4, 13])

$$\rho(t) = \sum_{0<|n|\le N} \frac{1}{2\pi i n} e^{2\pi i n t} + \Theta_N \Psi_N(t), \qquad |\Theta_N| \le 1, \quad \Psi_N(t) = \frac{1}{\sqrt{1 + N^2 \sin^2 \pi}}.$$

Let now transform the value $A(T)$. We have $A(T) = A_1 + A_2$, where

$$A_1 = \sum_{xyz\le T^c} \big((xyz + 1)^\gamma - (xyz)^\gamma\big) = \sum_{n\le T^c} \tau_3(n)\big((n + 1)^\gamma - n^\gamma\big),$$

$$A_2 = \sum_{xyz\le T^c} \big(\rho(-(xyz + 1)^\gamma) - \rho(-(xyz)^\gamma)\big).$$

The following equality holds for the value $A_1$:

$$A_1 = \gamma \sum_{n\le T^c} \tau_3(n) n^{\gamma-1} + O\Big(\sum_{n\le T^c} \tau_3(n) n^{\gamma-2}\Big) = D_3(T^c)\gamma T^{(\gamma-1)c} - \gamma \int_1^{T^c} D_3(t)\, dt^{\gamma-1} + O(1),$$

where

$$D_3(t) = \sum_{n\le t} \tau_3(n) = tP_2(\ln t) + \Delta(t), \qquad \Delta(t) \ll t^{5/11},$$

$P_2(z)$ is a polynomial of the second degree (see [28]).

Hence we have

$$A_1 = TQ_2(\ln T) + \Delta_1(T), \qquad \Delta_1(T) \ll T^{1-6c/11},$$

$Q_2(z)$ is a certain polynomial of the second degree.

Let us transform now the expression $A_2$. To this end, we divide the intervals of varying of variables $x, y, z$ into intervals of the form $X < x \le X_1$, $X_1 \le 2X$; $Y < y \le Y_1$, $Y_1 \le 2Y$; $Z < z \le Z_1$, $Z_1 \le 2Z$.

We obtain not more than $\ln^3 T$ parallelepipeds of the indicated form in the three-dimensional space $(x, y, z)$.

Using the representation of the function $\rho(t)$, which is given above, we find that for some $X, X_1, Y, Y_1, Z, Z_1$ and any $N \ge 2$, the following inequality holds:

$$\begin{aligned} A_2 &\ll \ln^3 T \sum_{X<x\le X_1} \sum_{\substack{Y<y\le Y_1 \\ xyz\le T^c}} \sum_{Z<z\le Z_1} \big(\rho(-(xyz)^\gamma) - \rho(-(xyz+1)^\gamma)\big) \\ &\ll \ln^3 T \sum_{n=1}^{N} \frac{1}{n} \sum_{X<x\le X_1} \sum_{\substack{Y<y\le Y_1 \\ xyz\le T^c}} \sum_{Z<z\le Z_1} \big(e^{-2\pi i n(xyz)^\gamma} - e^{-2\pi i n(xyz+1)^\gamma}\big) \\ &\quad + \ln^3 T \sum_{X<x\le X_1} \sum_{\substack{Y<y\le Y_1 \\ xyz\le T^c}} \sum_{Z<z\le Z_1} \big(\Psi_N((xyz)^\gamma) + \Psi_N((xyz+1)^\gamma)\big) \\ &= (A_{21} + A_{22}) \ln^3 T. \end{aligned}$$

First, we estimate the value $A_{22}$. We have

$$A_{22} \ll \sum_{\substack{XYZ\le n\le X_1Y_1Z_1 \\ n\le T^c}} \tau_3(n)\big(\Psi_N(n^\gamma) + \Psi_N(n+1)^\gamma\big) \ll T^\varepsilon \sum_{\substack{XYZ\le n\le X_1Y_1Z_1 \\ n\le T^c}} \Psi_N(n^\gamma).$$

Set $N_0 = [N \ln N]$. Then

$$\Psi_N(t) = \sum_{0<|n|\le N_0} c_n e^{2\pi i n t} + O\left(\frac{\ln N}{N}\right), \qquad |c_n| \ll \frac{\ln N}{N} e^{-|n|/N}.$$

Further, set $XYZ = V$. We have

$$A_{22} \ll T^\varepsilon \sum_{V\le n\le 8V} \Psi_N(n^\gamma) = T^\varepsilon \left| \sum_{0<|m|\le N_0} c_m \sum_{V\le n\le 8V} e^{2\pi i m n^\gamma} \right| + O\left(T^\varepsilon \frac{V \ln N}{N}\right).$$

To estimate $A_{22}$, it is necessary to estimate the trigonometrical sum

$$S(V) = \sum_{V\le n\le 8V} e^{2\pi i m n^\gamma}.$$

We use the following lemma.

**Lemma 1.** *We have the inequality*

$$\sum_{V\le n\le 8V} e^{2\pi i m n^\gamma} \ll \min(V, m^{-1}V^{1-\gamma}+m^{1/2}V^{\gamma/2}).$$

**Proof.** See [26].

We obtain from this lemma that

$$A_{22} \ll T^\varepsilon \sum_{0<|m|\le N_0} \frac{\ln N}{N}(m^{-1}V^{1-\gamma}+m^{1/2}V^{\gamma/2}) + VN^{-1}\ln N \ll T^\varepsilon(VN^{-1}+N^{1/2}V^{\gamma/2}).$$

Choose the parameter $N$ by the condition $N = V^{1-\gamma}T^{2\varepsilon}$. Then we have

$$A_{22} \ll V^\gamma T^{-\varepsilon} + V^{1/2}T^{2\varepsilon}.$$

Since $V \le T^c$, the contribution of $A_{22}$ in the sum $A_2$ does not exceed

$$V^{1/2}T^{2\varepsilon} < \frac{T}{\ln^4 T}$$

for $c/2+2\varepsilon<1$, i.e., for $c<2$.

Thus, it remains to estimate the value $A_{21}$. Consider the following part of this sum:

$$A'_{21} = \sum_{0<|m|\le N} \frac{1}{2\pi i n} \sum_{X<x\le X_1} \sum_{Y<y\le Y_1} \sum_{Z<z\le Z_1} e^{2\pi i n(xyz+\alpha)^\gamma},$$

where $XYZ = V$, $N = V^{1-\gamma}T^\varepsilon$ and the constant $\alpha$ can assume two values 0 or 1.

Without loss of generality, we can assume that $X \le Y \le Z$. Then $X \le V^{1/3} \le Z$.

Note that all possible values of $X, Y, Z$ can be divided into the following four cases:

(1) $Z \ge V^{11/18}$;

(2) at least one of the values $X$ or $Y$ belongs to the closed interval $[V^{2/9+\mu}, V^{1/3}]$, where $\mu$ is an arbitrary small constant, $0<\mu<10^{-10}$;

(3) $1 \le X \le Y \le V^{2/9+\mu}$, $V^{1/3} \le Z \le V^{11/18}$;

(4) $1 \le X \le V^{2/9+\mu}$, $V^{1/3} \le Y \le Z \le V^{11/18}$.

Consider case (1). Let us estimate the single trigonometrical sum

$$S_0 = \sum_{Z<z\le Z_1} e^{2\pi i f(z)},$$

where $Z < Z_1 = \min(2Z, T^c(xy)^{-1})$, $f(z) = n(uz+\alpha)^\gamma$, $u \ge 1$. By Lemma 1, we have

$$|S_0| \ll \min(Z, n^{-1}u^{-\gamma}Z^{1-\gamma} + n^{1/2}u^{\gamma/2}Z^{\gamma/2}).$$

Hence

$$\begin{aligned}|A_{21}| &\ll \sum_{n<N} \sum_{X<x\le 2X} \sum_{Y<y\le 2Y} \frac{1}{n}|S_0| \ll \sum_{n<N} n^{-1}XY(n^{-1}V^{-\gamma}Z + n^{1/2}V^{\gamma/2}) \\ &\ll \sum_{n\le N} n^{-2}V^{1-\gamma} + n^{-1/2}V^{1+\gamma/2}Z^{-1} \ll V^{1-\gamma} + N^{1/2}V^{1+\gamma/2}Z^{-1} \\ &\ll V^{1-\gamma} + V^{3/2}Z^{-1}T^\varepsilon.\end{aligned}$$

Note that for the validity of the asymptotical formula for the value $A(T)$, it is sufficient to obtain an estimate of the form

$$|A'_{21}| \ll T \ln^{-4} T.$$

We have, from the estimate of $A_{21}$ founded above, that this estimate satisfies the latter condition for $1 < c < 9/8$ and $Z > V^{11/18}$.

This completes the consideration of case (1).

Consider case (2). For defineteness, we shall assume that the value $X$ belongs to the closed interval $[V^{2/9+\mu}, V^{1/3}]$. If $V^{2/9+\mu} \le Y \le V^{1/3}$, then the proof is analogous.

Let $\tau_0(u)$ denote the number of solutions to the equation $u = yz$, $Y < y \le Y_1 \le 2Y$, $Z < y \le Z_1 \le 2Z$. Then we have

$$S = \sum_{X<x\le X_1} \sum_{\substack{Y<y\le Y_1 \\ xyz\le T^c}} \sum_{Z<z\le Z_1} e^{2\pi i n(xyz+\alpha)^\gamma} = \sum_{YZ<u\le Y_1Z_1} \tau_0(u) \sum_{X<x\le X'_1} e^{2\pi i f(x,u)},$$

where $f(x,u) = n(xu+\alpha)^\gamma$, $X'_1 = \min(X_1, T^c u^{-1})$, $\alpha$ is a constant equal to one of the values 0 or 1.

Further, since $X'_1 \le X_1 \le 2X$, the sum $S$ can be represented in the form

$$S = \sum_{YZ<u\le Y_1Z_1} \tau_0(u) \sum_{X<x\le 2X} e^{e\pi i f(x,u)} \frac{1}{2X} \sum_{k=0}^{2X-1} \sum_{X<t\le X'_1} e^{2\pi i \frac{k(x-t)}{2X}}.$$

Passing to inequalities in this equality, we obtain

$$\begin{aligned} |S| &\le \frac{1}{2X} \sum_{k=0}^{2X-1} \sum_{YZ<u\le 4YZ} \tau_0(u) \left| \sum_{X<x\le 2X} e^{2\pi i \left(f(x,u)+\frac{kx}{2X}\right)} \right| \times \left| \sum_{X<t\le X'_1} e^{-2\pi i \frac{kt}{2X}} \right| \\ &\ll \max_{0\le k\le 2X} \left( \sum_{YZ<u\le 4YZ} \tau_0(u) \left| \sum_{X<x\le 2X} e^{2\pi i \left(f(x,u)+\frac{kx}{2X}\right)} \right| \right) \times \frac{1}{2X} \sum_{k=0}^{2X-1} \min\left(X, \frac{1}{\|k/(2X)\|}\right), \end{aligned}$$

where $\|\cdot\|$ is the distance from a real number to the nearest integer.

It is known (see, e.g., [8, 9]) that for $X \ge 2$,

$$\frac{1}{2X} \sum_{k=0}^{2X-1} \min\left(X, \frac{1}{\|k/(2X)\|}\right) \ll \ln X.$$

Then for a certain $k_0$, $0 \le k_0 \le 2X$, we have $|S| \le S' \ln X$, where

$$S' = \sum_{YZ<u\le 4YZ} \tau_0(u) \left| \sum_{X<x\le 2X} e^{2\pi i \left(f(x,u)+\frac{k_0 x}{2X}\right)} \right|.$$

Further, we set $h = V^{2/9}$. Let us divide the closed interval $[X, 2X]$ into closed intervals of the form $[X_m, X_{m+1}]$ of length $h$; moreover, set $X_m = X + mh$, $m \le 0$. The last closed interval can have a smaller length. The number of such closed intervals $\ll Xh^{-1}+1$. Denote by $S_m$ the part of the sum $S'$ corresponding to the interval $(X_m, X_{m+1}]$,

$$S_m = \sum_{YZ<u\le 4YZ} \tau_0(u) \Bigg| \sum_{X_m<x\le X_{m+1}} e^{2\pi i\left(f(x,u)+\frac{k_0x}{2X}\right)}\Bigg|.$$

Square $S_m$ and use the Cauchy inequality. We have

$$S_m^2 \le \sum_{YZ<u\le 4YZ} \bigl(\tau_0(u)\bigr)^2 \sum_{YZ<u\le 4YZ} \Bigg| \sum_{X_m<x\le X_{m+1}} e^{2\pi i\left(f(x,u)+\frac{k_0x}{2X}\right)}\Bigg|^2 \ll YZ\ln^3 T(YZh+T_m),$$

where

$$T_m = \sum_{1\le\xi\le h}\ \sum_{X_m<z\le X_{m+1}-\xi}\ \sum_{YZ<u\le 4YZ} e^{2\pi i\left(f(x+\xi,u)-f(x,u)+\frac{k_0\xi}{2X}\right)}.$$

To estimate the sum $T_m$, we need in the following assertion.

**Lemma 2.** *Let $H>0$, $U^2\gg A\gg 1$, $0<r-q\ll U$. Assume also that $f(x)$ and $\varphi(x)$ are two real-valued functions such that*

$$A^{-1}\ll f''(x)\ll A^{-1}, \qquad \varphi(x)\ll H$$

*on the interval $q<x\le r$, and let $\varphi(x)$ have a finite number of monotonicity parts. Then the following estimate holds:*

$$\sum_{q<x\le r} \varphi(x)e^{2\pi i f(x)} \ll H\biggl(\frac{U}{\sqrt{A}}+\sqrt{A}+\ln(U+1)\biggr).$$

**Proof.** See [9].

In the case of the sum $T_m$, we take $\varphi(x)\equiv 1$ and the function

$$F(u) = f(x+\xi,u)-f(x,u)$$

as $f(x)$ in this lemma. The parameters of Lemma 2 have the following values:

$$H=1, \qquad U=YZ, \qquad A=n^{-1}\xi^{-1}X^{1-\gamma}(YZ)^{2-\gamma}.$$

Then we obtain from this lemma that

$$T_m \ll n^{1/2}h^{5/2}V^{\gamma/2}X^{-1/2}+n^{-1/2}h^{3/2}V^{1-\gamma/2}X^{-1/2}+h^2\ln T.$$

This implies

$$|S| \ll \bigl(Vh^{-1/2}+n^{1/4}h^{1/4}V^{1/2+\gamma/4}X^{1/4}+n^{-1/4}h^{-1/4}V^{1-\gamma/4}X^{1/4}+X(\ln T)^{1/2}\bigr)(\ln T)^{5/2}.$$

Hence

$$A_{21} \ll \bigl(Vh^{-1/2}+N^{1/4}h^{1/4}V^{1/2+\gamma/4}X^{1/4}+h^{-1/4}V^{1-\gamma/4}X^{1/4}\bigr)(\ln T)^{5/2} \ll V^{8/9}T^{\varepsilon}.$$

Since $1 < c < 9/8$, we obtain from the latter inequality the following admissible estimate for the value $A_{21}$:

$$A_{21} \ll T(\ln T)^{-4}.$$

Case (2) is now clear.

Consider now case (3). Since

$$1 \le X \le Y \le V^{2/9+\mu}, \qquad XYZ = V,$$

we have $Z = V(XY)^{-1} \ge V^{5/9-2\mu}$.

To estimate the trigonometrical sum

$$S_0 = \sum_{Z<z\le Z_1} e^{2\pi i n(xyz+\alpha)^\gamma}$$

we need in the following assertion.

**Lemma 3.** *Let $k \ge 2$, $K = 2^{k-1}$, and let the function $f(x)$ have $k$th continuous derivative on the closed interval $[a,b]$; moreover, assume that*

$$0 < \lambda_k \le f^{(k)}(x) \le h\lambda_k$$

*for all $x \in [a,b]$.*

*Then the following estimate holds:*

$$\left|\sum_{a<n\le b} e^{2\pi i f(x)}\right| \ll (b-a)\lambda_k^{1/(2K-2)}h^{2/K} + (b-a)^{1-2/K}\lambda_k^{-1/(2K-2)},$$

*where the constant in the sign $\ll$ is absolute.*

**Proof.** See [16, p. 27, Theorem 5].

Set $k = 3$ in this lemma. We have

$$\lambda_3 \asymp nV^\gamma Z^{-3}, \qquad h \asymp 1.$$

Then, by Lemma 3, the following estimate holds:

$$S_0 \ll n^{1/6}V^{\gamma/6}Z^{1/2} + n^{-1/6}V^{-\gamma/6}Z^{5/6}.$$

Hence we obtain

$$\begin{aligned} A_{21} &\ll \sum_{n\le N} n^{-1}XY(n^{1/6}V^{\gamma/6}Z^{1/2} + n^{-1/6}V^{-\gamma/6}Z^{5/6}) \\ &\ll N^{1/6}V^{1+\gamma/6}Z^{-1/2} + V^{1-\gamma/6}Z^{-1/6} \ll V^{8/9+\mu}. \end{aligned}$$

Therefore the estimate of the value $A_{21}$ is sufficient for the asymptotical formula of $A(T)$.

Case (3) is now clear.

Finally, consider case (4). If $Y \le Z \le V^{7/18-\mu/2}$, then we have from the condition $V = XYZ \le V^{7/9-\mu}$ that $X \ge V^{2/9+\mu}$. Hence these values of $X, Y, Z$ refer to the already considered case (2). Therefore we shall assume that $V^{7/18-\mu/2} \le Z \le V^{11/18}$.

Let now $\tau_0(u)$ denote the number of solutions to the equation $u = xz$, $X < x \le X_1 \le 2X$, $Z < y \le Z_1 \le 2Z$. Then we have

$$S = \sum_{X<x\le X_1} \sum_{\substack{Y<y\le Y_1\\ xyz\le T^c}} \sum_{Z<z\le Z_1} e^{2\pi i n(xyz+\alpha)^\gamma} = \sum_{XZ<u\le X_1Z_1} \tau_0(u) \sum_{Y<y\le Y_1'} e^{2\pi i f(y,u)},$$

where $f(y,u) = n(yu+\alpha)^\gamma$, $Y_1' = \min(Y_1, T^c u^{-1})$.

Set $h = V^{2/9}$. Divide the closed interval $[Y, 2Y]$ into closed intervals of the form $[Y_m, Y_{m+1}]$ of length $h$; moreover, set $Y_m = Y + mh$, $m \le 0$. The number of such closed intervals does not exceed $Yh^{-1} + 1$.

Let us estimate the sum

$$S_m = \sum_{XZ<u\le 4XZ} \tau_0(u) \left| \sum_{Y_m<y\le Y_{m+1}} e^{2\pi i \left(f(y,u) + \frac{k_0 y}{2Y}\right)} \right|,$$

corresponding to the interval $(Y_m, Y_{m+1}]$. Square $S_m$ and use the Cauchy inequality. We obtain

$$|S_m|^2 \ll YZ \ln^3 T (YZh + T_m),$$

where

$$T_m = \sum_{1\le\xi\le h} \sum_{Y_m<y\le Y_{m+1}-\xi} \sum_{XZ<u\le 4XZ} e^{2\pi i\left(f(y+\xi,u) - f(y,u) + \frac{k_0\xi}{2Y}\right)}.$$

Let us transform the interior sum $T_m(y)$ with respect to the variable $u$ with the help of the following lemma by I. M. Vinogradov.

**Lemma 4.** *Let $H, U, A, q, r$ be real numbers such that*

$$H > 0, \qquad U \gg A \gg 1, \qquad 0 < r - q \le U.$$

*Let further $f(x)$ and $\varphi(x)$ be two real-valued functions whose powers are bounded, and the following conditions hold on the closed interval $q \le x \le r$:*

$$A^{-1} \ll f''(x) \ll A^{-1}, \quad f'''(x) \ll \frac{1}{AU}, \qquad \varphi(x) \ll H, \quad \varphi'(x) \ll HU^{-1}, \quad \varphi''(x) \ll HU^{-2}.$$

*Then we have the formula*

$$\sum_{q\le x\le r} \varphi(x) e^{2\pi i f(x)} = \sum_{f'(q)\le n\le f'(r)} b_n Z_n + O\big(HT_q + HT_r + H\ln(U+1)\big),$$

*where*

$$Z_n = \frac{1+i}{\sqrt{2}} \frac{\varphi(x)}{\sqrt{f''(x_n)}} e^{2\pi i(-nx_n - f(x_n))}$$

*if $x_n$ is defined from the equation $f'(x_n) = n$; moreover, $b_n = 1$ if $n$ is different from $f'(q)$ and $f'(r)$, and $b_n = \frac{1}{2}$ if $n$ is equal to one of these numbers. In all the cases, we have $Z_n \ll H\sqrt{A}$. Finally,*

$$T_q = \begin{cases} 0 & \text{for integer } f'(q) \\ \min\left(\dfrac{1}{f'(q)}, \sqrt{A}\right) & \text{in the opposite case,} \end{cases}$$

*and*

$$T_r = \begin{cases} 0 & \textit{for integer } f'(r) \\ \min\left(\dfrac{1}{f'(r)}, \sqrt{A}\right) & \textit{in the opposite case.} \end{cases}$$

**Proof.** See [9].

Let $F(y,u) = f(y+\xi,u) - f(y,u)$. Let us use Lemma 4 for the transformation of the interior sum. We have

$$T_m(y) = \sum_{XZ<u\le 4XZ} e^{2\pi i F(y,u)}.$$

In the notation of this lemma, we have

$$H=1, \qquad U \asymp XZ, \qquad A \asymp \frac{1}{|F''_{uu}|} \asymp n^{-1}\xi^{-1}V^{1-\gamma}XZ = A_1.$$

Set $\nu_1 = F'_u(y,u)|_{u=4XZ}$, $\nu_2 = F'_u(y,u)|_{u=XZ}$. Further, we determine the values $u_\nu = u(\nu,y)$ by the relation $F'_u(y,u_\nu) = \nu$.

In the notation of Lemma 4, we have

$$T_m(y) = K(y) + O\bigl(n^{1/2}\xi^{1/2}V^{-(1-\gamma)/2}(XZ)^{-1/2} + \ln T\bigr),$$

where

$$K(y) = \sum_{\nu_1<\nu\le\nu_2} e^{2\pi i(F(y,u_\nu)-\nu u_\nu+1/8)}|F''_{uu}(y,u_\nu)|^{-1/2}.$$

Hence

$$T_m = \sum_{1\le\xi\le h}\sum_{Y_m<y\le Y_{m+1}-\xi} K(y) + O\bigl(hn^{1/2}\xi^{1/2}V^{-(1-\gamma)/2}(XZ)^{-1/2}\bigr).$$

Let us transform the sum

$$K = \sum_{Y_m<y\le Y_{m+1}-\xi} K(y).$$

First, we change the order of summation with respect to $\nu$ and $y$. Let $\nu_1=\nu_1(y)$ and $\nu_2=\nu_2(y)$ be the limits of varying of the variable $\nu$. Let $y_1=y_1(\nu)$ and $y_2=y_2(\nu)$ be the inverse functions to $\nu_1=\nu_1(y)$ and $\nu_2=\nu_2(y)$. Then we obtain

$$K = \sum_{\nu_1(Y_{m+1}-\xi)<\nu\le\nu_2(Y_m)}\ \sum_{y'_1<y\le y'_2} |F''_{uu}(y,u_\nu)|^{-1/2} e^{2\pi i(F(y,u_\nu)-\nu u_\nu+1/8)},$$

where $y'_1 = \max\bigl(Y_m, y_1(\nu)\bigr)$, $y'_2 = \min\bigl(Y_{m+1}-\xi, y_2(\nu)\bigr)$.

Denote by $L(\nu)$ the sum

$$L(\nu) = \sum_{y'_1<y\le y'_2} |F''_{uu}(y,u_\nu)|^{-1/2} e^{2\pi i(F(y,u_\nu)-\nu u_\nu+1/8)}.$$

Then

$$K = \sum_{\nu_1(Y_{m+1}-\xi)<\nu\le\nu_2(Y_m)} L(\nu).$$

The parameter $\nu$ in the sum $K$ has the order $\asymp n\xi V^{\gamma-1}+1$. Denote by $H(y)$ the function in the exponential of the trigonometrical sum $L(\nu)$, i.e.,

$$H(y)=F(y,u_\nu)-\nu u_\nu+\frac{1}{8}.$$

Further, we estimate the derivatives of the function $H(y)$. To this end, we need in the following assertion.

**Lemma 5.** *Let the function $g(x,y,z)=n(xyz)^\gamma$, $\frac{1}{2}<\gamma<1$, be given; assume that $F(x,y,z)$ coincides either with $g(x,y,z)$ or with the function $n(xyz+1)^\gamma$. Let further*

$$X\le Y\le Z,\qquad X<x\le 2X,\quad Y<y\le 2Y,\quad Z<z\le 2Z,$$
$$0\le r\le X^{0.9},\quad 0\le s\le Y^{0.9},\quad 0\le t\le Z^{0.9}.$$

*Then we have*

$$\begin{aligned}F^{(k+l+m)}_{u^kv^lw^m}(u,v,w)\big|_{\substack{u=x+r\\ v=y+s\\ w=z+t}}&=g^{(k+l+m)}_{x^ky^lz^m}(x,y,z)\big(1+O(X^{-0.1})\big)\\&=\frac{g(x,y,z)}{x^ky^lz^m}\,\frac{\gamma(\gamma-1)\cdots(\gamma-k+1)}{k!}\,\frac{\gamma(\gamma-1)\cdots(\gamma-l+1)}{l!}\\&\quad\times\frac{\gamma(\gamma-1)\cdots(\gamma-m+1)}{m!}\big(1+O(X^{-0.1})\big).\end{aligned}$$

**Proof.** See [13, p. 44].

By the definition of the value $\nu$ in Lemma 4, we have the equality $F'_u=\nu$. Using this equality, we find that

$$H'_y=(F-\nu u)'_y=F'_y+F'_uu'_y-\nu u'_y=F'_y+(F'_u-\nu)u'_y=F'_y.$$

Further, we have

$$H''_{yy}=(F'_y)'_y=F''_{yu}u'_y+F''_{yy}.$$

Let us find $u'_y$. We obtain from the equality $\nu=F'_u\big(y,u(\nu)\big)$ for a fixed integer value of the parameter $\nu$ that $\big(F'_u\big(y,u(\nu)\big)\big)'_y=0$, i.e.,

$$F'_{uy}+F''_{uu}u'_y=0,\qquad u'_y=-\frac{F''_{uy}}{F''_{uu}}.$$

Thus, we have

$$H''_{yy}=-F''_{yu}\frac{F''_{uy}}{F''_{uu}}+F''_{yy}=\frac{G}{F''_{uu}},$$

where $G=G(y,\nu)$ is the Hessian of the function $F\big(y,u(\nu)\big)$,

$$G=\begin{vmatrix}F''_{zz} & F''_{zu}\\ F''_{zu} & F''_{uu}\end{vmatrix}.$$

Let us compute the Hessian $G$. Since, by the Lagrange theorem, we have

$$F(y,u) = f(y+\xi,u) - f(y,u) = \xi f'_v(v,u)\big|_{v=z+\xi_1},$$

where $\xi_1 \in (y, y+\xi)$, in order to compute the values of the functions $F''_{yy}, F''_{uy}, F''_{uu}$, it is necessary to find $f'''_{y^3}, f'''_{uy^2}, f'''_{u^2y}$. They are found by Lemma 5.

We have

$$f'''_{y^3} = \frac{n(uy)^\gamma}{y^3}\,\frac{\gamma(\gamma-1)(\gamma-2)}{3!}\big(1+O(Y^{-0.1})\big),$$
$$f'''_{uy^2} = \frac{n(uy)^\gamma}{uy^2}\gamma\frac{\gamma(\gamma-1)}{2}\big(1+O(Y^{-0.1})\big),$$
$$f'''_{u^2y} = \frac{n(uy)^\gamma}{u^2y}\gamma\frac{\gamma(\gamma-1)}{2}\big(1+O(Y^{-0.1})\big).$$

Hence

$$G = \begin{vmatrix} F''_{yy} & F''_{yu} \\ F''_{yu} & F''_{uu} \end{vmatrix} = \xi^2 \begin{vmatrix} f'''_{y^3} & f'''_{uy^2} \\ f'''_{uy^2} & f'''_{u^2y} \end{vmatrix} = n^2\xi^2(uy)^\gamma \begin{vmatrix} \dfrac{\gamma(\gamma-1)(\gamma-2)}{6y^3} & \dfrac{\gamma^2(\gamma-1)}{2uy^2} \\ \dfrac{\gamma^2(\gamma-1)}{2uy^2} & \dfrac{\gamma^2(\gamma-1)}{2u^2y} \end{vmatrix} \big(1+O(Y^{-0.1})\big)$$
$$= \left(\frac{\gamma^4(\gamma-1)^2}{4u^2y^4} - \frac{1}{12}\,\frac{\gamma^3(\gamma-1)^2(\gamma-2)}{u^2y^4}\right) n^2\xi^2(uy)^\gamma\big(1+O(Y^{-0.1})\big).$$

It follows there from

$$|G| \asymp n^2\xi^2V^{\gamma-2}Y^{-2}.$$

Compute now the order of the value $H'''_{yyy}$. We have

$$H'''_{yyy} = \frac{G'_yF''_{uu} - GF'''_{uuy}}{(F''_{uu})^2} \asymp GY^{-1}|F''_{uu}|^{-1}.$$

Let us apply Lemma 3 for $k=3$ in order to estimate the sum $L(\nu)$. We obtain

$$L(\nu) \ll |F''_{uu}|^{-1/2}\big(hG^{1/6}Y^{-1/6}|F''_{uu}|^{-1/6} + h^{3/4}G^{-1/6}Y^{1/6}|F''_{uu}|^{1/6}\big).$$

Thus we find

$$T_m \ll \sum_{1\le\xi\le h} n\xi V^{1-\gamma}|F''_{uu}|^{-1/2}\big(hG^{1/6}Y^{-1/6}|F''_{uu}|^{-1/6} + h^{3/4}G^{-1/6}Y^{1/6}|F''_{uu}|^{1/6}\big).$$

Substituting the estimates of the values $G$, $F''_{uu}$, $Y$ and the value of the parameter $h$, we obtain the admissible estimate of the value $A_{21} \ll V^{8/9}$.

The theorem is proved.

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